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PERFECT SYSTEMS FOR SPECHT MODULES OF $G(m,1,n)$

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ABSTRACT

The perfect systems play a very important role in giving combinatorial constructions of representations of the Weyl groups. In this paper, we present an algorithm to construct the perfect systems for the complex root systems of type B_n^m. We then use this algorithm to find a basis for the Specht modules of the imprimitive complex reflection groups $G(m,1,n) = W(B_n^m)$. In particular, application of the algorithm conforms with known results in the representation theory of the generalized symmetric groups. Thus this completes the combinatorial construction of the irreducible representations of $W(B_n^m)$ in terms of root systems.

KEYWORDS

Perfect systems, Specht modules, Complex reflection groups, Irreducible representations.

1. INTRODUCTION

There are well-known constructions of irreducible representations and of irreducible modules, called Specht modules, for the symmetric groups S_n which are based on elegant combinatorial concepts connected with Young tableaux, etc. (see, e.g. [10]). Morris [11] described a possible extension of this work to Weyl groups in general. In recent years, a further development of these ideas has appeared in Halicioglu and Morris [8] and Halicioglu [9], where for Weyl groups, the symmetric groups were taken as role models. The familiar concepts of Young tableaux, tabloids, etc., which are so crucial in the development of the representation theory of the symmetric groups, are seen to have equally familiar counterparts in the context of root systems.

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Later on, the present author [2] has extended this alternative approach to deal with the imprimitive complex reflection groups G(m,1,n), where the Young tableaux method given in [1] for generalized symmetric groups have been further generalised. For the construction of a basis for the Specht modules of Weyl groups, Halicioglu [9] has considered the root systems of simply laced type only (i.e., A_n, D_n, E_6, E_7, E_8) and also parabolic subsystems only. The present author [4] extended the work of [9] to deal with the root systems of type of C_n.
In this paper, we construct the perfect systems for the complex root systems of type B_n^m, wherefore we present an algorithm which is a modification of Algorithm 3.1 of [4] and results in a basis for the Specht modules of $G(m,1,n)$.

2. SPECHT MODULES

We first establish the basic notation and state some results which are required later. We refer the reader to [8], [9] and [2] for much of the undefined terminology and quoted results. As a convention, throughout this paper, we assume that ξ is a primitive m-*th* root of unity.

2.1. Let $V = C^n$, the complex vector space of dimension n with standard unitary inner product (. , .) and the standard basis $\{e_1, e_2, \ldots, e_n\}$. Let S_n be the group of all $n \times n$ permutation matrices, and let $A(m,1,n)$ be the group of all diagonal $n \times n$ matrices with $\xi^{s_i}, s_i \in Z$ in the *(i, i)* position. Then S_n normalizes $A(m,1,n)$. We let $G(m,1,n) = A(m,1,n) \times S_n$ (semi-direct product). $G(m,1,n)$ is an *(imprimitive complex)* reflection group in V generated by unitary reflections, and $G(1,1,n) = W(A_{n-1})$ (Weyl group of type A_{n-1}) and $G(2,1,n) = W(B_n)$ (Weyl group of type B_n). The group $G(m,1,n)$ has the following presentation [7]:

$$G(m,1,n) = \langle r_1, \ldots, r_{n-1}, w_1, \ldots, w_n \mid r_i^2 = (r_i r_{i+1})^3 = (r_i r_j)^2 = e, |i-j| \geq 2,$$
$$w_i^m = e, w_i w_j = w_j w_i, r_i w_i = w_{i+1} r_i, r_i w_j = w_j r_i, j \neq i, i+1 \rangle.$$

If $\sigma \in G(m,1,n)$, then we may write $\sigma = \tau \prod_{i=1}^{n} w_i^{s_i}$, where $\tau \in W(A_{n-1}), 1 \leq s_i \leq m$ (see [1]). Define $\phi: G(m,1,n) \to W(A_{n-1})$ by $\phi(\sigma) = \tau$. Then ϕ is an epimorphism and $\ker \phi = A(m,1,n)$. Define the sign of $\sigma \in G(m,1,n)$ to be $\text{sgn}(\sigma) = (-1)^{l(\phi(\sigma))}$ where $l(\phi(\sigma))$ is the length of $\phi(\sigma)$.

We define $f(\sigma)=\sum_{i=1}^{n} s_i$ for $\sigma=\tau\prod_{i=1}^{n} w_i^{s_i}\in G(m,1,n)$, which is required later. Let $\pi=\gamma\prod_{i=1}^{n} w_i^{\rho_i}\in G(m,1,n)$, where $\gamma\in W(A_{n-1})$, $1\le\rho_i\le m$, then we have $\sigma\pi=\tau\gamma\prod_{i=1}^{n} w_i^{(\rho_i+s_{\gamma(i)})}$. Hence, $f(\sigma\pi)\equiv f(\sigma)+f(\pi)$ (mod m) for any $\sigma,\pi\in G(m,1,n)$. Furthermore, we assume that $f(\tau)=0$ for all $\tau\in W(A_{n-1})$. Thus, $f(\sigma^{-1})=mk-f(\sigma)$ for any $\sigma\in G(m,1,n)$, where $k\in Z$.

2.2. Let $\Phi=(R,f)$ be a root system with $W(\Phi)$. Let S be a subset of R and g be a map such that $g=f\,|\,S$. The pair $\Psi=(S,g)$ is called a subsystem of Φ if Ψ is a root system. A reflection subgroup $W(\Psi)$ of $W(\Phi)$ corresponding to the subsystem $\Psi=(S,g)$ of Φ is the subgroup generated by the $s_{a,g(a)}$ with $a\in S$. Let $\pi=(B,\theta)$ be a root graph. If a root system Φ is the pre-root system obtained from a root graph π as described in 4.10 (i) of [6], then π is called a simple system in Φ. If Φ is a root system with simple system π, then the graph associated to π is called a Cohen (Dynkin) diagram of Φ.

2.3. A root system for $G(m,1,n)$ may be defined as follows (see [6]). Let $\mu_m=\left\{\xi^l\,\middle|\,\ell\in N,\ \xi \text{ is a primitive } m-th \text{ root of unity}\right\}$ Put

$$R(m,1,n)=\mu_m\left\{\pm(e_i-\xi^l e_j),\ e_k \mid i,j,k,l\in N,\ i\neq j, 1\le i,j,k\le n\right\}$$

and let $f_{m,1,n}:R(m,1,n)\to N\setminus\{1\}$ be defined by

$$f_{m,1,n}(a)=\begin{cases} m & \text{if } \alpha\in\mu_m\{e_k \mid 1\le k\le n\} \\ 2 & \text{otherwise,}\end{cases}$$

then we have that $\Phi=\Phi(m,1,n)=(R(m,1,n),f_{m,1,n})$ is a root system with $W(\Phi)=G(m,1,n)$. Let $\pi=B_n^m=\{\alpha_i=e_i-e_{i+1}\ (i=1,\dots,n-1),\alpha_n=e_n\}$. Then $\pi=B_n^m$ is a root graph and so, a simple system for $\Phi=\Phi(m,1,n)$ with $W(B_n^m)=G(m,1,n)$.

The positive system in Φ determined by $\pi=B_n^m$ is obtained to be $\Phi^+=P\cup P'\cup Q$, where

$$P=\left\{e_i-\xi^l e_j \mid 1\le i<j\le n,\ 1\le l\le m\right\}$$
$$Q=\left\{e_k \mid 1\le k\le n\right\} \text{ and}$$
$$P'=-\lambda P \text{ with } \lambda=\begin{cases}\left\{\xi^l \mid 1\le l\le m-1\right\} \text{if } m \text{ is odd}\\ \left\{\xi^l \mid 1\le l\le \frac{m}{2}-1\right\} \text{ if } m \text{ is even (see [3]).}\end{cases}$$

If Ψ is a subsystem of Φ, then a simple system J of Ψ can be chosen such that $J\subset\Phi^+$ (see [3], Theorem 1).
Furthermore, the set $D_\Psi=\left\{w\in W(\Phi)\mid w(\alpha)\in\Phi^+ \text{ for all } \alpha\in J\right\}$ is a distinguished set of coset representatives for $W(\Psi)$ in $W(\Phi)$, that is, every element of $W(\Phi)$ can be uniquely expressed in the form $d_\Psi w_\Psi$ where $d_\Psi\in D_\Psi$ and $w_\Psi\in W(\Psi)$ (see [3], Theorem 2).

