

AN EXPLICIT CHARACTERIZATION OF DUAL SPHERICAL CURVE

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ABSTRACT

In this paper, a differential equation characterizing the dual spherical curves and an explicit solution of this differential equation is given. Without the precondition on the dual torsion, which is the torsion is nowhere zero, a necessary and sufficient condition for a dual curve to be spherical curve is given. Finally, the dual spherical version of the planar evolute in plane kinematics is presented.

1. INTRODUCTION

As a rigid body moves in space lines embedded in the body trace ruled surfaces. These lines may be the axes of the joints of spatial mechanisms or manipulators or the line of action of the end-of-arm tooling of a manipulator. The integral invariants of line trajectories seeks to characterize the shape of the trajectory ruled surface and relate it to the motion of body carrying the line that generates it.

Here the presentation of ruled surface is based on work by W. Blaschke and E. Study [2,5,9,11] from the point of view of the theory of invariants of a transitive transformation group. The fundamental idea is to replace points by lines as fundamental concepts of geometry. Points are defined by the totality of straight lines passing through them. Oriented lines in a Euclidean three-space E^3 may be represented by unit vectors with three components over the ring of dual members. A differentiable curve on the dual unit sphere depending on a one-real parameter "t" corresponds to a ruled surface in E^3 . This correspondence is one-to-one and allows the geometry of ruled surfaces to be represented by the geometry of dual spherical curves on a dual unit sphere. It allows a complete generalization of the mathematical expression for the spherical point geometry to the spatial line geometry by means of dual number extension, i.e. replacing all ordinary quantities by the corresponding dual number quantities [5,11,12].

Dual spherical geometry, expressed with the help of dual unit vectors, is closely analogous to real spherical geometry, expressed with the help of real unit vectors. Therefore, the properties of elementary real spherical geometry can also be carried over by analogy into the geometry of lines in E^3 .

References [1,2,5,9,11,12] contain the necessary basic concepts about the dual elements and one-to-one correspondance between ruled surfaces and one-parameter dual spherical motions.

2. BASIC CONCEPTS

If a and a^* are real numbers and $\varepsilon \neq 0$, $\varepsilon^2 = 0$, the combination

$$A = a + \varepsilon a^*, \quad (2.1)$$

is called a dual number. Hence ε is a dual unit. Dual numbers are considered as polynomials in ε , subject to the defining relation $\varepsilon^2 = 0$. W. K. Clifford defined the dual numbers and showed that they form an algebra, not a field. The pure dual numbers εa^* are zero divisors, $(\varepsilon a^*)(\varepsilon b^*) = 0$. No number εa^* has an inverse in the algebra. But the other laws of the algebra of dual numbers are the same as the laws of algebra of complex numbers. This means dual numbers form a ring over the real number field. For example, two dual numbers A and $B = b + \varepsilon b^*$ are added componentwise.

$$A + B = (a + b) + \varepsilon(a^* + b^*), \quad (2.2)$$

and they are multiplied by

$$AB = ab + \varepsilon(a^*b + ab^*). \quad (2.3)$$

For the equality of A and B we have

$$A = B \Leftrightarrow a = b, \text{ and } a^* = b^*. \quad (2.4)$$

An oriented line in E^3 may be given by two points on it, x and y . If μ is any non-zero constant, the parametric equation of the line is:

$$y = x + \mu a, \quad (2.5)$$

where a is a unit vector along the line. The moment of a with respect to the origin coordinates is

$$a^* = x \times a = y \times a. \quad (2.6)$$

This means that a and a^* are not independent of the choice of the points on the line. The two vectors a and a^* are not independent of one another; they satisfy the following equations:

$$\langle a, a \rangle = 1, \langle a, a^* \rangle = 0. \quad (2.7)$$

The six components a_i, a_i^* ($i=1,2,3$) of the vectors a and a^* are Plucker homogeneous line coordinates. Hence the two vectors a and a^* determine the oriented line. A point z is on this line if and only if

$$z \times a = a^*. \quad (2.8)$$

The set of oriented lines in E^3 is in one-to-one correspondance with pairs of vectors subject to the conditions (2.7), and so we may expect to represent it as a certain four-dimensional set in \mathcal{R}^6 of sixtuples of real numbers; we may take the space D^3 of triples of dual numbers with coordinates:

$$X_i = x_i + \varepsilon x_i^* \quad (i = 1,2,3). \quad (2.9)$$

Each line in E^3 is represented by the dual vector in D^3

$$\mathbf{A} = \mathbf{a} + \varepsilon \mathbf{a}^*; \tag{2.10}$$

$$\langle \mathbf{A}, \mathbf{A} \rangle = \langle \mathbf{a}, \mathbf{a} \rangle + 2\varepsilon \langle \mathbf{a}, \mathbf{a}^* \rangle = 1. \tag{2.11}$$

Theorem 2.1.(E.Study). The oriented lines in E^3 are in one-to-one correspondence with points of the dual unit sphere $\langle \mathbf{A}, \mathbf{A} \rangle = 1$ in D^3 .

