Commun. Fac. Sci. Univ. Ank. Series A1
V. 53.(No.1) pp.1-6 (2004)

SEMI-OPEN AND SEMI-CLOSED SETS IN DIGITAL TOPOLOGICAL SPACES

S.I.NADA

Mathematics Department, Faculty of Science,Minoufia University, Shebeen El-Kohm,Egypt

(Received Oct.31,2003; Accepted Jan.05, 2004)

ABSTRACT

The semi-open and semi-closed sets are classified in Khalimsky n-space, the space which appropriate the digital picture. The separation axioms, namely semi-T_1 and semi-T_2 are studied.

1. INTRODUCTION

The purpose of this paper is to study semi-open and semi-closed sets [2,7], in the digital n-space which is the product of digital lines with the Khalimský topology. Even though these digital spaces have been studied by many authors [1,4,5,6], it appears that these concepts have not been studied previously in this setting. The notions of the semi-openness and semi-closedness have been applied to compactness, connectedness, separation axioms, and continuity [3,10]. Some of these notions will be extended to digital n-space. In order to make this work self-contained, we begin by recalling some definitions. Let (X,τ) be a topological space, a subset $A \subset X$ is a semi-open if it is contained in the closure of its interior, symbolically $A \subset A^{o-}$. Also A is called semi-closed if $A^{o-} \subset A$.

2. SEMI-OPEN AND SEMI-CLOSED SETS IN THE KHALIMSKY N-SPACE:

The Khalimsky line is the set of integers Z which is equipped with the topology τ generated by the sub-base $S = \{\{2n-1,2n,2n+1\}, n \in Z\}$. If x is odd, then {x} is open in τ and if x is even, then {x-1,x,x+1} is open in τ. Of particular interest is the

Khalimsky plane Z^2, where Z^2 is endowed with the product topology τ^2. Two distinct lattice points *(a,b)* and *(c,d)* are 8-adjacent if max $\{|a-c|,|b-d|\}=1$. It is obvious that if $(a,b) \in Z^2$, where a and b are both even, then $\{(a,b)\}^o = \emptyset$. If *a* and *b* are both odd, then $\{(a,b)\}^o=\{(a,b)\}$. If *a* is even and *b* is odd, the smallest open set containing *(a,b)* is $\{(a-1,b), (a,b),(a+1,b)\}$ and lastly if *a* odd and *b* is even, then the smallest open set containing *(a,b)* is the set $\{(a,b-1), (a,b), (a,b+1)\}$. The digital n-space, the product of *n* copies of the Khalimsky line, Z^n with the product topology τ^n, has some important results. In τ^n, points all of whose coordinate are odd, are open, while points all of whose coordinates are even, are closed. These points are called pure points. The other points which are mixed, are neither open nor closed. We chose the (3^n-1)-adjacent points for each $x \in Z^n$. For such *x*, denote by *N(x)* the minimal open neighbourhood of *x* and denote by A_{op} the set of open points of *A*.

The following theorem classifies the semi-open sets in Khalimsky n-space, but let us start with the next lemma which will be used in proving our classification theorem.

Lemma 2.1. [6]. If $y \in N(x)$, then $x \in \overline{\{y\}}$. The converse is also true.

Theorem 2.2. $A \subseteq Z^n$ is semi-open if and only if for each $x \in A$, $N(x) \cap A_{op} \neq \emptyset$.

Proof. Suppose that $A \subseteq Z^n$ is semi-open and $x \in A$, if $\{x\}$ is open, then we are through. If on the other hand, $\{x\}$ is not open but $x \in A \subseteq A^{o-}$, then $A^o \cap N(x)$ is non-empty and hence must contain an open point $z \in Z^n$ because $A^o \cap N(x)$ is open. Since $N(x)$ is an open set and z is an open point, z is an adjacent to x this because x is not an open point and thus $z \in A_{op} \cap N(x)$.

For the converse, suppose that $x \in A$; if $\{x\}$ is open, then $x \in A^{o-}$ and we are through. If on the other hand $\{x\}$ is not open and $N(x) \cap A_{op} \neq \emptyset$, then there is an open point z, $z \in N(x)$. using the above lemma we see that $x \in \{z\}^-$. But $\{z\}$ is an

open subset of A,i.e. $\{z\} \subseteq A^o$, it follows that $x \in \{z\}^- \subset A^{o-}$, thus A is semi-open.

It is easy to show that the complement of every semi-open subset is semi-closed and vice versa. So, if we transform each implication in theorem (2.2), to its contrapositive, we obtain the logically equivalent next statement.

Theorem 2.3. $A \subseteq Z^n$ is semi-closed if and only if for each $x \in Z^n \setminus A$, $N(x) \cap (Z^n \setminus A)_p \neq \emptyset$.

The previous two theorems give an indication of which sets are semi-open but not semi-closed and vice versa.

Examples 2.4.

1. $\{2n+1\}$ is semi-open but not semi-closed.
2. $\{2n\}$ is semi-closed but not semi-open.
3. $\{2n,2n+1\}$ is both semi-open and semi-closed at the same time.
4. $\{2n,2n+7,2n+9\}$ is neither semi-open nor semi-closed..