2.4. Let Φ be a root system with simple system $\pi=\left\{\alpha_i=e_i-e_{i+1}(i=1,\dots,n-1),\alpha_n=e_n\right\}$ and corresponding reflection group $W=W(B_n^m)$. Let Φ^+ be the positive system determined by π as in 2.3. Let Ψ be a subsystem of Φ with simple system $J\subset\Phi^+$ and Cohen diagram Δ.
By [5], if $\Psi=\sum_{i=1}^{m}\sum_{j=1}^{s_i}B^{m_i}_{\lambda_j^{(i)}}$, with $m_i=\begin{cases}1 \text{ if } i=1\\ m \text{ if } i=2,\dots,m\end{cases}$ and $\sum_{j=1}^{s_i}(\lambda_j^{(1)}+1)+\sum_{i=2}^{m}\sum_{j=1}^{s_i}\lambda_j^{(i)}=n$, where $B^{m_i}_{\lambda_j^{(i)}}$ are the indecomposable components of Ψ, then let $J^{(i)}_{\lambda_j^{(i)}}$ be a simple system in $B^{m_i}_{\lambda_j^{(i)}}(j=1,\dots,s_i;\ i=1,\dots,m)$ and $J=\sum_{i=1}^{m}J^{(i)}$, where $J^{(i)}\sum_{j=1}^{s_i}J^{(i)}_{\lambda_j^{(i)}}$ $(i=1,\dots,m)$. Let $\Psi^\perp$ be the largest subsystem in Φ orthogonal to Ψ and let $J^\perp\subset\Phi^+$ be the simple system of $\Psi^\perp$. Let Ψ' be a subsystem of Φ which is contained in $\Phi\setminus\Psi$, with simple system $J'\subset\Phi^+$ and Cohen diagram Δ'.
Then $\Psi'=\sum_{i=1}^{m}\sum_{j=1}^{r_i}B^{m_i}_{\mu_j^{(i)}}$, with $m_i=\begin{cases}m \text{ if } i=1\\ 1 \text{ if } i=2,\dots,m\end{cases}$ and

$\sum_{j=1}^{r_1}\mu_j^{(1)}+\sum_{i=2}^{m}\sum_{j=1}^{r_i}(\mu_j^{(i)}+1)=n$, where $B^{m_i}_{\mu_j^{(i)}}$ are the indecomposable components of Ψ', and let $J'^{m_i}_{\mu_j^{(i)}}$ be a simple system in $B^{m_i}_{\mu_j^{(i)}}$ $(j=1,\dots,r_i;\ i=1,\dots,m)$ and $J'=\sum_{i=1}^{m}J'^{(i)}$, where $J'^{(i)}=\sum_{j=1}^{r_i}J'^{(i)}_{\mu_j^{(i)}}\ (i=1,\dots,m)$. Let $\Psi'^{\perp}$ be the largest subsystem in Φ orthogonal to Ψ' and let $J'^{\perp}\subset\Phi^{+}$ be the simple system of $\Psi'^{\perp}$. Let J stand for the regular ordered m-set

$$\left\{\left(J^{(1)}_{\lambda_1^{(1)}},\dots,J^{(1)}_{\lambda_{s_1}^{(1)}};\ J'^{(1)}_{\mu_1^{(1)}},\dots,J'^{(1)}_{\mu_{r_1}^{(1)}}\right),\dots,\left(J^{(m)}_{\lambda_1^{(m)}},\dots,J^{(m)}_{\lambda_{s_m}^{(m)}};J'^{(m)}_{\mu_1^{(m)}},\dots,J'^{(m)}_{\mu_{r_m}^{(m)}}\right)\right\},$$

where in addition the elements in each $J^{(i)}_{\lambda_j^{(i)}}$ and $J'^{(i)}_{\mu_j^{(i)}}$ are ordered, and put $\tau_\Delta=\{w\bar{J}\mid w\in W\}$. The regular ordered m-set of pairs $\bar{J}=\left\{\left(J^{(1)};\ J'^{(1)}\right),\dots,\left(J^{(m)};\ J'^{(m)}\right)\right\}$ is called a useful system in Φ if $W(J)\cap W(J')=\langle e\rangle$ and $W(J^{\perp})\cap W(J'^{\perp})=\langle e\rangle$. The elements of τ_Δ are called Δ-tableaux, the $J^{(k)}$ and $J'^{(k)}$ $(1\le k\le m)$ are called the row and column of the k-th constituent of $\bar{J}$ respectively. This is a natural extension of the concept of a $\lambda^{[m]}$-tableau in [1] (for a fuller explanation, see [2], Remark 2.5).

2.5. Two Δ-tableaux $\bar{J}$ and $\bar{K}$ are row equivalent, written $\bar{J}\sim\bar{K}$, if there exists $w\in W(J)$ such that $\bar{K}=w\bar{J}$. The equivalence class which contains the Δ-tableau $\bar{J}$ is $\{\bar{J}\}$ and is called a Δ-tabloid. Let τ_Δ be the set of all Δ-tabloids, then by 2.3 we have $\tau_\Delta=\left\{\{\overline{dJ}\}\mid d\in D_\Psi\right\}$. Let C be the complex field and let M^Δ be the CW-module whose basis elements are the Δ-tabloids. Now, define $K_{\bar{J}}\in CW$ by $K_{\bar{J}}=\sum_{\sigma\in W(J')}\xi^{-f(\sigma)}(\text{sgn }\sigma)\sigma$, where $f(\sigma)$ and sgn σ are defined as in 2.1. Then we call $e_{\bar{J}}=K_{\bar{J}}\{\bar{J}\}$ the Δ-polytabloid associated with $\bar{J}$. The Specht module $S^{\Delta,\Delta'}$ is the submodule of M^Δ generated by $e_{\overline{wj}}$, where $w\in W$. A useful system $\bar{J}$ in Φ is called a good system if $d\Psi\cap\Psi'=\phi$ for $d\in D_\Psi$ then one of the following conditions is satisfied:

1. $\{\overline{dJ}\}$ appears in $e_{\bar{J}}$,

2. There exists an i $(1 \le i \le n)$ which occurs as an index of some roots in the k-th constituent of $\overline{dJ}$ such that i occurs as an index of some roots in the l-th constituent of $\bar{J}$, where $k \neq \ell$.

If $\bar{J}$ is a good system in Φ, then $S^{\Delta,\Delta'}$ is an irreducible module of W over the complex field C (see [2], Corollary 3.14).

Let $\bar{J}$ be a good system in Φ, and $w \in W$. A Δ-tableau $\overline{wJ}$ is standard if $w \in D_{\Psi} \cap D_{\Psi'}$. A Δ-tabloid $\{\overline{wJ}\}$ is standard if there is a standard Δ-tableau in the equivalence class $\{\overline{wJ}\}$. A Δ-polytabloid $e_{\overline{wJ}}$ is standard if $\overline{wJ}$ is standard.

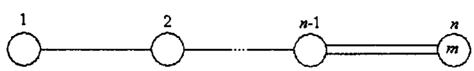
A good system $\bar{J}$ is called a very good system in Φ if, for all $d \in D_{\Psi} \cap D_{\Psi'}$ and $d' \in D_{\Psi}$, $d' = d\sigma\rho$ where $\sigma \in W(J')$, $\rho \in W(J)$ then $d \le d'$. If $\bar{J}$ is a very good system in Φ, then $\{ e_{\overline{dJ}} \mid d \in D_{\Psi} \cap D_{\Psi'} \}$ is linearly independent over C. The question arises whether this set is a C-basis for $S^{\Delta,\Delta'}$. In that case, we say that $\bar{J}$ is a perfect system in Φ if the set $\{e_{\overline{dJ}} \mid d \in D_{\Psi} \cap D_{\Psi'}\}$ is a basis for $S^{\Delta,\Delta'}$.

Thus, given a subsystem Ψ of Φ with simple system J, if we can determine a subsystem Ψ' in $\Phi \setminus \Psi$ with simple system J' such that $\bar{J}$ is a perfect system, not only is $S^{\Delta,\Delta'}$ an irreducible CW-module, but we also have a C-basis for $S^{\Delta,\Delta'}$ which consists of standard polytabloids. If J' uniquely exists, then we call J' the dual of J.

3. PERFECT SYSTEMS

We now construct the perfect systems in Φ, wherefore we present an algorithm which is a modification of Algorithm 3.1 of [4] and results in a suitable dual. In particular, application of the algorithm conforms with known results in the case of the generalized symmetric groups in [1].

Let $\Phi = B_n^m$ with simple system $\pi = \{\alpha_i = e_i - e_{i+1}\ (i = 1,...,n-1),\ \alpha_n = e_n\}$. Then the Cohen diagram for Φ is



where the node corresponding to $\alpha_i (i = 1,...,n)$ is denoted by *i*.

Definition 3.1 Let $\Phi = B_n^m$ with simple system π as above and let

$$\pi_t = \left\{\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_t} \mid \{i_1, ..., i_t\} \subseteq \{1, 2, ..., n\}\right\}$$

be a subset of π. For $m \in \{1,...,t\}$, the root $\alpha_{i_m} \in \pi_t$ is called the maximum root in π_t, written $\max\{\pi_t\} = \alpha_{i_m}$, if $\max\{i_1,...,i_t\} = i_m$.

For $\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq ... \lambda_{s_i}^{(i)} \geq 0$, $(i = 1, 2, ..., m)$ and $\sum_{j=1}^{s_1} (\lambda_j^{(1)} + 1) + \sum_{i=2}^{m} \sum_{j=1}^{s_i} \lambda_j^{(i)} = n$,

let $\Psi = \sum_{i=1}^{m} \sum_{j=1}^{s_i} B_{\lambda_j^{(i)}}^{m_i}$ be a subsystem of Φ with $m_i = \begin{cases} 1 & \text{if } i = 1 \\ m & \text{if } i = 2, 3, ..., m. \end{cases}$

Then $\lambda^{[m]} = \left((\lambda_1^{(1)} + 1, ..., \lambda_{s_1}^{(1)} + 1), ..., (\lambda_1^{(m)}, ..., \lambda_{s_m}^{(m)})\right)$ is an *m*-set of partitions of *n*. If we put $k_0^{(1)} = 0$, $k_j^{(1)} = \lambda_1^{(1)} + \lambda_2^{(1)} + ... + \lambda_j^{(1)} + j$ $(j = 1, ..., s_1)$ and $k_0^{(i)} = k_{s_1}^{(1)} + \sum_{v=2}^{i-1} \sum_{j=1}^{s_v} \lambda_j^{(v)}$, $k_j^{(i)} = k_0^{(i)} + \lambda_1^{(i)} + \lambda_2^{(i)} + ... + \lambda_j^{(i)} (j = 1, ..., s_i)$ for $i = 2, 3, ..., m$ then

