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COPOLYFORM MODULES

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ABSTRACT

A module M is called copolyform module if for any small submodule N of M, Hom(M,N/K)=0 for all submodules K of N. It is shown that rational numbers, and in general, fields of fractions of integral domains are copolyform modules and for a copolyform and lifting module M, S=End(M) is left and right principally projective ring, and if M is copolyform and $\sum-$ lifting module then S=End(M) is left and right semihereditary ring.

1. INTRODUCTION

Throughout this note all rings will have an identity and modules will be unital right modules. Let M be a module over a ring R. $N \leq M$ will stand for N a submodule of M. Let N be a submodule of M. N is said to be small in M and we write N << M whenever N+L=M for some submodule L of M implies M=L. A module M is called small if M is small in E(M), where E(M) is the injective hull of M. M is said to be hollow in case every proper submodule of M is small in M. In what follows Rad(M) will denote the radical of a module M and J(R) will be the Jacobson radical of a ring R. Rad(M) is the sum of all small submodules of M and intersection of maximal submodules of M. We call a module M copolyform if for any N << M, Hom(M,N/K) = 0 for all $K \leq N$.

In this note we study some properties of copolyform modules [5] in which they are defined as a dual of polyform modules [12]. Here we give a different approach and study some general properties of copolyform modules. Let **Z** and **Q** denote the integers and rational numbers, respectively.

2. PRELIMINARIES

In this section we study some small submodules of rationals. Some results of this section may be in the context. For the sake of completeness and as a preparotory section we give proofs in detail. We start with

Lemma 1. Let M be an R-module and $N \lneq M$ a proper submodule of M such that Hom(M,N/L)=0 for all $L \leq N$. Then N is small in M.

Proof. Let N be a proper submodule of M. Assume Hom(M,N/L)=0 for all $L \leq N$. We take $K \leq M$ such that M=K+N. If $K \neq M$, since $M/K \cong N/(K \cap N)$ then Hom(M,N/$(K \cap N)$) $\neq 0$.. Hence N<<M.

Lemma 2. Let R be a principal ideal domain with field of fractions K. Then for any R-submodule L of R, Hom(K,R/L)=0.

Proof. Let L be an R-submodule of R, $r/s \in K$, $t \in L$ and $f \in$ Hom(K,R/L). Set f(r/ts)=y+L for some $y \in R$. Then f(r/s)=ty+L= 0 since $ty \in L$. So f(r/s)=0 for all $r/s \in K$. Hence f=0.

Corollary 3. Let R be a principal ideal domain with field of fractions K. Then for any submodule N of R, Hom(K,N/L)=0 for all submodules L of N.

Proof. The same proof of Lemma 2 works here.

Corollary 4. Let R be a principal ideal domain with field of fractions K. Then every submodule N of R is small in K.

Proof. Clear from Lemma 1 and Corollary 3.

Definition 5. Let M be a module with a submodule N. M is called quasi-corational extension of N if Hom(M,N/L)=0 for all submodules L in N.

Corollary 6. Let R be a principal ideal domain with field of fractions K. Then K is quasi-corational extension of R.

Lemma 7. Let **Z** and **Q** denote the integers and rational numbers, respectively and let a/b, c/d $\in \mathbf{Q}$ and let N denote the submodule (a/b)**Z**+(c/d)**Z** of **Q**. Then Hom(**Q**,N/L) = 0 for all $L \leq N$.

Proof. Let L be a submodule of N, $f \in \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Q}, N/L)$ and $f(1) = (a/b)n + (c/d)m + L$ $= ((adn + bcm)/bd) + L$ for some $n,m \in \mathbf{Z}$. Set t=adn+bcm so that f(1)=(t/bd)+L. Take $0 \neq t_1 \in L \cap \mathbf{Z}$. Then there exists $u \in \mathbf{Z}$ such that $f(1/t_1) = (u/bd) + L$. Hence $f(1) = (t_1u/bd) + L$ and so $f(bd) = t_1u + L = 0$ since $t_1u \in L$. Hence f(1) =(t/bd)+L implies f(bd) = t+L = 0 or $t \in L$. Let $1/y \in \mathbf{Q}$ and set $f(1/y) = (t_2/bd) + L$ and $f(1/ty) = (t_3/bd) + L$ for some $t_2, t_3 \in \mathbf{Z}$. Then $f(1/y) = (tt_3/bd) + L$ and so $t_2 - tt_3 \in L$. Since $t \in L$ then $t_2 \in L$. It follows that if f(1/ybd)=(v/bd)+L for some $v \in \mathbf{Z}$ then $v \in L$ and f(1/y)=v+L. Thus f(1/y) = 0 for all $1/y \in \mathbf{Q}$. Let x/y be any element of **Q**. Then f(x/y)=f(1/y)x = 0. Hence f = 0.