3. SEMI-CONTINUOUS AND IRRESOLUTE FUNCTIONS

The main problem in image processing is to present a continuous picture in R^n on a computer screen using a finite set of points. So it is important to translate a continuous picture from R^3 or R^2 to Z^2 by using digital points. The greatest lower integer function explains how this translation might occur. Consider the real line with the usual topology ν. Recall that a function *f* from *X* to *Y* is semi-continuous if and only if every open set in *Y* has a semi-open inverse in *X*. The Khalimsky line was defined to be a quotient space of the real line in which $2n \to 2n$ and the open interval $]2n\text{-}2, 2n[\to 2n\text{-}1$. This usual function, of course, is continuous. However, it is obvious that

$$f: (R, \nu) \to (Z, \tau)$$
$$x \to [x]$$

is semi continuous, where $[x]$ denotes the greatest lower integer function.

One can easily show that *f* is not continuous for example $f^{-1}\{2n\text{-}1, 2n, 2n+1\} = [2n\text{-}1, 2n+2[$ which is not open in ν.

Recall that *f* is irresolute [8] if and only if the inverse image of a semi-open subset in τ is again a semi-open subset in ν.

Theorem 3.1. *f* is an irresolute function.

Proof. By using theorem (2.2) one can see that any semi-open subset in τ may be represented as a union of semi-open subsets in the form $U = \{2n, 2n\text{-}1\}$ (or $\{2m\text{-}1\}$). Since $f^{-1}\left(\underset{i}{\cup} A_i\right) = \underset{i}{\cup} f^{-1}A_i$ and the union of semi-open subsets is semi-open, we only prove that $f^1(U)$ is semi–open in ν. $f^1(U)=[2n,2n+1[\cup[2n\text{-}1,2n[$ (or $=[2m\text{-}1,2m[$). Hence $(f^1 (U)^{o-}) = [2n\text{-}1,2n+1]$ (or $= [2m\text{-}1,2m]$) and consequently $f^1 (U)$ is a subset of $(f^1 (U))^{o-}$ i.e. *f* is irresolute.

4. SEMI-T_1 AND SEMI-T_2

A space X is called semi-T_1 if and only if $\{x\}$ is semi-closed for each $x \in X$ and X is called semi-T_2 if and only if for each two distinct points x and y in X, there exist two disjoint semi-open sets U_x and V_y of x and y, respectively.

Theorem 4.1. (Z^n, τ^n) is a semi T_1-space.

Proof. The product of semi-closed subsets is also semi-closed because $\prod_i \overline{A_i}^{o} = \overline{\prod_i A^o}$. Therefore, all we need to prove is that (Z, τ) is semi-T_1 or equivalently that each singleton in (Z, τ) is semi-closed [3]. For $x \in Z$, $x = 2n$ or $x = 2n+1$. Then $\overline{\{2n\}}^{o} = \{2n\}^o = \emptyset$ or $\overline{\{2n+1\}}^{o} = \{2n, 2n+1, 2n+2\}^o = \{2n+1\}$. In either cases, $A = \{x\}$ is semi-closed.

Of course τ is not T_1-space, for example $\{2n+1\}$ is not closed in τ, and consequently it is not T_2-space. However one can get,

Theorem 4.2. (Z^n, τ^n) is a semi-T_2-space.

Proof. As we said before, we need only to show that (Z, τ) is semi-T_2. Let $x, y \in Z$. Then either they are adjacent or not. If they are adjacent, then one of them is odd and the other is even. Let $x=2n$ and $y=2n+1$ then $\{2n+1\}$ and $\{2n-1, 2n\}$ are disjoint semi-open sets containing x and y, respectively. By a similar technique one can prove that any numbers can be separated by two disjoint semi-open sets.

We noticed that τ with the usual order (totally ordered) on Z is not an ordered topological space in the sense of Nachbin's definition [9], because if it was, it would be Hausdorff [9] which is not as mentioned above.

ACKNOWLEDGMENT

I am greatly indebted to Prof.T.Y.Kong from whom I heard for the first time about digital topology. Thanks to Dr.A.Kozae and Dr. A.Geasa for many insightful discussions.

ÖZET

Bu çalışmada, uygun dijital görüntüye sahip n-boryutlu Khalimsky uzayında yarı-açık ve yarı-kapalı kümelerin sınıflandırılması yapılmıştır.

REFERENCES

[1] Arenas, P.G. (1999) Alexandroff spaces, Acta Math.Univ.Semenianae vol. LXVIII, 1, 17-25.

[2] Crossly, S.G. and Hildebrand, S.K. (1971). Semi-closed sets and semi-continuity in topological spaces, Texas J.Sci. (2+3) (22), 123-126.

[3] Crossly,S.G. and Hildebrand S.K. (1972). Semi-topological properties, Fund. Math. 74, 233-264.

[4] Kong, T.Y., Kopperman, R. and Mayer, P.A. (1991). Topological approach to digital topology, American Mathematical Monthly, Vol.98, No.10 December

[5] Kong, T.Y., Rosenfield, A. (1989). A digital topology: Introduction and survey, computer version, Graphics and Image Processing, 84, 357-393.

[6] Kopperman,R. The Khalimsky Line as a Foundation of Digital Topology (not published)

[7] Levine,N. (1963). Semi-open sets and semi-continuity in topological spaces, Amer.Math.Monthly, 70, 63-41.

[8] Mahmoud, R.A. and Abd El-Monsef, M.E. (1989). β-topological invariant, J.Pakistan Academy of Science, 285-296.

[9] Nachbin,L. (1965). Topology and Order, Van Nostrand, Princeton.