By using this correspondence, one can derive the properties of the spatial motion of a line. Hence the geometry of ruled surface is represented by the geometry of curves on the dual unit sphere in D^3 .

3. THE BLASCHKE FRAME

A ruled surface is a surface swept out by a straight line L moving along a curve $\mathbf{z} = \mathbf{z}(t)$. The various position of the generating lines L are called the rulings of the surface. Such a surface, thus, has a parameterization in the ruled form:

$$\mathbf{M}(t, u) = \mathbf{z}(t) + u\mathbf{a}(t), \quad u \in \mathfrak{R} \tag{3.1}$$

where $\mathbf{z} = \mathbf{z}(t)$ is called the base curve, $\mathbf{a} = \mathbf{a}(t)$ is the unit vector giving the direction of generating line, and t is the motion parameter. The base curve is not unique, since any curve of the form:

$$\mathbf{C}(t) = \mathbf{z}(t) + \eta(t) \mathbf{a}(t), \tag{3.2}$$

may be used as its base curve, $\eta(t)$ is a smooth function. If there exist a common perpendicular to two preceding rulings on $M = M(u, t)$, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the striction curve. In (3.2) if

$$\eta(t) = -\frac{\langle \mathbf{z}', \mathbf{a}' \rangle}{\|\mathbf{a}'\|^2}, \quad (\prime = \frac{d}{dt}),$$

then $\mathbf{C}(t)$ is called the striction curve on the rulings, and it is unique.

E. Study Theorem allows us to form equation (3.1) by the dual vector function given as:

$$\mathbf{X}(t) = \mathbf{x}(t) + \varepsilon \mathbf{z}(t) \times \mathbf{x}(t) = \mathbf{x}(t) + \varepsilon \mathbf{x}^*(t). \tag{3.3}$$

Thus, the ruled surface is represented by the dual vector $\mathbf{X} = \mathbf{X}(t)$, also is dual unit as:

$$\langle \mathbf{X}, \mathbf{X} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + 2\varepsilon \langle \mathbf{x}, \mathbf{z} \times \mathbf{x} \rangle + \varepsilon^2 \langle \mathbf{z} \times \mathbf{x}, \mathbf{z} \times \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle = 1. \tag{3.4}$$

Then, the ruled surface can be represented by dual curve on dual unit sphere in D^3 .

We now define an orthonormal moving frame along this dual curve as follows:

$$\mathbf{A}_1 = \mathbf{X}(t), \quad \mathbf{A}_2 = \frac{\mathbf{A}_1'}{\|\mathbf{A}_1'\|}, \quad \mathbf{A}_3 = \mathbf{A}_1 \times \mathbf{A}_2, \tag{3.5}$$

and we can show that $\langle \mathbf{A}_1, \mathbf{A}_1 \rangle = \langle \mathbf{A}_2, \mathbf{A}_2 \rangle = \langle \mathbf{A}_3, \mathbf{A}_3 \rangle = 1$. It is necessary to suppose that $\mathbf{x}(t)$ is not a constant vector, i.e., that the ruled surface is not a cylinder. And we set $\mathbf{x}^*(t) \neq 0$, in this case the ruled surface is not a cone. By hypothesis \mathbf{A}_2

is well defined, differentiation of $\langle \mathbf{A}_1, \mathbf{A}_1 \rangle = 1$, shows that $\langle \mathbf{A}_1, \mathbf{A}_2 \rangle = 0$. The remaining orthogonality relations $\langle \mathbf{A}_1, \mathbf{A}_3 \rangle = \langle \mathbf{A}_3, \mathbf{A}_2 \rangle = 0$ follows from (3.5). The frame in (3.5) is called the Blaschke frame and it has an intuitive interpretation in E^3 . $\langle \mathbf{A}_1, \mathbf{A}_2 \rangle = 0$ means that the two lines are orthogonal. \mathbf{A}_1 , \mathbf{A}_2 and \mathbf{A}_3 are therefore three concurrent mutually orthogonal lines in E^3 . Their point of intersection is the central point on the ruling \mathbf{A}_1 . $\mathbf{A}_3(t)$ is the limit position of the common perpendicular to $\mathbf{A}_1(t)$ and $\mathbf{A}_1(t+dt)$, and is called the central tangent of the ruled surface $\mathbf{A}_1 = \mathbf{A}(t)$ at the central point. The line $\mathbf{A}_2 = \mathbf{A}(t)$ is called the central normal of $\mathbf{A}_1 = \mathbf{A}(t)$ at the central point. By construction, the Blaschke formula is