We record the following theorem which is well known (in [8] page 108,Example.5)

Theorem 8. Let N be a finitely generated **Z**-submodule of **Q**. Then N is small in **Q**.
Proof. Let N be a finitely generated **Z**-submodule of **Q**. An induction on the generators of N and applying the same proof of Lemma 7 may complete the proof.

There are small submodules of **Q** that are not finitely generated as **Z**-modules.

Lemma 9. Let N denote the **Z**-submodule $\sum_p (1/p)\mathbf{Z}$ of **Q**, where p ranges over all prime integers. Then N is small in **Q**.
Proof. Let N = $\sum_p (1/p)\mathbf{Z}$ and L a submodule of **Q** such that **Q**=N+L. We prove **Q**=L. Since **Z** is small in **Q**, to complete the proof we assume $\mathbf{Q} \neq \mathbf{Z}+\mathrm{L}$ and get a contradiction. Now assume $\mathbf{Q} \neq \mathbf{Z}+\mathrm{L}$. Then $\mathrm{N} \neq (\mathrm{N} \cap \mathrm{L})+\mathbf{Z}$. Since $\mathrm{p}[((1/\mathrm{p})\mathbf{Z})/\mathbf{Z}]=0$ then N/**Z** is semisimple. So $\mathrm{N}/((\mathrm{N} \cap \mathrm{L})+\mathbf{Z})$ is semisimple as a homomorphic image of N/**Z**. Hence there exists a maximal submodule H of N with $(\mathrm{N} \cap \mathrm{L})+\mathbf{Z} \leq \mathrm{H} \leq \mathrm{N}$. Since $\mathbf{Q}/(\mathrm{H}+\mathrm{L})=(\mathrm{N}+\mathrm{L})/(\mathrm{H}+\mathrm{L}) \cong \mathrm{N}/\mathrm{H}$ and N/H is simple then H+L is maximal submodule of **Q**. This is the desired contradiction.

Theorem 10. Let S_0 denote the **Z**-module **Z** and let S_n denote the **Z**-submodule $\sum_p (1/p^n)\mathbf{Z}$ of **Q**, where p ranges over all prime integers and n = 1,2,3,... . Then

(1) S_n (n = 0,1,2,3,...) are small submodules of **Q** and S_n (n = 1,2,3,...) are not finitely generated **Z**-modules.

(2) Hom(**Q**, S_n/K) = 0 for every n = 0,1,2,3... and all submodules K of S_n.

Proof (1) : We proceed induction on n. For n = 0 (1) follows from Lemma 2 and Corollary 4, for n = 1, (1) follows from Lemma 9.

Assume n > 1 and S_k is small in **Q** for all k with k < n. Suppose that **Q** = L + S_n for some $\mathrm{L} \leq \mathbf{Q}$. Since S_{n-1} is small in **Q**, to complete the proof we assume $\mathbf{Q} \neq \mathrm{L}+\mathrm{S}_{n-1}$ and get a contradiction. So assume $\mathbf{Q} \neq \mathrm{L}+\mathrm{S}_{n-1}$. Then $\mathrm{S}_n \neq \mathrm{S}_n \cap \mathrm{L}+\mathrm{S}_{n-1}$.. Since $\mathrm{p}[(1/\mathrm{p}^n)\mathbf{Z}/(1/\mathrm{p}^{n-1})\mathbf{Z}]=0$, $\mathrm{S}_n/\mathrm{S}_{n-1}$, is semisimple, and then $\mathrm{S}_n/((\mathrm{S}_n \cap \mathrm{L})+\mathrm{S}_{n-1})$ is semisimple as a homomorphic image of semisimple module S_n/S_{n-1}. Hence there exists a maximal submodule H_n of S_n containing $\mathrm{S}_n \cap \mathrm{L}+\mathrm{S}_{n-1}$.. It is easy to check that H_n + L is a maximal submodule of **Q**. This is the desired contradiction. This completes the proof of (1).