[10] Noiri, T. (1973). Remarks on semi-open mappings, Bull.Cal.Math.Soc., 5 (65),197-201.

Commun. Fac. Sci. Univ. Ank. Series A1
V. 53.(No.1) pp.7-19 (2004)

THE *P*-COCKCROFT PROPERTY OF THE GRAPH PRODUCT

A.S. ÇEVİK AND F. (AÇIL) KİRAZ
Department of Mathematics, Faculty of Science and Art, Balıkesir University, Balıkesir, TURKEY

(Received Jun. 31, 2003; Revised March 15,2004; Accepted March 14, 2004)

ABSTRACT

In [2], Baik-Howie-Pride defined a set of the generating pictures of $\pi_2(P)$ where P is a presentation of a graph product of the vertex groups. In this paper, as an application of this, we discuss necessary and sufficient conditions for the presentation P to be p-Cockcroft, where p is a prime or 0. In addition we examine some special cases of this result.

KEYWORDS p-Cockcroft property; graph product; graph group; efficiency.

1. INTRODUCTION

Let

$$P = < \mathbf{x} \, ; \mathbf{r} > \qquad (1)$$

be a group presentation. Let F denotes the free group on $\mathbf{x}$ and let N denotes the normal closure of $\mathbf{r}$ in F. The quotient $G = F/N$ is the group defined by P.

If we regard P as a 2-complex with one 0-cell, a 1-cell for each $x \in \mathbf{x}$, and a 2-cell for each $R \in \mathbf{r}$ in the standart way, then G is just the fundamental group of P. We then define the second homotopy group $\pi_2(P)$ of P, which is a left $\mathbf{Z}G$-module. The elements of $\pi_2(P)$ can be represented by geometric configurations called *spherical pictures*. These are described in detail in [20] and we refer the reader these for details. Moreover, by [20], there are certain operations on spherical pictures.

Suppose $\mathbf{X}$ is a collection of spherical pictures over P. Then, by [20], one can define the additional operation on spherical pictures. Allowing this additional operation leads to the notion of *equivalence* (*rel* $\mathbf{X}$) of spherical pictures. Then, by [20], *the elements* $< \mathbf{P} >$ ($\mathbf{P} \in \mathbf{X}$) *generate* $\pi_2(P)$ *as a module if and only if every spherical picture is equivalent (rel* $\mathbf{X}$) *to the empty picture.* If the elements $< \mathbf{P} >$ ($\mathbf{P} \in \mathbf{X}$) generate $\pi_2(P)$ then we say that $\mathbf{X}$ *generates* $\pi_2(P)$.

2000 Mathematics Subject Classification: *20F05; 20F06; 20F32; 20F55; 29K25; 57M05.*

For any picture **P** over P and for any $R \in \mathbf{r}$, the *exponent sum* of R in **P**, denoted by $exp_R(\mathbf{P})$ is the number of discs of **P** labelled by R, minus the number of discs labelled by R^{-1}. It is clear that if pictures $\mathbf{P}_1$ and $\mathbf{P}_2$ are equivalent, then $exp_R(\mathbf{P}_1) = exp_R(\mathbf{P}_2)$ for all $R \in \mathbf{r}$.

Definition 1.1. Let *P be as in* (1), *and let n be a non-negative integer. Then P is said to be n*-Cockcroft *if* $exp_R(\mathbf{P}) \equiv 0 \ (mod\ n)$, (*where congruence* (mod 0) *is taken to be equality*) *for all* $R \in \mathbf{r}$ *and all spherical pictures* **P** *over P. A group G is said to be n-Cockcroft if it admits an n-Cockcroft presentation. The case n = 0 is usually just called Cockcroft.*

The reader can find some examples and details about Cockcroft property, for example, in [11], [13], [14], [17] and [19] and about p-Cockcroft property, for example, in [8] and [19].

We remark that to verify the n-Cockcroft property holds, it is enough to check for pictures $\mathbf{P} \in \mathbf{X}$, where **X** is a set of generating pictures.

A *graph* Γ consist of two disjoint set $\mathbf{v} = \mathbf{v}(\Gamma)$ (vertices) and $\mathbf{e} = \mathbf{e}(\Gamma)$ (edges) and three functions

$$\iota : \mathbf{e} \to \mathbf{v}, \qquad \tau : \mathbf{e} \to \mathbf{v} \qquad \text{and} \qquad ^{-1} : \mathbf{e} \to \mathbf{e}$$

satisfying: $\iota(e) = \tau(e^{-1})$, $(e^{-1})^{-1} = e$, $e^{-1} \neq e$ for all $e \in \mathbf{e}$. We call $\iota(e)$ and $\tau(e)$ the *initial* and *terminal* point of $e \in \mathbf{e}$, respectively. An orientation $\mathbf{e}^+$ of Γ consists of a choice of exactly one edge from edge pair e, e^{-1} ($e \in \mathbf{e}$). We will call to pair ($\mathbf{v}$, $\mathbf{e}^+$) with the functions ι, τ as an *oriented graph* with oriented edge set $\mathbf{e}^+$. A graph Γ is called *simple* if whenever $\iota(e_1) = \iota(e_2)$ and $\tau(e_1) = \tau(e_2)$ then $e_1 = e_2$ for all $e_1, e_2 \in \mathbf{e}$. A simple graph Γ is called *complete* if for any two distinct vertices u and v, there is an edge e with $\iota(e) = u$, $\tau(e) = v$. The details and applications of these can be found, for instance in [3].