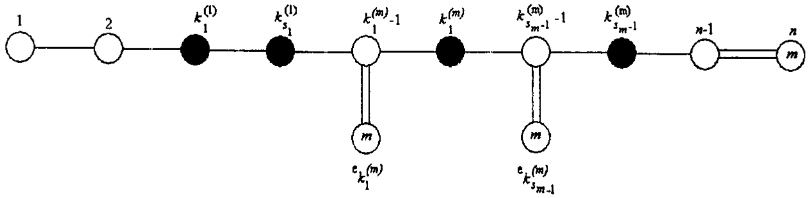
$$J_{k_j^{(1)}}^{(1)} = \left\{ \alpha_{k_{j-1}^{(1)}+1}, \alpha_{k_{j-1}^{(1)}+2}, ..., \alpha_{k_j^{(1)}-1} \right\}$$
$$= \left\{ e_{k_{j-1}^{(1)}+1} - e_{k_{j-1}^{(1)}+2}, e_{k_{j-1}^{(1)}+2} - e_{k_{j-1}^{(1)}+3}, ..., e_{k_j^{(1)}-1} - e_{k_j^{(1)}} \right\}$$

is a simple system for $B_{\lambda_j^{(1)}}^{m_1}$ and therefore $J^{(1)} = \sum_{j=1}^{s_1} J_{k_j^{(1)}}^{(1)}$ is a simple system for $\sum_{j=1}^{s_1} B_{\lambda_j^{(1)}}^{(1)}$, and

$$J_{k_j^{(i)}}^{(i)} = \left\{ \alpha_{k_{j-1}^{(i)}+1}, \alpha_{k_{j-1}^{(i)}+2}, ..., \alpha_{k_j^{(i)}-1}, e_{k_j^{(i)}} \right\}$$
$$= \left\{ e_{k_{j-1}^{(i)}+1} - e_{k_{j-1}^{(i)}+2}, e_{k_{j-1}^{(i)}+2} - e_{k_{j-1}^{(i)}+3}, ..., e_{k_j^{(i)}-1} - e_{k_j^{(i)}}, e_{k_j^{(i)}} \right\}$$

is a simple system for $B^{m_i}_{\lambda^{(i)}_j}$, where $i = 2,3,...,m$ and therefore $J^{(i)} = \sum_{j=1}^{s_i} J^{(i)}_{k^{(i)}_j}$ is a simple system for $\sum_{j=1}^{s_i} B^{(i)}_{\lambda^{(i)}_j}$ $(i = 2,3,...,m)$.

Thus, $J = \sum_{i=1}^{m} J^{(i)}$ is a simple system for Ψ. Note that for any $i \in \{1,2,...,m\}$, if $s_i = 0$ then we write $k^{(i)}_j = 0$ and $J^{(i)} = \phi$, and in the subsystems $\Psi = \sum_{i=1}^{m}\sum_{j=1}^{s_i} B^{m_i}_{\lambda^{(i)}_j}$, for any i and j if $\lambda^{(i)}_j = 0$ then we write $B^{m_i}_{\lambda^{(i)}_j} = \phi$. The Cohen diagram for Ψ is



that is, the nodes $k^{(i)}_1, ..., k^{(i)}_{s_i}$, $(i = 1,2,...,m-1)$, $k^{(m)}_1, ..., k^{(m)}_{s_m - 1}$ have been deleted and the nodes $e_{k^{(i)}_1}, ..., e_{k^{(i)}_{s_i}}$ $(i = 2,3,...,m-1)$, $e_{k^{(m)}_1}, ..., e_{k^{(m)}_{s_m-1}}$ have been added. If we do not consider the nodes $e_{k^{(i)}_1}, ..., e_{k^{(i)}_{s_i}}$ $(i = 2,3,...,m-1)$, $e_{k^{(m)}_1}, ..., e_{k^{(m)}_{s_m-1}}$ which have been added to the Cohen diagram of Φ then we may look upon the simple system J of the subsystem $\Psi = \sum_{i=1}^{m}\sum_{j=1}^{s_i} B^{m_i}_{\lambda^{(i)}_j}$ as a subset of π.

Now, let $\Psi = \sum_{i=1}^{m}\sum_{j=1}^{s_i} B^{m_i}_{\lambda^{(i)}_j}$ $\left(m_i = \begin{Bmatrix} 1 & \text{if } i = 1 \\ m & \text{if } i = 2,3,...,m \end{Bmatrix} \right)$ be a subsystem of Φ with simple system J such that

$$\pi_J = J \setminus \{\alpha \in J \mid \alpha = e_k,\ 1 \le k \le n-1\} \subseteq \pi$$

where $J = \sum_{i=1}^{m} J^{(i)}$ with $J^{(i)} = \sum_{j=1}^{s_i} J^{(i)}_{\lambda^{(i)}_j}$ $(i = 1,...,m)$ being a simple system for $\sum_{j=1}^{s_i} B^{m_i}_{\lambda^{(i)}_j}$ $(i = 1,...,m)$.

For $J^{(i)} = \sum_{j=1}^{s_i} J^{(i)}_{\lambda^{(i)}_j}$ $(i = 2,3,\dots,m)$, let

$$J^{(i)*}_{\lambda^{(i)}_j} = J^{(i)}_{\lambda^{(i)}_j} \setminus \left\{ \alpha \in J^{(i)}_{\lambda^{(i)}_j} \mid \alpha = e_k,\ 1 \le k \le n \right\}\ (i = 2,3,\dots,m),$$

where $j = 1,\dots,s_i$ for $i = 2,3,\dots,m$, and let $J^{(i)*} = \sum_{j=1}^{s_i} J^{(i)*}_{\lambda^{(i)}_j}$ $(i = 2,3,\dots,m)$.

Let $\pi \setminus \pi_J$ be the set of all the deleted nodes from the Cohen diagram of Φ. Let π^*_J be the subset of $\pi \setminus \pi_J$ such that each element in π^*_J is connected to two components $J^{(i)}$ and $J^{(j)}$ of J for all pairs $i \ne j$ in $\{2,3,\dots,m\}$, i.e.

$\pi^*_J = \left\{ \beta \in \pi \setminus \pi_J \mid (\beta, \alpha^{(i)}) \ne 0,\ (\beta, \alpha^{(j)}) \ne 0 \text{ for some } \alpha^{(i)} \in J^{(i)}, \alpha^{(j)} \in J^{(j)} \right\}$
where $i, j = 2,3,\dots,m$ with $i \ne j$.

Let $\pi^J = (\pi \setminus \pi_J) \setminus \pi^*_J$. For $i = 2,3,\dots,m$, let $\pi^{J^{(i)}} = \left\{ \beta^{(i)}_1, \beta^{(i)}_2, \dots, \beta^{(i)}_{t_i} \right\}$ be the subset of π^J such that each element in $\pi^{J^{(i)}}$ is connected to two components of $J^{(i)}$ (i.e. for each element $\gamma \in \prod^{J^{(i)}}$ $(i = 2,3,\dots,m)$ there exist two components of $J^{(i)}$ $(i = 2,3,\dots,m)$ which have a node connected to γ).

Suppose that $\pi^{J^{(1)}} = \pi^J \setminus \bigcup_{i=2}^{m} \pi^{J^{(i)}} = \left\{ \beta^{(1)}_1, \dots, \beta^{(1)}_{t_1} \right\}$. For $v = 1,2,\dots,t_1$ let

$$\Pi^{(1)}_v = \bigcup \left\{ J^{(1)}_{\lambda^{(1)}_j} \mid (\alpha^{(1)}_{\lambda^{(1)}_j}, \beta^{(1)}_v) \ne 0 \text{ for some } \alpha^{(1)}_{\lambda^{(1)}_j} \in J^{(1)}_{\lambda^{(1)}_j} \right\},$$

that is, the components of $J^{(1)}$ which have a node connected to $\beta^{(1)}_v$.

For $u = 1,2,\dots,t_i$ $(i = 2,3,\dots,m)$ let

$$\Pi^{(i)}_u = \bigcup \left\{ J^{(i)*}_{\lambda^{(i)}_j} \mid (\alpha^{(i)}_{\lambda^{(i)}_j}, \beta^{(i)}_u) \ne 0 \text{ for some } \alpha^{(i)}_{\lambda^{(i)}_j} \in J^{(i)*}_{\lambda^{(i)}_j} \right\},$$

$(i = 2,3,\dots,m)$ that is, the components of $J^{(i)*}$ which have a node connected to $\beta^{(i)}_u$ $(i = 2,3,\dots,m)$.

Algorithm 3.2 (a) For each $\beta^{(1)}_v$ $(v = 1,2,\dots,t_1)$ let $\Phi^{(1)}_v$ be the root system with simple system $\Pi^{(1)}_v \cup \left\{ \beta^{(1)}_v \right\}$ and let $n^{(1)}_v$ be the length of the longest positive root in $\Phi^{(1)}_v$.

(i) Define recursively $D_{v,t}^{(1)}$ $(t=1,\dots,n_v^{(1)})$ as follows: Put $D_{v,1}^{(1)}=\left\{\beta_v^{(1)}\right\}$ and $D_{v,t+1}^{(1)}=\left\{\tau_\alpha(\beta)\mid(\alpha,\beta)<0,\ \alpha\in\prod_v^{(1)},\ \beta\in D_{v,t}^{(1)}\right\}$.