(2) : The case $n = 0$ follows from Lemma 2. Assume Hom($\mathbf{Q}$,S_n/K) is nonzero for some positive integer n and $K \le S_n$. Let f be a nonzero element in Hom($\mathbf{Q}$,S_n/K). Suppose that $S_n \ne S_{n-1} + K$. Since S_n/S_{n-1} is semisimple, $S_n/(S_{n-1}+K)$ is semisimple as a homomorphic image of S_n/S_{n-1}. Then $\mathbf{Q}$/Ker f is isomorphic to a submodule of a semisimple module and so $\mathbf{Q}$ has a maximal submodule. This is a contradiction. Hence $S_n = S_{n-1} + K$ and so $S_n/K \cong S_{n-1}/(S_n/S_{n-1} \cap K) \ne 0$. Now assume $S_{n-1} \ne S_{n-2} + S_{n-1} \cap K$. Then by the same reasoning $S_n/(S_{n-2} + S_{n-1} \cap K)$ is semisimple and $\mathbf{Q}$ has a maximal submodule. It follows that $S_{n-2} + S_{n-1} \cap K = S_{n-1}$ and $S_n = S_{n-2} + K$. We continue this way and get $S_1 = S_1 \cap K + S_0 = S_1 \cap K + \mathbf{Z}$, and so $S_n = K + \mathbf{Z}$. Then we may replace S_n/K by $\mathbf{Z}/v\mathbf{Z}$ (for some $v \in \mathbf{Z}$) in Hom($\mathbf{Q}$,S_n/K). By Lemma 2 Hom($\mathbf{Q}$,$\mathbf{Z}/v\mathbf{Z}$)=0. This contradicts the assumption

Let N be a small submodule of $\mathbf{Q}$ and $f \in \mathrm{Hom}(\mathbf{Q}, N/K)$ for some submodule K of N. Let α denote the homomorphism from N/K onto (N+Z)/(K+Z) defined by $\alpha(n+K) = n + (K + \mathbf{Z})$ where $n + K \in N/K$. By Lemma 2, and since $\mathrm{Ker}\alpha = (K + \mathbf{Z})/K \cong \mathbf{Z}/(K \cap \mathbf{Z})$, we $\mathrm{Hom}(\mathbf{Q}, \mathbf{Z}/(K \cap \mathbf{Z})) = 0$. Hence to prove f is zero homomorphism, without loss of generality, we may assume in Lemma 12 and Theorem 13 that N and K contain $\mathbf{Z}$ in case K is a nonzero submodule of N. For an easy reference we record Lemma 11.

Lemma 11. Let N be a submodule of $\mathbf{Q}$ and $ab/c^i \in N$ for some $ab/c^i \in \mathbf{Q}$ with (a,c) = 1. Then a, ab and b/c^j are in N for all j with $1 \le j \le i$.

Lemma 12. Let N be a small $\mathbf{Z}$-submodule of $\mathbf{Q}$ and let $f \in$ Hom($\mathbf{Q}$,N/K) be a nonzero homomorphism for some $K \le N$ and $t \in \mathbf{Z}$ such that $\mathrm{Ker} f \cap \mathbf{Z} = t\mathbf{Z}$. Then

(1) If f(1) is nonzero then there exist an integer a and a positive integer k such that $f(1) = a/t^k + K$ and $a/t^k \in N$.

(2) If $y \in \mathbf{Z}$ with (y,t) = 1 and $f(1/y) = b/t^l + K$ for some integer b and positive integer l with $b/t^l \in N$ then k = l.

Proof. (1): Let N be a small submodule of $\mathbf{Q}$ and $f \in \mathrm{Hom}(\mathbf{Q}, N/K)$ be a nonzero homomorphism for some $K \le N$. Then $f(1) \ne 0$. Hence $f(1) = x/b + K$ for some $x/b \in N$ with $x \ne 0$. Then f(b) = 0. There exist positive integers k and y such that $b = t^k y$ with (t,y) = 1 and so $f(1) = c/t^k + d/y + K$ for some $c,d \in \mathbf{Z}$. Since f(t) = 0 and (t,y) = 1, $dt^k/y \in K$, and so by Lemma 11, $d/y \in K$. Hence $f(1) = c/t^k + K$. We choose a with (a,t) = 1 and k as small as to $f(1) = a/t^k + K$.