1.1. Graph Product

Let Γ be a simple oriented graph with a vertex set **v** and edge **e** (thus **e** is a collection of 2-element subsets of **v**). For each $v \in \mathbf{v}$, let G_v be a *vertex group* given by a presentation $P_v = \langle \mathbf{x}_v; s_v \rangle$ where the elements of $\mathbf{s}_v$ are cyclically reduced words on $\mathbf{x}_v$. For each $e \in \mathbf{e}$ with $\iota(e) = u$ and $\tau(e) = v$, let G_e be an *edge group* given by a presentation $P_e = \langle \mathbf{x}_u, \mathbf{x}_v ; \mathbf{s}_u, \mathbf{s}_v, \mathbf{r}_e \rangle$ where the elements of $\mathbf{r}_e$ are cyclically reduced words on $\mathbf{x}_u \cup \mathbf{x}_v$ each involving at least one $\mathbf{x}_u$-symbol and $\mathbf{x}_v$- symbol and each $\mathbf{r}_e$ $(e \in \mathbf{e})$ consists of all words $[x,y] = xyx^{-1}y^{-1}$ $(x \in \mathbf{x}_{\iota(e)}, y \in \mathbf{x}_{\tau(e)})$.

Let

$$P = \langle \mathbf{x} ; \mathbf{s}, \mathbf{r} \rangle \tag{2}$$

be a presentation where $\mathbf{x} = \bigcup_{v \in \mathbf{v}} \mathbf{x}_v$, $\mathbf{s} = \bigcup_{v \in \mathbf{v}} \mathbf{s}_v$, $\mathbf{r} = \bigcup_{e \in \mathbf{e}} \mathbf{r}_e$. The group $G = G(\Gamma)$ defined by P is called a *graph product* of the vertex groups G_v for all $v \in \mathbf{v}$ ([2], [5], [15], [16]).

A graph product has two extreme cases:

- If the edge set **e** is empty then G is the *free product* of the groups G_v $(v \in \mathbf{v})$.
- If Γ is complete and each G_v is finite then G is the *direct product* of the groups G_v $(v \in \mathbf{v})$.

If all the vertex groups G_v $(v \in \mathbf{v})$ are infinite cyclic then G is called a *graph group* (see [2], [9], [10], [22], [23]).

The main result of this paper is the following:

Theorem 1.2. (Main Theorem) *Let p be a prime or* 0 *and let P be a presentation as in* (2). *Then P is p-Cockcroft if and only if*

i) each P_v $(v \in \mathbf{v})$ *is p-Cockcroft,*

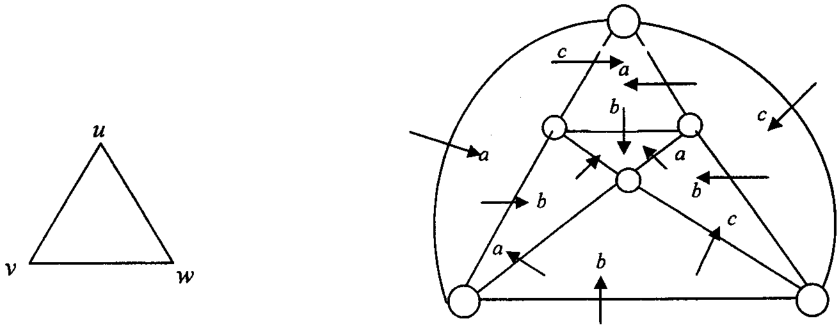
ii) for each $v \in \mathbf{v}$, $exp_x(S) \equiv 0 \ (mod\ p)$ *where* $x \in \mathbf{x}_v$, $S \in \mathbf{s}_v$.

2. PRELIMINARIES

In this section we exhibit the generating pictures of $\pi_2(P)$ where P is a presentation as in (2), in order to prove the main theorem. We may refer to [2] for the details of this material.

Let Γ be an oriented graph. For each triangle $\{u, v, w\}$ (that is a 3-element subset of **v** for which $\{u, v\}$, $\{v, w\}$, $\{w, u\} \in$ **e**) in Γ (see Figure 1-(*a*)), we have a collection of spherical pictures of the form depicted in Figure 1-(*b*) where $a \in \mathbf{x}_u$, $b \in \mathbf{x}_v$, $c \in \mathbf{x}_w$. Let **Z** be the union of all these collections over all triangles of Γ.

For each $e \in$ **e** with $\iota(e) = u$, $\tau(e) = v$, let $S = x_1x_2 \ldots\ldots x_n$ ($x_i \in \mathbf{x}_u$, $i = 1, 2, \ldots\ldots, n$ where $n \in \mathbf{Z}^+$) be a relator in $\mathbf{s}_u$. Then for each $y \in \mathbf{x}_v$, we have a spherical picture $\mathbf{P}_{S,y}$ over P of the form as depicted in Figure 2-(*a*).



(*a*) (*b*)

FIGURE 1

Similarly, for each $x \in \mathbf{x}_u$, we get a spherical picture $\mathbf{P}_{T,x}$ over P where $T = y_1y_2 \ldots\ldots y_m$ ($y_i \in \mathbf{x}_v$, $j = 1,2,\ldots, m$ where $m \in \mathbf{Z}^+$) is a relator in $\mathbf{s}_v$ (Figure 2-(*b*)).