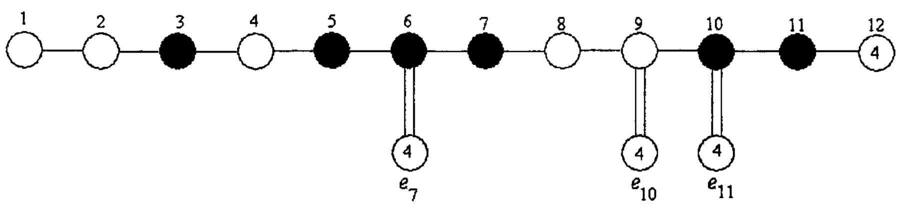
(ii) For each $\beta_v^{(1)}$ $(v=1,\dots,t_1)$ such that $\beta_v^{(1)}\neq e_n$, let $\max\left\{\prod_v^{(1)}\cup\left\{\beta_v^{(1)}\right\}\right\}=\alpha_i=e_i-e_{i+1}\ (1\le i\le n-1)$ be the maximum root in $\prod_v^{(1)}\cup\left\{\beta_v^{(1)}\right\}$. For this maximum root, choose $e_{i+1}\in\Phi$, and for $t=1,\dots,n_v^{(1)}$ put

$$D_{v,t}=\left\{\alpha\in D_{v,t}^{(1)}\mid(\alpha,e_{i+1})<0\right\},\text{ and}$$
$$D'_{v,t}=\left\{\tau_\alpha(e_{i+1})\mid\alpha\in D_{v,t}\right\}.$$

(b) For each $\beta_u^{(i)}$ $(u=1,2,\dots,t_i)$, where $i=2,3,\dots,m$, let $\Phi_u^{(i)}$ be the root system with simple system $\prod_u^{(i)}\cup\left\{\beta_u^{(i)}\right\}$ and let $n_u^{(i)}$ be the length of the longest positive root in $\Phi_u^{(i)}$.

Define recursively $D_{u,t}^{(i)}$ $(t=1,\dots,n_u^{(i)})$, where $i=2,3,\dots,m$, as follows: Put $D_{u,1}^{(i)}=\left\{\beta_u^{(i)}\right\}$ and $D_{u,t+1}^{(i)}=\left\{\tau_\alpha(\beta)\mid(\alpha,\beta)<0,\ \alpha\in\prod_u^{(i)},\ \beta\in D_{u,t}^{(i)}\right\}(i=2,3,\dots,m)$.

Example 3.3 Let $\Phi=B_{12}^4$ with simple system
$\pi=\left\{\alpha_i=e_i-e_{i+1}\ (i=1,2,\dots,11),\ \alpha_{12}=e_{12}\right\}$. Let $\Psi=B_2^{4_1}+B_1^{4_1}+B_1^{4_2}+B_3^{4_4}+B_1^{4_4}+B_1^{4_4}$ be a subsystem of B_{12}^4 with simple system $J=J^{(1)}+J^{(2)}+J^{(4)}=\left\{e_1-e_2,\ e_2-e_3,\ e_4-e_5\right\}\cup\left\{e_7\right\}\cup\left\{e_8-e_9,\ e_9-e_{10},\ e_{10},e_{11},e_{12}\right\}$.
Then the Cohen diagram for Ψ is



The set $\pi\setminus\pi_J=\left\{\alpha_3,\alpha_5,\alpha_6,\alpha_7,\alpha_{10},\alpha_{11}\right\}$ and since $(\alpha_7,e_7)\neq0$ and $(\alpha_7,\alpha_8)\neq0$ for $e_7\in J^{(2)}$ and $\alpha_8\in J^{(4)}$ then $\pi_j^*=\left\{\alpha_7\right\}$ and so $\pi^J=(\pi\setminus\pi_J)\setminus\pi_j^*=\left\{\alpha_3,\alpha_5,\alpha_6,\alpha_{10},\alpha_{11}\right\}$. For $i=2,3,4$ we have $\prod^{J^{(2)}}=\phi,\ \prod^{J^{(3)}}=\phi$ and $\prod^{J^{(4)}}=\left\{\alpha_{10},\alpha_{11}\right\}$ and so $\pi^{J^{(1)}}=\pi^J\setminus\bigcup_{i=2}^4 J^{(i)}=\left\{\alpha_3,\alpha_5,\alpha_6\right\}$.

(i) Consider $\alpha_3 \in \pi^{J^{(1)}}$ as a deleted node. Then by part (a) of Algorithm 3.2, $\Pi_3^{(1)} = \{e_1 - e_2, e_2 - e_3, e_4 - e_5\}$, and for $\max\{\Pi_3^{(1)} \cup \{\alpha_3\}\} = e_4 - e_5$, choose $e_5 \in \Phi$. Then

$$D_{3,1}^{(1)} = \{e_3 - e_4\} \qquad D'_{3,1} = \phi$$
$$D_{3,2}^{(1)} = \{e_2 - e_4, e_3 - e_5\} \qquad D'_{3,2} = \{e_3\}$$
$$D_{3,3}^{(1)} = \{e_1 - e_4, e_2 - e_5\} \qquad D'_{3,3} = \{e_2\}$$
$$D_{3,4}^{(1)} = \{e_1 - e_5\} \qquad D'_{3,4} = \{e_1\}$$

(ii) Consider $\alpha_5 \in \pi^{J^{(1)}}$ as a deleted node. Then by part (a) of Algorithm 3.2, $\Pi_5^{(1)} = \{e_4 - e_5\}$, and for $\max\{\Pi_5^{(1)} \cup \{\alpha_5\}\} = e_5 - e_6$, choose $e_6 \in \Phi$. Then

$$D_{5,1}^{(1)} = \{e_5 - e_6\} \qquad D'_{5,1} = \{e_5\}$$
$$D_{5,2}^{(1)} = \{e_4 - e_6\} \qquad D'_{5,2} = \{e_4\}$$

(iii) Consider $\alpha_6 \in \pi^{J^{(1)}}$ as a deleted node. Then by part (a) of Algorithm 3.2, $\Pi_6^{(1)} = \phi$, and for $\max\{\Pi_6^{(1)} \cup \{\alpha_6\}\} = e_6 - e_7$, choose $e_7 \in \Phi$. Then

$$D_{6,1}^{(1)} = \{e_6 - e_7\} \qquad D'_{6,1} = \{e_6\}$$

(iv) Consider $\alpha_{10} \in \pi^{J^{(4)}}$ as a deleted node. Then by part (b) of Algorithm 3.2 we have

$$\Pi_{10}^{(4)} = \{e_8 - e_9, e_9 - e_{10}\},$$
$$D_{10,1}^{(4)} = \{e_{10} - e_{11}\}$$
$$D_{10,2}^{(4)} = \{e_9 - e_{11}\}$$
$$D_{10,3}^{(4)} = \{e_8 - e_{11}\}$$

(v) Consider $\alpha_{11} \in \pi^{J^{(4)}}$ as a deleted node. Then by part (b) of Algorithm 3.2 we have

$$\Pi_{11}^{(4)} = \phi \text{ and } D_{11,1}^{(4)} = \{e_{11} - e_{12}\}$$

On the other hand, the subsystem $\Psi = B_2^{4_1} + B_1^{4_1} + B_1^{4_2} + B_3^{4_4} + B_1^{4_4} + B_1^{4_4}$ corresponds to the 4-set of partitions $\lambda^{[4]} = (321, 1, 0, 311)$ of 12. Thus the subsystem $\Psi = B_2^{4_1} + B_1^{4_1} + B_1^{4_2} + B_3^{4_4} + B_1^{4_4} + B_1^{4_4}$ is represented by the rows of the $\lambda^{[4]}$-tableau

$$t = \begin{pmatrix} 1 & 2 & 3 & & & 8 & 9 & 10 \\ 4 & 5 & & ,7, & \phi, & 11 & & \\ 6 & & & & & 12 & & \end{pmatrix}$$

Then its row stabilizer R_t is $W_{\{1,2,3\}}^1 \times W_{\{4,5\}}^1 \times W_{\{6\}}^1 \times W_{\{7\}}^4 \times W_{\{8,9,10\}}^4 \times W_{\{11\}}^4 \times W_{\{12\}}^4$ and its column stabilizer C_t is $W_{\{1,4,6\}}^4 \times W_{\{2,5\}}^4 \times W_{\{3\}}^4 \times W_{\{7\}}^1 \times W_{\{8,11,12\}}^1 \times W_{\{9\}}^1 \times W_{\{10\}}^1$, as in [1]. Now, put $J'^{(1)} = D_{3,3}^{(1)} + D'_{3,2} + D_{5,2}^{(1)} + D'_{5,1} + D'_{6,1}$, $J'^{(2)} = \phi$ and $J'^{(4)} = D_{10,3}^{(4)} + D_{11,1}^{(4)}$ then $J'^{(1)} + J'^{(2)} + J'^{(4)} = \{e_1 - e_4, e_4 - e_6, e_6, e_2 - e_5, e_5, e_3\}$ $\cup \{e_8 - e_{11}, e_{11} - e_{12}\}$ is linearly independent over C. If we put $J' = J'^{(1)} + J'^{(2)} + J'^{(4)}$, then the corresponding subsystem Ψ' is $B_3^{4_1} + B_2^{4_1} + B_1^{4_1} + B_2^{4_4}$ with simple system J'. The subsystem $\Psi' = B_3^{4_1} + B_2^{4_1} + B_1^{4_1} + B_2^{4_4}$ is represented by the columns of the $\lambda^{[4]}$-tableau t, and so $R_t \cong W(J)$ and $C_t \cong W(J')$. Thus, known results [1] in the representation theory of the generalized symmetric groups gives that $\bar{J} = \{(J^{(1)}; J'^{(1)}), (J^{(2)}; J'^{(2)}), (\phi; \phi), (J^{(4)}; J'^{(4)})\}$ is a perfect system in B_{12}^4.

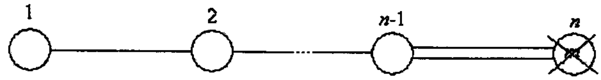
Remark 3.4 Let Ψ be a subsystem of Φ with simple system J as given earlier. Then the subsystem Ψ is represented by the rows of the $\lambda^{[m]}$-tableau as in Remark 2.5 of [2]. Since the Algorithm 3.2 enables us to construct the subsystem Ψ' such that its simple system J' is represented by the columns of the $\lambda^{[m]}$-tableau, the remainder of the paper shows that this is true in general. Furthermore, this work can be translated to the language of $\lambda^{[m]}$-tableaux in the generalized symmetric groups context, that is, the key concepts (i.e. the useful systems, good systems, very good systems and perfect systems) of this paper are reduced to the standard $\lambda^{[m]}$-tableaux.