(2): Assume first that $k \gneq l$. Since $a/t^{k-1} \in K$ and (a,t) = 1 and $k-1 \ge 1$, by Lemma 11, $b/t^{\ell} \in K$. Hence f(1) = 0. Now suppose that $k \lneq l$. Since $f(1) = yb/t^{\ell} + K$,

$yb/t^{\ell-1} \in K$. By (b,t) = 1 and then by Lemma 11, $1/t^{\ell-i} \in K$ for all $1 \le i \le \ell$. Since $k \le \ell - 1$, we have ya/t^k+K. Now f(y) = f(1)y = ya/t^k + K implies f(y) = 0. Hence f(1) = 0. This is a contradiction. Thus k = ℓ.

Theorem 13. Let N be any small **Z**-submodule of **Q**. Then Hom(**Q**,N/K) = 0 for all $K \le N$.
Proof. Let N be a small submodule of **Q** and $f \in$ Hom(**Q**,N/K) for some $K \le N$ and $\mathrm{Ker}f \cap K = t\mathbf{Z}$ as in Lemma 12. By Lemma 12, there exists a positive integer k such that for $x/y \in \mathbf{Q}$ $f(x/y) = m/t^k + K$ for some $m \in \mathbf{Z}$ with $m/t^k \in N$. It follows that $f(\mathbf{Q}) \le (S_n + K)/K$ for some positive integer n. Since $(S_n + K)/K \cong S_n/(S_n \cap K)$ $(S_n + K)$, by Lemma 10 (2), f = 0.

3. COPOLYFORM MODULES

Definition 14. Let M be a module. M is called comonoform module if for any $N \lneq M$, Hom$_R$(M,N/L)=0 for all $L \le N$.

Definition 15. Let M be a module. We call M a copolyform module if for any small submodule N of M, Hom$_R$(M,N/L)=0 for all $L \le N$. In comparing with Lemma 1 copolyform modules are those modules that satisfy the converse statement of Lemma 1.

We call a ring R comonoform (copolyform) ring provided R is comonoform (copolyform) right R-module. It is clear from definitions that a ring R is copolyform if and only if J(R) = 0. Every comonoform module is copolyform. For the ring of integers **Z**, J(**Z**) = 0 and $\mathbf{Z} \cong 2\mathbf{Z}$ then **Z** is copolyform but not comonoform.

A module M is comonoform if and only if M is quasi-corational extension of every submodule N with $0 \le N \lneq M$, and M is copolyform if and only if M is quasi-coratinal extension of every small submodule in M. By Theorem 13, **Q** is quasi-corational extension of every small submodule.

Corollary 16. Let M be a copolyform module. A submodule N of M is small in M if and only if Hom(M,N/L)=0 for all $L \le N$.
Proof. By definitions and Lemma 1.

We note that for a module M and a submodule N of M whenever $N \ll M$ implies $N \ll E(M)$. The converse is not true in general. There may happen a module M with a submodule N such that N is small in E(M) but N is not small in M. Namely 2**Z** is not small in **Z** but by Corollary 3 it is small in **Q**.

Lemma 17. Let M be a module.
(1) If M is comonoform then M is hollow.

(2) If M is hollow and copolyform then M is comonoform.
Proof. Clear from definitions.

Definition 18. Let M be an R-module. We set $Z^*(M)=\{m\in M{:}mR \text{ is small}\}$ (see,namely [6]). We remark that it is known (and easy to prove) that $Z^*(M)=0$ implies $Z^*(E(M))=0$, and if $M=M_1\oplus M_2$ then $Z^*(M)=Z^*(M_1)\oplus Z^*(M)$. By definition, $Z^*(M)=M\cap Rad(E(M))$ and $Rad(M)\subseteq Z^*(M)$. So if M is a module with $Z^*(M)=0$ then M is a copolyform module.