Let $Y_{e,u} = \{\mathbf{P}_{S,y} : S \in \mathbf{s}_u, y \in \mathbf{x}_v\}$ and $Y_{e,v} = \{\mathbf{P}_{T,x} : T \in \mathbf{s}_v, x \in \mathbf{x}_u\}$ be the sets of these spherical pictures. Also for each $e \in$ **e** in Γ, let us define $Y_e = Y_{e,u} \cup Y_{e,v}$ and $\mathbf{Y} = \bigcup_{e \in \mathbf{e}} Y_e$.

Let X_v be a collection of spherical pictures over P_v such that $\pi_2(P_v)$ is generated by X_v and let $\mathbf{X} = \bigcup_{v \in \mathbf{V}} X_v$.

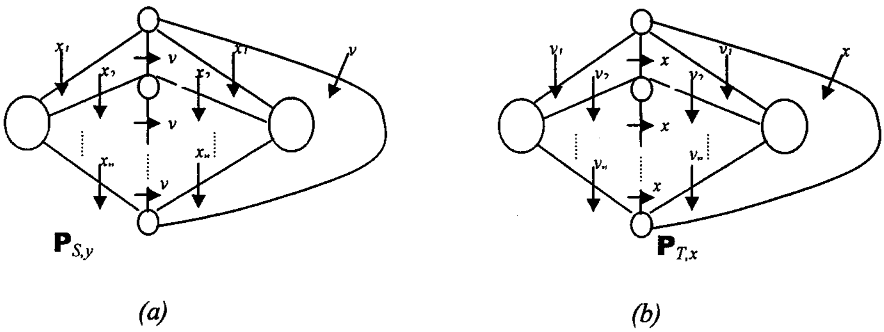


FIGURE 2

The proof of the following result can be found in [2] and [3].

Theorem 2.1. *Let P be a presentation as in* (2). *Then* $\pi_2(P)$ *is generated by*

$\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$.

3. PROOF OF THE MAIN THEOREM

Throughout this section the notations will be the same as in the previous ones.

Let p be a prime or 0. In this part of the proof, let us suppose that the presentation P, as given in (2), is p-Cockcroft for any prime p. By Theorem 2.1, since $\pi_2(P)$ is generated by $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ and, by the assumption, since the exponent sum of the discs of each spherical picture defined in the sets $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ is equivalent to zero by *mod p* then, by Definition 1.1, this gives that each P_v ($v \in \mathbf{v}$) is p-Cockcroft and then since each of the relators S is defined in the presentation P_v, so we get

$$exp_x(S) \equiv 0 \ (mod\, p),$$

for all $x \in \mathbf{x}_v$, $S \in \mathbf{s}_v$, as required.

For the other part of the proof let us assume that conditions *i)* and *ii)* hold for the presentation P. By Theorem 2.1, for a presentation P as in (2), since $\pi_2(P)$ is generated by the union of the sets **X**, **Y** and **Z** then we need to calculate the exponent sum of the pictures in the sets **X**, **Y** and **Z** under these assumptions. Now by the definition of **X**, the exponent sum of the discs $\mathbf{s}_v$ ($v \in \mathbf{v}$) of each spherical picture is equivalent to zero by *mod p*. Additionaly, since the exponent sum of the discs $\mathbf{r}_e$ ($e \in \mathbf{e}$) of the spherical pictures in the set **Z** is equal to zero (see Figure 1-(*b*)) then it is enough to check that the exponent sum of the discs of spherical pictures in the set **Y**.

By Figure 2, the spherical pictures $\mathbf{P}_{S,y}$ and $\mathbf{P}_{T,x}$ ($S \in \mathbf{s}_u$, $T \in \mathbf{s}_v$, $y \in \mathbf{x}_v$, $x \in \mathbf{x}_u$) in the set **Y** consist of the discs S, T, $[y,x_i]$ ($x_i \in \mathbf{x}_u$, $i = 1, 2, \ldots, n$ where $n \in \mathbf{Z}^+$), $[x,y_j]$ ($y_j \in \mathbf{x}_v$, $j = 1, 2, \ldots, m$ where $m \in \mathbf{Z}^+$). Since the conditions *i), ii)* hold for P and each of the relators S, T is defined in the presentation P_v then we have

$$exp_{x_i}(S) \equiv 0 \ (mod\ p), \qquad \forall x_i \in \mathbf{x}_u \ \text{ for } \ i = 1,2, \ldots, n$$

and

$$exp_{y_j}(T) \equiv 0 \ (mod\ p), \qquad \forall y_j \in \mathbf{x}_v \ \text{ for } \ j = 1,2, \ldots, m.$$

Also it is easy to see that

$$\exp_{[y,x_i]}(\mathbf{P}_{S,y}) = \exp_{x_i}(S) \quad \text{and} \quad \exp_{[x,y_j]}(\mathbf{P}_{T,x}) = \exp_{y_j}(T)$$

where $\forall x_i \in \mathbf{x}_u$, $\forall y_j \in \mathbf{x}_v$, $i=1, 2, \ldots, n$, $j=1, 2, \ldots, m$ and $n, m \in \mathbf{Z}^+$. Therefore we get

$$\exp_{[y,x_i]}(\mathbf{P}_{S,y}) \equiv 0 \ (mod\ p) \quad \text{and} \quad \exp_{[x,y_j]}(\mathbf{P}_{T,x}) \equiv 0 \ (mod\ p).$$

Moreover, for the discs S, T in the spherical pictures $\mathbf{P}_{S,y}$ and $\mathbf{P}_{T,x}$, it is clear that

$$exp_S(\mathbf{P}_{S,y}) = 1 - 1 = exp_T(\mathbf{P}_{T,x}) = 0.$$

Hence, since the above processing can be made for all the spherical pictures in the set **Y** then we get P is p-Cockcroft where p is a prime or 0, as required.