Let $\Psi=\sum_{i=1}^{m}\sum_{j=1}^{s_i}B_{\lambda_j^{(i)}}^{m_i}$ $\left(m_i=\begin{cases}1 & \text{if } i=1\\ m & \text{if } i=2,3,\ldots,m\end{cases}\right)$ be a subsystem of Φ with simple system $J=\sum_{i=1}^{m}J^{(i)}$ as above. We now apply Algorithm 3.2 step by step to determine a subsystem Ψ' in $\Phi\setminus\Psi$ with simple system $J'=\sum_{i=1}^{m}J'^{(i)}$ such that

$$\bar{J}=\left\{(J^{(1)};J'^{(1)}),\ \ldots,\ (J^{(m)};J'^{(m)})\ \right\}$$

is a perfect system in Φ. For this, we consider the following possible cases:

(1) Let $\Psi=B_{n-1}^{m_1}$ be a subsystem of Φ with simple system $J=J^{(1)}=\{\alpha_i=e_i-e_{i+1}\ (i=1,2,\ldots,n-1)\}$. Then the Cohen diagram for $B_{n-1}^{m_1}$ is



Consider $\alpha_n\in\pi^{J^{(1)}}$ s a deleted node. Then by applying part (a) of Algorithm 3.2 we obtain

$$\Pi_n^{(1)}=\{\alpha_i=e_i-e_{i+1}\ \ (i=1,2,\ldots,n-1)\}$$
$$D_{n,1}^{(1)}=\{\alpha_n\}=\{e_n\}$$
$$D_{n,t}^{(1)}=\{\ e_{n-t+1}\}\ \ (t=2,3,\ldots,n).$$

Since $\alpha_n=e_n$, we do not consider the part (a) (ii) of Algorithm 3.2.

On the other hand, the subsystem $\Psi=B_{n-1}^{m_1}$ corresponds to the m-set of partitions $\lambda^{[m]}=(n,0,0,\ldots,0)$ of n. Thus the subsystem $\Psi=B_{n-1}^{m_1}$ is represented by the row of the $\lambda^{[m]}$-tableau

$$t=(12\ldots n,\phi,\ldots,\phi).$$

Then its row stabilizer R_t is $W_{\{1,\ldots,n\}}^{1}$ and its column stabilizer C_t is $W_{\{1\}}^{m}\times W_{\{2\}}^{m}\times\ldots\times W_{\{n\}}^{m}$, as in [1]. Now, put $J'^{(1)}=\sum_{t=1}^{n}D_{n,t}^{(1)}$ then $J'^{(1)}$ is linearly independent over C. If we put $J'=J'^{(1)}$ then J' is a simple system for $\Psi'=nB_1^{m_1}$ with $m_1=m$. The subsystem $\Psi'=nB_1^{m_1}$ with $m_1=m$ is represented by the columns of the $\lambda^{[m]}$-tableau t, and so $R_t\cong W(J)$ and $C_t\cong W(J')$. It follows that

$\bar{J} = \{(J^{(1)}; J'^{(1)}), (\phi;\phi), ..., (\phi;\phi)\}$ is a perfect system in Φ. Then we have the following Lemma.

Lemma 3.5 Let $\Psi = B^{m_1}_{n-1}$ be a subsystem of Φ with simple system given by system $J = J^{(1)} = \{\alpha_i = e_i - e_{i+1} \ (i = 1,2,...,n-1)\}$ and let Ψ' be the subsystem of Φ with simple system $J' = J'^{(1)} = \sum_{t=1}^{n} D^{(1)}_{n,t}$. Then $\bar{J} = \{(J^{(1)}; J'^{(1)}), (\phi;\phi), ..., (\phi;\phi)\}$ is a perfect system in Φ.

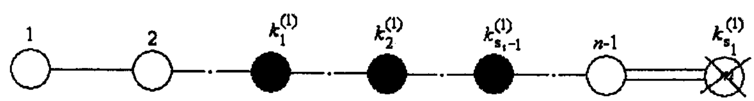
(2) For $s_1 \geq 2$, $\lambda^{(1)}_1 \geq \lambda^{(1)}_2 \geq ... \geq \lambda^{(1)}_{s_1} \geq 0$ and $\sum_{j=1}^{s_1}(\lambda^{(1)}_j + 1) = n$, let $\Psi = \sum_{j=1}^{s_1} B^{m_1}_{\lambda^{(1)}_j} \ (m_1 = 1)$ be a subsystem of Φ.

Put $k^{(1)}_0 = 0$, $k^{(1)}_j = \lambda^{(1)}_1 + \lambda^{(1)}_2 + ... + \lambda^{(1)}_j + j \ (j = 1,...s_1)$ then

$$J^{(1)}_{k^{(1)}_j} = \{\alpha_{k^{(1)}_{j-1}+1}, \alpha_{k^{(1)}_{j-1}+2}, ..., \alpha_{k^{(1)}_j - 1}\}$$

is a simple system for $B^{m_1}_{\lambda^{(1)}_j}$ and therefore $J = J^{(1)} = \sum_{j=1}^{s_1} J^{(1)}_{k^{(1)}_j}$ is a simple system for $\Psi = \sum_{j=1}^{s_1} B^{m_1}_{\lambda^{(1)}_j}$.

The Cohen diagram for Ψ is



that is, the nodes $k^{(1)}_1, k^{(1)}_2, ..., k^{(1)}_{s_1}$ have been deleted.

By part (a)(i) of Algorithm 3.2, for $v = 1,2,...,s_1$ we obtain $\Pi^{(1)}_{k^{(1)}_v}$ and $D^{(1)}_{k^{(1)}_v,t} \ (1 \leq t \leq \lambda^{(1)}_v + 1)$ as follows:

$\Pi^{(1)}_{k^{(1)}_v} = J^{(1)}_{k^{(1)}_v} \cup J^{(1)}_{k^{(1)}_{v+1}} \ (1 \leq v \leq s_1 - 1)$ and $\Pi^{(1)}_{k^{(1)}_{s_1}} = J^{(1)}_{k^{(1)}_{s_1}}$,

$$D^{(1)}_{k^{(1)}_v,t} = \left\{ \sum_{j=k^{(1)}_v+i-t}^{k^{(1)}_v+i-1} \alpha_j \mid k^{(1)}_v + i - 1 \leq k^{(1)}_{v+1} - 1 \text{ for } i \in \{1,...,t\} \right\} \begin{pmatrix} 1 \leq v \leq s_1 - 1 \\ 1 \leq t \leq \lambda^{(1)}_v + 1 \end{pmatrix},$$

$$D^{(1)}_{k^{(1)}_{s_1},t} = \left\{ e_{k^{(1)}_{s_1}+1-t} \right\} \left(1 \le t \le \lambda^{(1)}_{s_1} + 1\right).$$

By part (a)(ii) of Algorithm 3.2, for $1 \le v \le s_1 - 1$,

$$\max\left\{ \Pi^{(1)}_{k^{(1)}_v} \cup \{ \alpha_{k^{(1)}_v} \} \right\} = \begin{cases} \alpha_{k^{(1)}_{v+1}-1} & \text{if } J^{(1)}_{k^{(1)}_v} \neq \phi \text{ and } J^{(1)}_{k^{(1)}_{v+1}} \neq \phi \\ \alpha_{k^{(1)}_v} & \text{if } J^{(1)}_{k^{(1)}_v} \neq \phi \text{ and } J^{(1)}_{k^{(1)}_{v+1}} = \phi \\ & \text{or if } \Pi^{(1)}_{k^{(1)}_v} = \phi. \end{cases}$$

For $1 \le v \le s_1 - 1$, if $J^{(1)}_{k^{(1)}_v} \neq \phi$ and $J^{(1)}_{k^{(1)}_{v+1}} \neq \phi$ then choose $e_{k^{(1)}_{v+1}} \in \Phi$ and consider $D^{(1)}_{k^{(1)}_v,t}$, where $1 \le t \le \lambda^{(1)}_v$. For $i \in \{1,2,\ldots,\lambda^{(1)}_{v+1}+1\}$, we have $k^{(1)}_v + i \le k^{(1)}_{v+1}$ $(1 \le v \le s_1 - 1)$. If $\lambda^{(1)}_v = \lambda^{(1)}_{v+1}$ $(1 \le v \le s_1 - 1)$ then $k^{(1)}_v + i \le k^{(1)}_{v+1}$ for all $i \in \left\{1,2,\ldots,\lambda^{(1)}_v\right\}$, and so $(\alpha, e_{k^{(1)}_{v+1}}) = 0$ for all $\alpha \in D^{(1)}_{k^{(1)}_v,t}$ $(1 \le t \le \lambda^{(1)}_v)$. Thus $D_{k^{(1)}_v,t} = \phi$ $(1 \le t \le \lambda^{(1)}_v)$ and so $D'_{k^{(1)}_v,t} = \phi$ $(1 \le t \le \lambda^{(1)}_v)$.

If $\lambda^{(1)}_v > \lambda^{(1)}_{v+1}$ $(1 \le v \le s_1 - 1)$ then for $i = 1,2,\ldots,\lambda^{(1)}_{v+1}$ we have $k^{(1)}_v + i < k^{(1)}_{v+1}$ and $(\alpha, e_{k^{(1)}_{v+1}}) = 0$ for all $\alpha \in D^{(1)}_{k^{(1)}_v,t}$ $(1 \le t \le \lambda^{(1)}_{v+1})$ and so $D_{k^{(1)}_v,t} = \phi$ $(1 \le t \le \lambda^{(1)}_{v+1})$ and $D'_{k^{(1)}_v,t} = \phi$ $(1 \le t \le \lambda^{(1)}_{v+1})$.