Lemma 19. Let R be a ring and E(R) denote the injective hull of R. Then $R\oplus E(R)$ is copolyform module if and only if $Z^*(R)=0$.
Proof. Assume $R\oplus E(R)$ is copolyform module. Let $x\in Z^*(R)$. Then xR is small in E(R). It is clear that xR is small in $R\oplus E(R)$. Now define $R\oplus E(R)\xrightarrow{h}R\xrightarrow{g}xR$; $(r,t)\to r\to xr$ where $r\in R$ and $t\in E(R)$. Set f=gh. By hypothesis, f=0. Hence x=0. For the converse, assume $Z^*(R)=0$. Then $Z^*(E(R))=0$ and so small submodules of R and E(R) are zero. Let N be a submodule of $M=R\oplus E(R)$ and π_1 and π_2 denote the projections of M on R and E(R), respectively. Since homomorphic images of small submodules are small, $\pi_1(N)$ and $\pi_2(N)$ are zero as small submodules of R and E(R), respectively. Hence N is zero. This completes the proof.

There are submodules and homomorphic images of copolyform modules which are not copolyform.

Example 20. (i). Let M denote the Prüfer p-group $\mathbf{Z}(p^\infty)$ for some prime integer p. It is known that for any submodule N with $N\neq M$, $M/N\cong M$. Let N be a submodule of M with $N\neq M$ and L any submodule of N and $f\in Hom(M,N/L)$. Set K=Ker(f). Assume $f\neq 0$. Then M/K is isomorphic to a submodule of N/L which is Noetherian. This is a contradiction since $M\cong M/K$. Then M is copolyform. Let $t\in\mathbf{Z}$ with $t\geq 4$ and $N_t=(1/p^t+\mathbf{Z})\mathbf{Z}$ denote the submodule of M such that $p^tN_t=0$. Let m and n be positive integers such that m<n<t. Then there exists a non-zero homomorphism f from N_t to N_n/N_m defined by $f(a/p^t+\mathbf{Z})=a/p^n+N_m$ where $a/p^t+\mathbf{Z}\in N_t$. Hence N_t is not copolyform.

(ii). Let M denote the **Z**-module **Z** and N the submodule $p^m\mathbf{Z}$ of M for some prime integer p and some integer m>1, and let t be an integer with t>1 and set $L=p^{mt}\mathbf{Z}$. Then M is copolyform **Z**-module and **pZ/L** is the unique maximal submodule of M/L and N/L is small in M/L. Now define f from M/L to N/L by $f(x+L)=p^mx+L$, where $x+L\in M/L$. It is clear that f is a nonzero homomorphism and so M/L is not copolyform.

In [3] it is proved that for a module M with a projective cover (P,f) M is copolyform if and only if J(End(P)) = 0. Now we prove

Theorem 21. Let M be a module and $x \in Rad(M)$. Assume M/xR has a projective cover (P,f). Then M is copolyform if and only if M/xR is copolyform.

Proof. Let N denote the submodule xR of M with $x \in Rad(M)$ and (P,f) be a projective cover of M/N. Then N and Kerf are small in M and P, respectively, and f can be lifted to a map g from P to M. It can be easily checked that g is onto and Kerg is small in P. Hence (P,g) is a projective cover of M. By the preceeding remark, M is copolyform if and only if J(End(P)) = 0 if and only if M/N is copolyform.

Corollary 22. Let R be a right perfect ring. Then a module M is copolyform if and only if M/N is copolyform for every submodule N of Rad(M).

Proof. By Remark (3),page 317 of [2] every module M over a right perfect ring has a projective cover and Rad(M) is small in M.

Lemma 23. Let M be a copolyform module.Then every direct summand of M is copolyform.

Proof. Assume that $M = M_1 \oplus M_2$ and M is copolyform module. Let $N << M_1$ and $f \in$ Hom(M_1,N/K) for some $K \leq N$. Then N<<M. Now define $f_1 : M \to N/K$, $f_1(m_1 + m_2) = f(m_1)$, where $m_1 \in M_1, m_2 \in M_2$. Then $f_1 \in$Hom(M,N/K). By assumption f=0.

Definition 24. Let M be a module. M is called lifting(orD_1-)module whenever for any submodule N of M there is a submodule A of M contained in N such that $M = A \oplus B$ for some submodule B of M with $N \cap B$ small in B[9]. We say that M is finitely $\sum$- lifting if every finite direct sum of copies of M is lifting.