Hence the result. ◊

4. APPLICATIONS OF THE MAIN THEOREM

Our aim in this section is to investigate what we get by changing some situations in the main theorem and then trying to get some consequences of it. In fact we will obtain some well-known results by applying these variations.

Let Γ be a graph with vertex set **v** and edge set **e** and let $G = G(\Gamma)$ be a graph product.

First let us suppose that all the vertex groups G_v $(v \in \mathbf{v})$ are infinite cyclic. Then a presentation of a group G_v is given by $P_v = < \mathbf{x}_v \, ; \quad >$. Thus the presentation P, as in (2), becomes

$$P = < \mathbf{x} \, ; \quad >. \tag{3}$$

Corollary 4.1. *Let p be a prime or* 0 *and let P be a presentation as in* (3). *Then P is p-Cockcroft.*

Proof. By the definition of **X** and **Y**, it is easy to see that they are equal to the empty sets. Then by Theorem 2.1, $\pi_2(P)$ is generated by only the set **Z**. Additionally, since all the spherical pictures are Cockcroft in the set **Z** (see Figure 1-(b)) then P is p-Cockcroft, as required.◊

Now assume that the edge set **e** is empty in Γ. Then the set $\mathbf{r}_e$ $(e \in \mathbf{e})$ in the presentations P_e $(e \in \mathbf{e})$ will be empty. Therefore $\mathbf{r} = \varnothing$ and the group G becomes the free product of the groups G_v $(v \in \mathbf{v})$. Thus, by [18], G is given by a presentation

$$P = < \mathbf{x} \, ; \mathbf{s} > \tag{4}$$

Corollary 4.2. *Let p be a prime or* 0 *and let P be a presentation as in* (4). *Then P is p-Cockcroft if and only if each* P_v $(v \in \mathbf{v})$ *is p-Cockcroft.*

Proof. It is clear that $\mathbf{Y} = \varnothing$ and $\mathbf{Z} = \varnothing$. Then $\pi_2(P)$ is generated by only the set

$\mathbf{X} = \bigcup_{v \in \mathbf{v}} X_v$. First assume that P is p-Cockcroft for some prime or 0. Therefore, since

$\pi_2(P)$ is generated by the union of the sets X_v $(v \in \mathbf{v})$ then we get each P_v is

p-Cockcroft.

Now suppose that each P_v ($v \in \mathbf{v}$) is p-Cockcroft. Since $\mathbf{X} = \bigcup_{v \in \mathbf{V}} X_v$ and $\pi_2(P)$ is generated by the set $\mathbf{X}$ then P is p-Cockcroft, as required. ◊

Finally let us suppose that the oriented edge set is $\{e_1, e_2, e_3\}$ and the vertex set is $\{u, v, w\}$ in Γ as shown in Figure 3. Also let $P_u = < x \,;\, x^{p_1} >$, $P_v = < y \,;\, y^{p_2} >$, $P_w = < z \,;\, z^{p_3} >$ be the presentations of the vertex groups G_u, G_v and G_w, respectively where p_1, p_2, p_3 are distinct primes. Thus

$$P = < x,y,z \,;\; x^{p_1}, y^{p_2}, z^{p_3}, [x,y], [y,z], [z,x] > \qquad (5)$$

is a presentation of the graph product of the groups G_u, G_v, G_w.

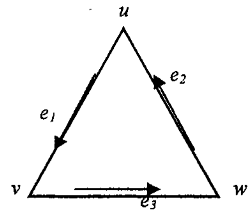


FIGURE 3 : The graph Γ

Remark 4.3. *It is well known that the presentation P, as in* (5), *is actually presenting the direct product of the cyclic groups of order p_1, p_2 and p_3, respectively where p_1, p_2, p_3 are distinct primes.*

Let P be a presentation as in (5). As a consequence of Section 2, we can define the generating pictures of $\pi_2(P)$ as depicted in Figure 4. It is easy to see that set $\mathbf{Z}$ is empty for this group (we may refer [1] for the details of this).

It is clear that the proof of the following lemma is a immediate consequence of Theorem 2.1.

Lemma 4.4. *Let P be a presentation as in* (5). *Then $\pi_2(P)$ is generated by* $\mathbf{X} \cup \mathbf{Y}$.

Now as an application of Theorem 1.2 we can obtain the following result.