For $i = \lambda^{(1)}_{v+1} + 1$ $(1 \le v \le s_1 - 1)$ we have $k^{(1)}_v + i = k^{(1)}_{v+1}$ and $(\alpha, e_{k^{(1)}_{v+1}}) < 0$ for some $\alpha \in D^{(1)}_{k^{(1)}_v,t}$ $(\lambda^{(1)}_{v+1} + 1 \le t \le \lambda^{(1)}_v)$, and so

$$D_{k^{(1)}_v,t} = \left\{ e_{k^{(1)}_{v+1}-t} - e_{k^{(1)}_{v+1}} \right\} \ (\lambda^{(1)}_{v+1} + 1 \le t \le \lambda^{(1)}_v)$$

$$D'_{k^{(1)}_v,t} = \left\{ e_{k^{(1)}_{v+1}-t} \right\} \ (\lambda^{(1)}_{v+1} + 1 \le t \le \lambda^{(1)}_v).$$

For $1 \le v \le s_1 - 1$, if $J^{(1)}_{k^{(1)}_v} \neq \phi$ and $J^{(1)}_{k^{(1)}_{v+1}} = \phi$ or if $\Pi^{(1)}_{k^{(1)}_v} \neq \phi$ then choose $e_{k^{(1)}_v+1} \in \Phi$ and so

$$D'_{k^{(1)}_v,t} = \left\{ e_{k^{(1)}_v+1-t} \right\} \ (1 \le t \le \lambda^{(1)}_v + 1).$$

On the other hand, the subsystem $\Psi = \sum_{j=1}^{s_1} B^{m_1}_{\lambda^{(1)}_j}$ corresponds to the m-set of partitions

$$\lambda^{[m]} = \left((\lambda_1^{(1)}+1,\ldots,\lambda_{s_1}^{(1)}+1),0,0,\ldots,0\right)$$

of n. Thus the subsystem $\Psi = \sum_{j=1}^{s_1} B^{m_1}_{\lambda_j^{(1)}}$ is represented by the rows of the $\lambda^{[m]}$-tableau

$$t = \left(\begin{matrix} 1 & 2 & \ldots & . & k_1^{(1)} \\ k_1^{(1)}+1 & k_1^{(1)}+2 & \ldots & k_2^{(1)} & \\ . & . & \ldots & & \\ k_{s_1-1}^{(1)}+1 & k_{s_1-1}^{(1)}+2 & \ldots & n & \end{matrix}\quad , \phi, \ \phi, \ \ldots, \phi\right)$$

If we put

$$J'^{(1)} = \sum_{v=1}^{s_1-1}\left\{ D^{(1)}_{k_v^{(1)},\lambda_v^{(1)}+1} + \sum_{t=1}^{\lambda_v^{(1)}} D'_{k_v^{(1)},t}\right\} + \sum_{t=1}^{\lambda_{s_1}^{(1)}+1} D_{k_{s_1}^{(1)},t} \ ,$$

then $J'^{(1)}$ is represented by the columns of the $\lambda^{[m]}$-tableau t and so $J'^{(1)}$ is linearly independent over C.

If Ψ' is a subsystem of Φ with simple system $J' = J'^{(1)}$ then $R_t \cong W(J)$ and $C_t \cong W(J')$, where R_t (resp. C_t) is the row (resp. column) stabilizer of the $\lambda^{[m]}$-tableau t.

It follows that $\bar{J} = \{(J^{(1)}; J'^{(1)}), (\phi;\phi), \ldots, (\phi;\phi)\}$ is a perfect system in Φ. We have therefore proved the following lemma.

Lemma 3.6 For $s_1 \geq 2$, $\lambda_1^{(1)} \geq \lambda_2^{(1)} \geq \ldots \geq \lambda_{s_1}^{(1)} \geq 0$ and $\sum_{j=1}^{s_1}(\lambda_j^{(1)}+1) = n$, let $\Psi = \sum_{j=1}^{s_1} B^{m_1}_{\lambda_j^{(1)}}$ $(m_1 = 1)$ be a subsystem of Φ. Let $k_0^{(1)} = 0$ and $k_j^{(1)} = \lambda_1^{(1)} + \ldots + \lambda_j^{(1)} + j (j = 1,2,\ldots,s_1)$. For $j = 1,\ldots,s_1$, let $J^{(1)}_{k_j^{(1)}} = \left\{\alpha_{k_{j-1}^{(1)}+1}, \alpha_{k_{j-1}^{(1)}+2}, \ldots, \alpha_{k_j^{(1)}-1}\right\}$ be a simple system for $B^{m_1}_{\lambda_j^{(1)}}$ and let $J = J^{(1)} = \sum_{j=1}^{s_1} J^{(1)}_{k_j^{(1)}}$ be a simple system for Ψ.

Let Ψ' be the subsystem of Φ with simple system

$$J' = J'^{(1)} = \sum_{v=1}^{s_1-1}\left\{ D^{(1)}_{k_v^{(1)},\lambda_v^{(1)}+1} + \sum_{t=1}^{\lambda_v^{(1)}} D'_{k_v^{(1)},t}\right\} + \sum_{t=1}^{\lambda_{s_1}^{(1)}+1} D^{(1)}_{k_{s_1}^{(1)},t}\,.$$

Then $\bar{J} = \{(J^{(1)};\, J'^{(1)}),\ (\phi;\phi),\ \ldots,(\phi;\phi)\}$ is a perfect system in Φ.

(3) Let $i \in \{2,3,\ldots,m\}$. For $\lambda_1^{(i)} \ge \lambda_2^{(i)} \ge \ldots \ge \lambda_{s_i}^{(i)} > 0$ and $\sum_{j=1}^{s_1} \lambda_{k_j}^{(i)} = n$, let

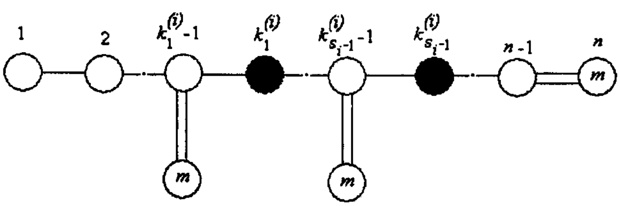
$\Psi = \sum_{j=1}^{s_1} B^{m_i}_{\lambda_j^{(i)}}$ $(m_i = m)$ be a subsystem of Φ.

If we put $k_0^{(1)} = 0,\ k_j^{(1)} = \lambda_1^{(i)} + \lambda_2^{(i)} + \ldots + \lambda_j^{(i)}$ $(j = 1,2,\ldots,s_i)$ then

$$J^{(i)}_{k_j^{(i)}} = \left\{\alpha_{k_{j-1}^{(i)}+1},\ \alpha_{k_{j-1}^{(i)}+2},\ \ldots,\alpha_{k_j^{(i)}-1},e_{k_j^{(i)}}\right\}$$

is a simple system for $B^{m_i}_{\lambda_j^{(i)}}$ and therefore $J = J^{(i)} = \sum_{j=1}^{s_i} J^{(i)}_{k_j^{(i)}}$ is a simple system for

$\Psi = \sum_{j=1}^{s_i} B^{m_i}_{\lambda_j^{(i)}}$. The Cohen diagram for Ψ is



that is, the nodes $k_1^{(i)}, k_2^{(i)}, \ldots, k_{s_i-1}^{(i)}$ have been deleted. By applying part (b) of Algorithm 3.2, for $u = 1,2,\ldots,s_i - 1$ we have

$$D^{(i)}_{k_u^{(i)},t} = \left\{ \sum_{j=k_u^{(i)}+l-t}^{k_u^{(i)}+l-1} \alpha_j \mid k_u^{(i)} + l - 1 \le k_{u+1}^{(i)} - 1 \ \text{ for } \ i \in \{1,2,\ldots,t\}\right\}$$

where $1 \le t \le \lambda_u^{(i)}$.

On the other hand, the subsystem $\Psi = \sum_{j=1}^{s_i} B^{m_i}_{\lambda_j^{(i)}}$ corresponds to the m-set of partitions

$$\lambda^{[m]} = \left(0,\ \ldots,0,(\lambda_1^{(i)},\ldots,\lambda_{s_i}^{(i)}),\ 0,\ldots,0\right)$$

of n. Thus the subsystem $\Psi=\sum_{j=1}^{s_i} B_{\lambda_j^{(i)}}^{m_i}$ is represented by the rows of the $\lambda^{[m]}$-tableau

$$t=\begin{pmatrix} & 1 & 2 & \cdots & & k_1^{(i)} & & \\ \phi,\ldots,\phi, & k_1^{(i)}+1 & k_1^{(i)}+2 & \cdots & k_2^{(i)} & & , & \phi,\ldots,\phi \\ & \cdot & \cdot & \cdots & & & & \\ & k_{s_i-1}^{(i)}+1 & k_{s_i-1}^{(i)}+2 & \cdots & n & & & \end{pmatrix}$$

Now, for $u=1,2,\ldots,s_i-1$, put

$$J'^{(i)}=\sum_{u=1}^{s_i-1} D_{k_u^{(i)},\lambda_u^{(i)}}^{(i)}.$$

Then $J'^{(i)}$ is represented by the columns of the $\lambda^{[m]}$-tableau t and so $J'^{(i)}$ is linearly independent over C. If Ψ' is a subsystem of Φ with simple system $J'=J'^{(i)}$ then $R_t\cong W(J)$ and $C_t\cong W(J')$, where R_t (resp. C_t) is the row (resp. column) stabilizer of the $\lambda^{[m]}$-tableau t.

It follows that $\overline{J}=\{(\phi;\phi),\ldots,(\phi;\phi),(J^{(i)};J'^{(i)}),(\phi;\phi),\ldots,(\phi;\phi)\}$ is a perfect system in Φ. Then we have the following lemma.