Lemma 25. Let M be a copolyform module and S = End(M) the ring of endomorphisms of M.

(1) If M is lifting then S is left and right principally projective ring.
(2) If M is finitely $\sum$- lifting then S is left and right semihereditary.

Proof (1) Let $f \in S$. Since M is lifting, there exists a direct summand M_1 of M such that $M_1 \leq f(M)$ and $M = M_1 \oplus M_2$ and $f(M) \cap M_2 << M_2$ for some submodule M_2 of M. It is easy to show that $f(M) \cap M_2$ is small in M and $f(M) = M_1 \oplus (f(M) \cap M_2)$. We consider the map αf from M onto $f(M) \cap M_2$ is the composition of f with α where α is the canonical projection from f(M) onto $f(M) \cap M_2$. Since $(\alpha f)(M) = f(M) \cap M_2$ is small in M by hypothesis, $\alpha f = 0$. It follows that $f(M) = M_1$. Thus f(M) is a direct summand of M for every $f \in S$. By (39.11 in [11]) S is a right principally projective ring.

To prove S is left principally projective we take $f \in S$. The same proof of the first paragraph shows that f(M) is a direct summand of M and so f(M) = e(M) for some idempotent e in S. Let β denote the map from S onto Sf defined by $\beta(s) = sf$ where $s \in S$. Then $(1-e)f(M) = 0$ or $S(1-e) \leq \operatorname{Ker}\beta$. Let $g \in \operatorname{Ker}\beta$. Then gf = 0 and so gf(M) = ge(M) = 0 implies ge = 0 and then $g(1-e) = g \in S(1-e)$. Thus $S(1-e) = \operatorname{Ker}\beta$. Now $S/\operatorname{Ker}\beta = S/S(1-e) \cong Se$ and $\beta(S) = Sf \cong S/\operatorname{Ker}\beta \cong Se$ prove that Sf is a projective left ideal of S. Thus S is a principally projective ring.

(2) Let $S^{n\times n}$ denote the ring of $n \times n$ matrices over S for positive integer n. By (1), $\operatorname{End}(M^n) \cong S^{n\times n}$ is a left and right principally projective ring. By (39.13 in [11]), S is left and right semi-hereditary.

Definition 26. Let M be a module with dual Krull dimension $k^0(M)$ [1] . Let be an ordinal. M is called α – atomic if $k^0(M) = \alpha$ and $k^0(N) \lneq \alpha$ for each submodule N with $0 \leq N \lneq M$, and M is α – coatomic if M/N is α – atomic for every submodule N of M with $N \lneq M$. In [5] It is shown that a module M is α – atomic if and only if M is α – coatomic for some ordinal α.

Lemma 27. Let M be an α – coatomic module for some ordinal α. Then M is copolyform.

Proof. Let M be an α – coatomic module. Then $k^0(M) = \alpha$. It is known that for each submodule N with $0 \leq N \lneq M$, $k^0(N) < \alpha$ and $k^0(M/N) = \alpha$. Let N be a small submodule of M and $f \in \operatorname{Hom}(M,N/K)$ for some submodule K of N. Then $f(M)=L/K \leq N/K$ for some $L \leq N$ and then by hypothesis, $M/\operatorname{Ker}(f) \cong f(M)$ implies $k^0(f(M)) = \alpha$ and $f(M)=L/K \leq N/K$ implies $\alpha = k^0(f(M)) \leq k^0(N/K) \lneq \alpha$. This leads to f=0 and so M is copolyform.

Lemma 28. Let M be a module with a maximal submodule N. Then $k^0(M) = k^0(N)$.

Proof. Let N be a maximal submodule of M. Since M/N is simple then $k^0(M/N) \leq 0$. By $k^0(M) = \max\{k^0(N), k^0(M/N)\}$ (see namely [1]), $k^0(M) = k^0(N)$.

Lemma 29. Let M be an α – coatomic module. Then M is simple or hollow module with Rad(M) = M.