Corollary 4.5. *Let P be a presentation as in* (5) *and let p be a prime. Then P is p-Cockcroft if each prime* p_i *is equal to p (i=1,2,3).*

Proof. Let us label the relations in the presentations P_u, P_v, P_w as follows:

$$x^{p_1} = R_1, y^{p_2} = R_2, z^{p_3} = R_3,$$
$$[x,y] = S_1, [y,z] = S_2, [z,x] = S_3 .$$

By Lemma 4.4, we must check the exponent sum of the discs in the sets **X** and **Y**. By Figure 4, the spherical pictures $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ in the set **X** consist of discs R_1, R_2, R_3 and the spherical pictures $\mathbf{P}_4$, $\mathbf{P}_5$, $\mathbf{P}_6$, $\mathbf{P}_7$, $\mathbf{P}_8$, $\mathbf{P}_9$ in the set **Y** consist of discs R_1, R_2, R_3, S_1, S_2 and S_3. Then the exponent sum of these discs are

$$exp_{R_1}(\mathbf{P}_1) = exp_{R_2}(\mathbf{P}_2) = exp_{R_3}(\mathbf{P}_3) = exp_{R_1}(\mathbf{P}_4) = exp_{R_2}(\mathbf{P}_5) = 1\text{-}1 = 0,$$

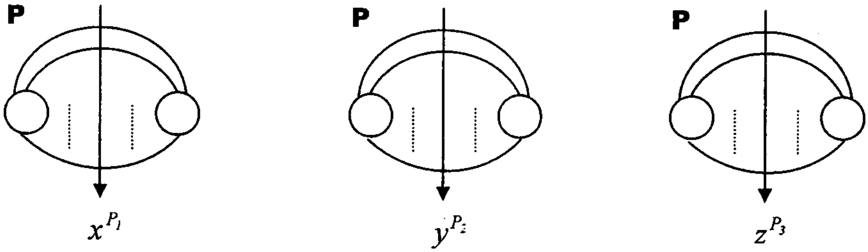
$$exp_{R_2}(\mathbf{P}_6) = exp_{R_3}(\mathbf{P}_7) = exp_{R_3}(\mathbf{P}_8) = exp_{R_1}(\mathbf{P}_9) = 1\text{-}1 = 0,$$

$$exp_{S_1}(\mathbf{P}_4) = p_1,\ exp_{S_1}(\mathbf{P}_5) = p_2,\ exp_{S_2}(\mathbf{P}_6) = p_2,$$

$$exp_{S_2}(\mathbf{P}_7) = p_3 = exp_{S_3}(\mathbf{P}_8),\ exp_{S_3}(\mathbf{P}_9) = p_1.$$

Therefore, by Theorem 1.2, P is p-Cockcroft if each of the presentation P_u, P_v, P_w is p-Cockcroft for the same prime p. Thus the primes p_1, p_2, p_3 must be equal to p, as required.

The set **X**:



The set **Y**:

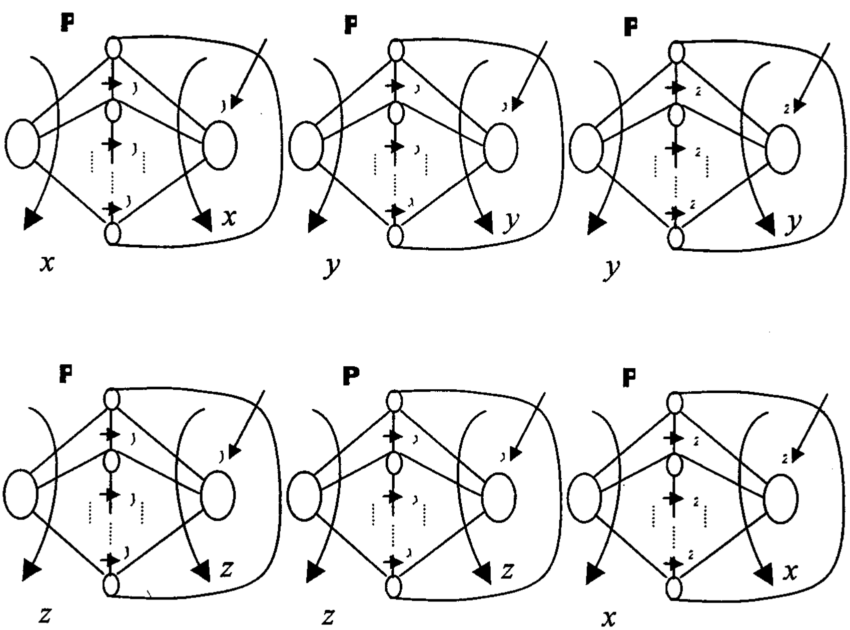


FIGURE 4

Remark 4.6. *Although Corollary* 4.5 *states the p-Cockcroft property of a sphecial case of the graph product, in fact, by Remark* 4.3, *it states that the p-Cockcroft property of the direct product of the cyclic groups of order* p_1, p_2, p_3 *respectively which was studied and taken too much attention by the authors* (*see, for example, in* [21]).

One can find the definition of *efficiency* for a presentation P as in (1), for instance, in [4], [6], [7] and [24]. In [12], Epstein (and later on Kilgour-Pride in [19]) showed that a presentation P, as given in (1), is *efficient* if and only if it is *p-Cockcroft* for some prime p.

As an application of Theorem 1.2, we can also give the following example which is used the term efficiency instead of the p-Cockcroft property for the presentations of the vertex groups.