Lemma 3.7 Let $i\in\{2,3,\ldots,m\}$. For $\lambda_1^{(i)}\geq\lambda_2^{(i)}\geq\ldots\geq\lambda_{s_i}^{(i)}>0$ and $\sum_{j=1}^{s_i}\lambda_j^{(i)}=n$, let

$\Psi=\sum_{j=1}^{s_i} B_{\lambda_j^{(i)}}^{m_i}$ $(m_i=m)$ be a subsystem of Φ.

Let $k_0^{(i)}=0$, $k_j^{(i)}=\lambda_1^{(i)}+\lambda_2^{(i)}+\ldots+\lambda_j^{(i)}$ $(j=1,2,\ldots,s_i)$. For $j=1,2,\ldots,s_i$ let

$$J_{k_j^{(i)}}^{(i)}=\left\{\alpha_{k_{j-1}^{(i)}+1},\ \alpha_{k_{j-1}^{(i)}+2},\ \ldots,\alpha_{k_j^{(i)}-1},e_{k_j^{(i)}}\right\}$$

is a simple system for $B_{\lambda_j^{(i)}}^{m_i}$ and $J=J^{(i)}=\sum_{j=1}^{s_i} J_{k_j^{(i)}}^{(i)}$ is a simple system for $\Psi=\sum_{j=1}^{s_i} B_{\lambda_j^{(i)}}^{(i)}$. Let Ψ' be the subsystem of Φ with simple system

$$J'=J'^{(i)}=\sum_{u=1}^{s_i-1} D_{k_u^{(1)},\lambda_u^{(i)}}^{(i)}.$$

Then $\bar{J} = \{(\phi;\phi), \ldots, (\phi;\phi), (J^{(i)}; J'^{(i)}), (\phi;\phi), \ldots, (\phi;\phi)\}$ is a perfect system in Φ.

(4) For $\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \ldots \geq \lambda_{s_i}^{(i)} > 0$ $(i = 1,2,\ldots,m)$ and $\sum_{j=1}^{s_i}(\lambda_{k_j}^{(1)} + 1) + \sum_{i=2}^{m}\sum_{j=1}^{s_i} \lambda_j^{(i)} = n$,

let $\Psi = \sum_{i=1}^{m}\sum_{j=1}^{s_i} B_{\lambda_j^{(i)}}^{m_i}$ be a subsystem of Φ with $m_i = \begin{cases} 1 & \text{if } i = 1 \\ m & \text{if } i = 2,3,\ldots,m. \end{cases}$

If we put $k_0^{(1)} = 0$, $k_j^{(1)} = \lambda_1^{(1)} + \ldots + \lambda_j^{(1)} + j$ $(j = 1,2,\ldots,s_1)$ and $k_0^{(i)} = k_{s_1}^{(1)} + \sum_{v=2}^{i-1}\sum_{j=1}^{s_v} \lambda_j^{(v)}$, $k_j^{(i)} = k_0^{(i)} + \lambda_1^{(i)} + \ldots + \lambda_j^{(i)}$ $(j = 1,2,\ldots,s_i)$ for $i = 2,3,\ldots,m$ then

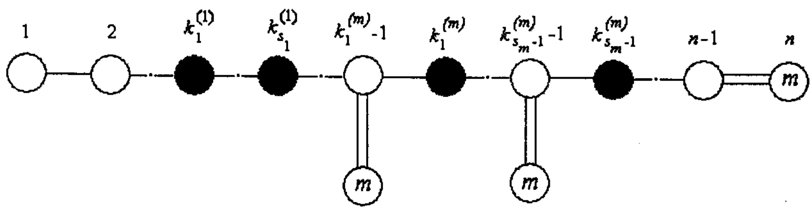
$$J_{k_j^{(1)}}^{(1)} = \left\{\alpha_{k_{j-1}^{(1)}+1}, \alpha_{k_{j-1}^{(1)}+2}, \ldots, \alpha_{k_j^{(1)}-1}\right\}$$

is a simple system for $B_{\lambda_j^{(1)}}^{m_1}$ and therefore $J^{(1)} = \sum_{j=1}^{s_1} J_{k_j^{(1)}}^{(1)}$ is a simple system for $\sum_{j=1}^{s_1} B_{\lambda_j^{(1)}}^{m_1}$, and

$$J_{k_j^{(i)}}^{(i)} = \left\{\alpha_{k_{j-1}^{(i)}+1}, \alpha_{k_{j-1}^{(i)}+2}, \ldots, \alpha_{k_j^{(i)}-1}, e_{k_j^{(i)}}\right\}$$

is a simple system for $B_{\lambda_j^{(i)}}^{m_i}$ $(i = 2,3,\ldots,m)$ and therefore $J^{(i)} = \sum_{j=1}^{s_i} J_{k_j^{(i)}}^{(i)}$ is a simple system for $\sum_{j=1}^{s_i} B_{\lambda_j^{(i)}}^{m_i}$ $(i = 2,3,\ldots,m)$. Thus $J = \sum_{i=1}^{m} J^{(i)}$ is a simple system for Ψ.

The Cohen diagram for Ψ is



that is, the nodes $k_1^{(i)},\dots,k_{s_i}^{(i)}$ $(i=1,2,\dots,m-1)$, $k_1^{(m)},\dots,k_{s_m-1}^{(m)}$ have been deleted. By part (a)(i) of Algorithm 3.2, for $\nu=1,2,\dots,s_1$ we obtain $\Pi_{k_\nu^{(1)}}^{(1)}$ and $D_{k_\nu^{(1)}}^{(1)}$ $(1\le t\le \lambda_\nu^{(1)}+1)$ as follows: $\Pi_{k_\nu^{(1)}}^{(1)}=J_{k_\nu^{(1)}}^{(1)}\cup J_{k_{\nu+1}^{(1)}}^{(1)}$ $(1\le \nu\le s_1-1)$ and $\Pi_{k_{s_1}^{(1)}}^{(1)}=J_{k_{s_1}^{(1)}}^{(1)}$,

$$D_{k_\nu^{(1)},t}^{(1)}=\left\{\sum_{j=k_\nu^{(1)}+i-t}^{k_\nu^{(1)}+i-1}\alpha_j \mid k_\nu^{(1)}+i-1\le k_{\nu+1}^{(1)}-1 \text{ for } i\in\{1,\dots,t\}\right\}\begin{pmatrix}1\le \nu\le s_1-1\\ 1\le t\le \lambda_\nu^{(1)}+1\end{pmatrix},$$

$$D_{k_{s_1}^{(1)},t}^{(1)}=\left\{\sum_{j=k_{s_1}^{(1)}+1-t}^{k_{s_1}^{(1)}}\alpha_j\right\}\left(1\le t\le \lambda_{s_1}^{(1)}+1\right).$$

By part (a)(ii) of Algorithm 3.2, for $1\le \nu\le s_1-1$,

$$\max\left\{\Pi_{k_\nu^{(1)}}^{(1)}\cup\{\alpha_{k_\nu^{(1)}}\}\right\}=\begin{cases}\alpha_{k_{\nu+1}^{(1)}-1} & \text{if } J_{k_\nu^{(1)}}^{(1)}\ne\phi \text{ and } J_{k_{\nu+1}^{(1)}}^{(1)}\ne\phi\\ \alpha_{k_\nu^{(1)}} & \text{if } J_{k_\nu^{(1)}}^{(1)}\ne\phi \text{ and } J_{k_{\nu+1}^{(1)}}^{(1)}=\phi\\ & \text{or if } \Pi_{k_\nu^{(1)}}^{(1)}=\phi.\end{cases}$$

For $1\le \nu\le s_1-1$, if $J_{k_\nu^{(1)}}^{(1)}\ne\phi$ and $J_{k_{\nu+1}^{(1)}}^{(1)}\ne\phi$ then choose $e_{k_{\nu+1}^{(1)}}\in\Phi$ and consider $D_{k_\nu^{(1)},t}^{(1)}$, where $1\le t\le \lambda_\nu^{(1)}$. For $i\in\{1,2,\dots,\lambda_{\nu+1}^{(1)}+1\}$, we have $k_\nu^{(1)}+i\le k_{\nu+1}^{(1)}$ $(1\le \nu\le s_1-1)$. If $\lambda_\nu^{(1)}=\lambda_{\nu+1}^{(1)}$ $(1\le \nu\le s_1-1)$ then $k_\nu^{(1)}+i\le k_{\nu+1}^{(1)}$ for all $i\in\left\{1,2,\dots,\lambda_\nu^{(1)}\right\}$, and so $(\alpha,e_{k_{\nu+1}^{(1)}})=0$ for all $\alpha\in D_{k_\nu^{(1)},t}^{(1)}$ $(1\le t\le\lambda_\nu^{(1)})$. Thus $D_{k_\nu^{(1)},t}=\phi$ $(1\le t\le \lambda_\nu^{(1)})$ and so $D'_{k_\nu^{(1)},t}=\phi$ $(1\le t\le\lambda_\nu^{(1)})$.

If $\lambda_\nu^{(1)}>\lambda_{\nu+1}^{(1)}$ $(1\le \nu\le s_1-1)$ then for $i=1,2,\dots,\lambda_{\nu+1}^{(1)}$ we have $k_\nu^{(1)}+i<k_{\nu+1}^{(1)}$ and $(\alpha,e_{k_{\nu+1}^{(1)}})=0$ for all $\alpha\in D_{k_\nu^{(1)},t}^{(1)}$ $(1\le t\le\lambda_{\nu+1}^{(1)})$ and so $D_{k_\nu^{(1)},t}=\phi$ $(1\le t\le \lambda_{\nu+1}^{(1)})$ and $D'_{k_\nu^{(1)},t}=\phi$ $(1\le t\le\lambda_{\nu+1}^{(1)})$.