Proof. Let M be an α -coatomic module for some ordinal α. Then $k^0(M) = \alpha$ and for any proper submodule N of M $k^0(N) < \alpha$. By Lemma 28, $k^0(M) = k^0(L)$ for all maximal submodules L of M. Hence any nonzero proper submodule of M can not be maximal in M. Thus for every proper submodule N of M $k^0(N) < k^0(M)$. Assume M is not simple and M = N + L for some proper submodules N and L. From $k^0(M) = \max\{k^0(N), k^0(M/N)\}$ and $k^0(L) = \max\{k^0(L \cap N), k^0(L/L \cap N)\}$ and $M/N \cong L/(L \cap N)$ and $k^0(N)<k^0(M)$ and $k^0(L)<k^0(M)$ it follows that $k^0(M)<k^0(M)$. This contradiction shows that if M is not simple and M = N + L for some submodules N and L then M = N or M = L. Hence M is simple or hollow module with Rad(M) = M.

Proposition 30. Let M be a projective module and S the ring End(M) of endomorphisms of M. Then the followings are equivalent.

(1) M is copolyform.
(2) S is copolyform.
(3) J(S) = 0.

Proof. Let M be a projective module and S = End (M). In the proof we use the fact that for $f \in S$, $f \in J(S)$ if and only if $f(M) << M$ [2, Lemma 17.11]. By definitions, (2) and (3) are equivalent. Suppose that M is copolyform module. Let $f \in J(S)$. Then $f(M) << M$. Hence f = 0. Thus J(S) = 0. This proves that (1) implies (3). As for (3) implies (1), assume J(S) = 0. Let $N << M$ and $f \in Hom(M,N/K)$ for some $K \leq N$. Since M is projective f lifts to an element g of S. Being $N << M$ and $g(M) \leq N$ then $g(M) << M$ or $g \in J(S)$. Hence g = 0 and so f = 0.

Let M be a module. M is called V-module by Hirano [7](or cosemisimple by Fuller [4]) if every proper submodule of M is an intersection of maximal submodules. The ring R is called V-ring if the right R-module R is V-module.

Theorem 31. Let R be a ring. Then the following are equivalent.

(1) R is a V-ring.
(2) Every R-module is copolyform.
(3) For every R-module M, $Z^*(M) = 0$

Proof. The equivalence of (1) and (3) is established in [10]. Clearly (3) implies (2). Assume (2) that every R-module is copolyform. Let M be a module and $x \in Z^*(M)$. Now we consider the module $\tilde{M} = E(M) \oplus R$ as a right R-module. xR is a small submodule of the injective hull E(M) of M and so is small in $\tilde{M}$. Define the map f: $\tilde{M} \to xR$ by f(m+r)=xr where $m + r \in \tilde{M}$, $m \in E(M)$ and $r \in R$. By (2), $\tilde{M}$ is copolyform and so f = 0 or x = 0. Hence $Z^*(M) = 0$ and (3) holds.

Example 32. We want to mention some relations of copolyform modules with some classes of modules. A module M is cosemisimple if and only if Rad(M/N) = 0 for all $N \leq M$ (see [2],page 122,Exer.14). In a cosemisimple module M, Rad(M) = 0 therefore every cosemisimple module is copolyform. M is said to be coatomic if, for any submodule N of M, Rad(M/N) = M/N implies M/N = 0 [13]. Hence every cosemisimple module is coatomic.

There are coatomic modules that are neither cosemisimple nor copolyform: Consider the ring $R=\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a,b,c \in \mathbf{Z}\}$ with usual matrix operations. Then $J(R)=\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} : b \in \mathbf{Z}\}$ and so R is not copolyform. Let I be any right ideal of R. It is easy to check that I is contained in a maximal right ideal in the form $\begin{bmatrix} n\mathbf{Z} & \mathbf{Z} \\ 0 & m\mathbf{Z} \end{bmatrix}$ where either n=1 and m is a prime integer or n is a prime integer and m=1. Hence R is coatomic.

There are copolyform modules which are not cosemisimple. Namely the ring $S=\begin{bmatrix} \mathbf{Z} & \mathbf{Q} \\ 0 & \mathbf{Q} \end{bmatrix}$. Then $J(S)=\begin{bmatrix} 0 & \mathbf{Q} \\ 0 & 0 \end{bmatrix}$. Set R=S/J(S). It is easily seen that the right ideal $I=\{\begin{bmatrix} 4n & 0 \\ 0 & 0 \end{bmatrix} + J(S) : n \in \mathbf{Z}\}$ is not an intersection of maximal right ideals of R. Hence R is not cosemisimple. But R is copolyform since J(R)=0.

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