Example 4.7. Let Γ be a graph with the oriented edge set is $\{e_1\}$ and the vertex set is $\{v_1, v_2\}$. Let

$$P_{v_1} = < a,b; a^n, aba^{-m}b^{-1} > \ and \ P_{v_2} = < c,d; c^t, cdc^{-k}d^{-1} >$$

be presentations of the vertex groups G_{v_1} and G_{v_2}, respectively where $(n, m-1) \neq 1$, $(t, k-1) \neq 1$ $(n, m, t, k \in Z^+)$ and p is any prime with $p \mid n$, $p \mid t$. Then

$$P = < a,b,c,d \,;\, a^n, aba^{-m}b^{-1}, c^t, cdc^{-k}d^{-1}, [a,c], [a,d], [b,c], [b,d] >$$

is a presentation of the graph product of the groups G_{v_1} and G_{v_2}. By [3], each of the presentations P_{v_1} *and* P_{v_2} is efficient (and so is p-Cockcroft by the last paragraph before example). Moreover the exponent sum of the each letter in the relators of the presentations P_{v_1} *and* P_{v_2} is congruent to zero by mod p. Then, by Theorem 1.2, we have that P is p-Cockcroft.

ACKNOWLEDGEMENT

The authors are grateful to the referee for her/his helpful suggestions and for reminding us the reference [24].

ÖZET Baik - Howie - Pride, vertex gruplarının graph çarpımının bir gösterimi *P* olmak üzere $\pi_2(P)$ ye ait üreteç resimlerinin bir kümesini tanımlamıştı, [2]. Bu çalışmada bunun bir uygulaması olarak, *p* asal veya sıfır olmak üzere *P* gösteriminin p- Cockcroft olması için gerek ve yeter koşullar ile birlikte bu sonuca ait bazı özel durumlar incelenmiştir.

REFERENCES

[1] F. Açil, *The Efficiency of the Group Presentations,* M. Sc. Thesis, Balikesir University, (1999).

[2] Y.G. Baik, J. Howie and S.J. Pride, *The Identity Problem for Graph Products of Groups,* Journal of Algebra **162** (1993), 168-177.

[3] Y.G. Baik, *Generators of the Second Homotopy Module of Group Presentations with Applications,* Ph.D. Thesis, University of Glasgow, (1992).

[4] Y.G. Baik and S.J. Pride, *On the Efficiency of Coxeter Groups,* Bull. of the London Math. Soc. **29**, (1997), 32-36.

[5] I.M. Chiswell, *The Euler Characteristic of Graph Products and of Coxeter groups,* "Discrete groups and geometry" *(Birmingham, 1991),* 36--46, L. M. S. Lec. Note Ser., Vol. **173**, *Cambridge Univ. Press, Cambridge,* 1992.

[6] A.S. Çevik, *The Efficiency on 2-generators of Semi-Direct Product of Groups,* Commun. Fac. Sci. Univ. Ank. Series Al, Vol. **48** (1999), 7-13.

[7] A.S. Çevik, *The Efficiency of Standard Wreath Product,* Proc. of the Edinburgh Math. Soc. **43**, (2000), 415-423.

[8] A.S. Çevik, *The p-Cockcroft Property of Central Extensions of Groups,* Commumications in Algebra **29** (3), (2001), 1085-1094.

[9] C. Droms, *Isomorphisms of Graph Groups,* Proc. Amer. Math. Soc. 100(3) (1987), 407-408.

[10] C. Droms, *Subgroups of Graph Groups*, Journal of Algebra **110** (1987), 519-522.

[11] M.N. Dyer, *Cockcroft 2-Complexes,* preprint, University of Oregon, 1992.

[12] D.B.A. Epstein, *Finite Presentations of Groups and 3-manifolds,* Quart. J.Math. Oxford Ser.(2), **12**, (1961), 205-212.

[13] N.D. Gilbert, J. Howie, *Threshold Subgroups for Cockcroft 2-complexes,* Communications in Algebra **23** (1), (1995), 255-275.

[14] N.D. Gilbert, J. Howie, *Cockcroft Properties of Graphs of 2-Complexes,* Proc.Royal Soc. of Edinburgh Section A-Mathematics (**124**), No. Pt2, (1994), 363-369.

[15] E.R. Green, *Graph Products of Groups,* Ph.D. Thesis, University of Leeds, 1990.

[16] E.R. Green, *Residual Properties of Graph Products,* Preprint, University of Leeds, 1990.

[17] J. Harlander, *Minimal Cockcroft Subgroups,* Glasgow Journal of Math. **36**, (1994), 87-90.

[18] D.L. Johnson, *Presentation of Groups,* L.M.S. Lect. Note Series, Vol. **22**, CUP, 1976.

[19] C.W. Kilgour, S.J. Pride, *Cockcroft Presentations,* Journal of Pure and Applied Algebra, Vol (**106**), No 3, (1996), 275-295.

[20] S.J. Pride, *Identities Among Relations of Group Presentations*, Group Theory From a "Geometrical Viewpoint", Tiresto 1990, (E.Ghys, A. Haefliger, A. Verjovsky, edts.), World Scientific Publishing, 1991, 687-717.

[21] E.F. Robertson, N. Ruskuc and J. Wiegold, *Generators and Relations of Direct Products of Semigroups*, Trans. Amer. Math. Soc. **350** (1998), 2665-2685.

[22] H. Servatius, *Automorphisms of Graph Groups,* Journal of Algebra **126**, (1989), 34-60.

[23] H. Servatius, C. Droms and B. Servatius, *The Finite Basis Extension Property and Graph Groups,* "Topology and Combinatorial Group Theory" (edt. P.Latiolais), L.M.S. Lect. Note Series, Vol. **1440**, CUP, 1990.

[24] J.G. Sunday, *Presentations of the Groups SL (2, m) and PSL (2, m),* Can. J. Math., **24**, (1972), 1129-1131.