For $i=\lambda_{\nu+1}^{(1)}+1$ $(1\le \nu\le s_1-1)$ we have $k_\nu^{(1)}+i=k_{\nu+1}^{(1)}$ and $(\alpha,e_{k_{\nu+1}^{(1)}})<0$ for some $\alpha\in D_{k_\nu^{(1)},t}^{(1)}$ $(\lambda_{\nu+1}^{(1)}+1\le t\le\lambda_\nu^{(1)})$, and so

$$D_{k_\nu^{(1)},t}=\left\{e_{k_{\nu+1}^{(1)}-t}-e_{k_{\nu+1}^{(1)}}\right\}(\lambda_{\nu+1}^{(1)}+1\le t\le\lambda_\nu^{(1)})$$

$$D'_{k_v^{(1)},t} = \left\{ e_{k_{v+1}^{(1)}-t} \right\} \ (\lambda_{v+1}^{(1)}+1 \le t \le \lambda_v^{(1)}).$$

For $1 \le v \le s_1 - 1$, if $J_{k_v^{(1)}}^{(1)} \ne \phi$ and $J_{k_{v+1}^{(1)}}^{(1)} = \phi$ or if $\Pi_{k_v^{(1)}}^{(1)} \ne \phi$ then choose $e_{k_v^{(1)}+1} \in \Phi$ and so

$$D'_{k_v^{(1)},t} = \left\{ e_{k_v^{(1)}+1-t} \right\} \ (1 \le t \le \lambda_v^{(1)}+1).$$

By part (a)(ii) of Algorithm 3.2, for $v = s_1$, $\max\left\{ \Pi_{k_{s_1}^{(1)}}^{(1)} \cup \left\{ \alpha_{k_{s_1}^{(1)}} \right\} \right\} = \alpha_{k_{s_1}^{(1)}}$ and choose $e_{k_{s_1}^{(1)}+1} \in \Phi$ and so $D'_{k_{s_1}^{(1)},t} = \left\{ e_{k_{s_1}^{(1)}+1-t} \right\} \ (1 \le t \le \lambda_{s_1}^{(1)}+1)$.

By part (b) of Algorithm 3.2, for $u = 1,2,\ldots s_i - 1$ $(i = 2,3,\ldots,m)$ we have

$$D_{k_u^{(i)},t}^{(i)} = \left\{ \sum_{j=k_u^{(i)}+l-t}^{k_u^{(i)}+l-1} \alpha_j \mid k_u^{(i)} + l - 1 \le k_{u+1}^{(i)} - 1 \text{ for } i \in \{1,2,\ldots,t\} \right\}$$

where $1 \le t \le \lambda_u^{(i)}$, for $i = 2,3,\ldots,m$.

On the other hand, the subsystem $\Psi = \sum_{i=1}^{m} \sum_{j=1}^{s_i} B_{\lambda_j^{(i)}}^{m_i}$ corresponds to the m-set of partitions

$$\lambda^{[m]} = \left((\lambda_1^{(1)}+1,\ldots,\lambda_{s_1}^{(1)}+1),\ldots,(\lambda_1^{(m)},\ldots,\lambda_{s_m}^{(m)}) \right) \text{ of } n.$$

Thus the subsystem $\Psi = \sum_{i=1}^{m} \sum_{j=1}^{s_i} B_{\lambda_j^{(i)}}^{m_i}$ is represented by the rows of the $\lambda^{[m]}$-tableau t below

$$\left(\begin{array}{cccccc} 1 & 2 & \cdot & \cdot & \cdot & k_1^{(1)} \\ k_1^{(1)}+1 & k_1^{(1)}+2 & \cdot & \cdot & k_2^{(1)} & \\ \cdot & \cdot & \cdot & \cdot & & \\ k_{s_1-1}^{(1)}+1 & k_{s_1-1}^{(1)}+2 & \cdot & k_{s_1}^{(1)} & & \end{array}, \ldots, \begin{array}{cccccc} k_{s_{m-1}}^{(m-1)}+1 & k_{s_{m-1}}^{(m-1)}+2 & \cdot & \cdot & \cdot & k_1^{(m)} \\ k_1^{(m)}+1 & k_1^{(m)}+2 & \cdot & \cdot & k_2^{(m)} & \\ \cdot & \cdot & \cdot & \cdot & & \\ k_{s_m-1}^{(m)}+1 & k_{s_m-1}^{(m)}+2 & \cdot & n & & \end{array} \right)$$

Now, for $v = 1,2,\ldots,s_1 - 1$, put

$$J'^{(1)} = \sum_{v=1}^{s_i-1} \left\{ D_{k_v^{(1)},\lambda_v^{(1)}+1}^{(1)} + \sum_{t=1}^{\lambda_v^{(1)}} D'_{k_v^{(1)},t} \right\} + \sum_{t=1}^{\lambda_{s_1}^{(1)}+1} D'_{k_{s_1}^{(1)},t},$$

and for $u = 1,2,\ldots,s_i - 1$ $(i = 2,3,\ldots,m)$, put

$$J'^{(i)} = \sum_{u=1}^{s_i-1} D^{(i)}_{k_u^{(i)},\lambda_u^{(i)}} \quad (i = 2,3,\ldots,m).$$

Then $\sum_{i=1}^{m} J'^{(i)}$ is represented by the columns of the $\lambda^{[m]}$-tableau t and so $\sum_{i=1}^{m} J'^{(i)}$ is linearly independent over C. If Ψ' is a subsystem of Φ with simple system $J' = \sum_{i=1}^{m} J'^{(i)}$ then $R_t \cong W(J)$ and $C_t \cong W(J')$, where R_t (resp. C_t) is the row (resp. column) stabilizer of the $\lambda^{[m]}$-tableau t.

It follows that $\bar{J} = \{(J^{(1)};J'^{(1)}),\ldots,(J^{(m)};J'^{(m)})\}$ is a perfect system in Φ. Then we have the following theorem.

Theorem 3.8 For $\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \ldots \geq \lambda_{s_i}^{(i)} \geq 0$ $(i = 1,2,\ldots,m)$ and $\sum_{j=1}^{s_i}(\lambda_j^{(i)} + 1) + \sum_{i=2}^{m}\sum_{j=1}^{s_i}\lambda_j^{(i)} = n,$ let $\Psi = \sum_{i=1}^{m}\sum_{j=1}^{s_i} B^{m_i}_{\lambda_j^{(i)}}$ be a subsystem of Φ with

$$m_i = \begin{cases} 1 & \text{if } i = 1 \\ m & \text{if } i = 2,3,\ldots,m. \end{cases}$$

Let $k_0^{(1)} = 0$, $k_j^{(1)} = \lambda_1^{(1)} + \ldots + \lambda_j^{(1)} + j$ $(j = 1,2,\ldots,s_1)$ and $k_0^{(i)} = k_{s_1}^{(1)} + \sum_{v=2}^{i-1}\sum_{j=1}^{s_v}\lambda_j^{(v)}$, $k_j^{(i)} = k_0^{(i)} + \lambda_1^{(i)} + \ldots + \lambda_j^{(i)}$ $(j = 1,2,\ldots,s_i)$ for $i = 2,3,\ldots,m$. For $j = 1,2,\ldots,s_i$, let

$$J^{(1)}_{k_j^{(1)}} = \left\{\alpha_{k_{j-1}^{(1)}+1},\ \alpha_{k_{j-1}^{(1)}+2},\ \ldots,\alpha_{k_j^{(1)}-1}\right\}$$

be a simple system for $B^{m_1}_{\lambda_j^{(1)}}$ and for $j = 1,2,\ldots,s_i$ $(i = 2,3,\ldots,m)$ let

$$J^{(i)}_{k_j^{(i)}} = \left\{\alpha_{k_{j-1}^{(i)}+1},\alpha_{k_{j-1}^{(i)}+2},\ldots,\alpha_{k_j^{(i)}-1},e_{k_j^{(i)}}\right\}$$

be a simple system for $B^{m_i}_{\lambda_j^{(i)}}$ $(i = 2,3,\ldots,m)$. Let $J = \sum_{i=1}^{m} J^{(i)}$ be a simple system for $\Psi = \sum_{i=1}^{m}\sum_{j=1}^{s_i} B^{m_i}_{\lambda_j^{(i)}}$ where $J^{(i)} = \sum_{j=1}^{s_i} J^{(i)}_{k_j^{(i)}}$ $(i = 1,2,\ldots,m)$.

Let Ψ' be the subsystem of Φ with simple system

$$J' = \sum_{i=1}^{m} J'^{(i)} = \sum_{v=1}^{s_i - 1} \left\{ D^{(1)}_{k_v^{(1)}, \lambda_v^{(1)}+1} + \sum_{t=1}^{\lambda_v^{(1)}} D'_{k_v^{(1)}, t} \right\} + \sum_{t=1}^{\lambda_{s_1}^{(1)}+1} D'_{k_{s_1}^{(1)}, t} + \sum_{i=2}^{m} \sum_{u=1}^{s_i - 1} D^{(i)}_{k_u^{(i)}, \lambda_u^{(i)}} .$$

Then $\bar{J} = \{(J^{(1)};\ J'^{(1)}) ,..., (J^{(m)};\ J'^{(m)})\}$ is a perfect system in Φ.

ÖZET

Weyl gruplarının reprezantasyonlarının kombinatoryal inşasını vermede mükemmel sistemler çok önemli bir rol oynar. Bu makalede, *G(m,1,n)* tipindeki kompleks kök sistemler ile ilgili mükemmel sistemleri elde etmek için bir algoritma veriyoruz. Bu algoritmayı kullanarak *G(m,1,n)* kompleks yansıma gruplarının Specht modülleri için bir baz buluyoruz. Bu algoritmanın uygulaması genelleştirilmiş simetrik grupların reprezantasyon teorisindeki bilinen sonuçlarla tam bir uyum göstermektedir. Böylece, kök sistemlere göre *G(m,1,n)* grubunun indirgenemez reprezantasyonlarının kombinatoryal inşası tamamlamış olur.

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