$$\frac{d}{dt} \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{pmatrix} = \begin{pmatrix} 0 & P & 0 \\ -P & 0 & Q \\ 0 & -Q & 0 \end{pmatrix} \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{pmatrix}, \quad (3.6)$$

where $P = p + \varepsilon p^* = \|\mathbf{A}'_1\|$, $Q = q + \varepsilon q^* = \frac{\det(\mathbf{A}, \mathbf{A}', \mathbf{A}'')}{p^2}$ are called the Blaschke integral invariants. The integrals $\int P dt$, and $\int Q dt$ are the dual arc-length of the dual curves $\mathbf{A}_1(t)$ and $\mathbf{A}_3(t)$, respectively.

One of the invariants of the ruled surface $\mathbf{X}=\mathbf{X}(t)$ is

$$\Sigma = \sigma + \varepsilon \sigma^* = \frac{Q}{P}, \quad P \neq 0, \quad (3.7)$$

known as the dual geodesic curvature [5]. The formula for Σ is obtained in terms of $\mathbf{X}=\mathbf{X}(t)$ and its derivatives with respect to t :

$$\Sigma = \frac{(\mathbf{X}, \mathbf{X}', \mathbf{X}'')}{\|\mathbf{X}'\|^2}. \quad (3.8)$$

By using Σ , we can easily obtain the Blaschke formulas for derivatives with respect to the dual arc-length $dS = \int P dt$ (denoted by dots):

$$\dot{\mathbf{X}} = \dot{\mathbf{A}}_1 = \mathbf{A}_2, \quad \dot{\mathbf{A}}_2 = -\mathbf{A}_1 + \Sigma \mathbf{A}_3, \quad \dot{\mathbf{A}}_3 = -\Sigma \mathbf{A}_2. \quad (3.9)$$

4. THE DUAL FRENET FRAME

We now derive the mathematical measurements of turning and twisting of the dual curve $\mathbf{X}=\mathbf{X}(S)$ on the dual unit sphere. The dual frenet frame of this dual curve is the set of dual unit vectors \mathbf{T} , \mathbf{N} and \mathbf{B} defined in the following way. The dual unit vector tangent to $\mathbf{X}(S)$ is tangent to the dual unit sphere. It is obtained by

$$\dot{\mathbf{X}} = \mathbf{T}. \quad (4.1)$$

Since $\|\mathbf{T}\| = 1$, derivative $\dot{\mathbf{T}}$ will be normal to \mathbf{T} , this is shown in (3.9). Hence $\dot{\mathbf{T}}$ is chosen as direction of the dual unit vector \mathbf{N} , so we have

$$\mathbf{N} = \frac{\dot{\mathbf{T}}}{\|\dot{\mathbf{T}}\|} = \frac{-\mathbf{A}_1 + \sum \mathbf{A}_3}{\sqrt{1 + \sum^2}}. \tag{4.2}$$

We define the dual function $K = k + \varepsilon k^* = \|\dot{\mathbf{T}}\|$ which measure the bend of $\mathbf{X}=\mathbf{X}(S)$ out of \mathbf{T} , to be the natural dual curvature function of $\mathbf{X}=\mathbf{X}(S)$. Thus, we get that

$$K = \sqrt{1 + \sum^2}. \tag{4.3}$$

The remaining dual unit vector \mathbf{B} is obtained by computing the identity:

$$\frac{d}{ds} \langle \mathbf{N}, \mathbf{T} \rangle = 0,$$

implies that

$$\langle \dot{\mathbf{N}}, \mathbf{T} \rangle = -K,$$

Choose the direction of \mathbf{B} such that the Frenet frame has positive orientation. By expanding

$$\begin{aligned} \frac{d}{ds} \langle \mathbf{N}, \mathbf{B} \rangle &= 0 \\ \langle -K \mathbf{T} + \mathbf{T} \mathbf{B}, \mathbf{B} \rangle + \langle \mathbf{N}, \dot{\mathbf{B}} \rangle &= 0, \end{aligned}$$

yields that:

$$\mathbf{T} = \tau + \varepsilon \tau^* = -\langle \mathbf{N}, \dot{\mathbf{B}} \rangle.$$

Hence

$$\dot{\mathbf{B}} = -\mathbf{T} \mathbf{N}. \tag{4.4}$$

Thus, we may express the Frenet formulas in the matrix form:

$$\begin{pmatrix} \dot{\mathbf{T}} \\ \dot{\mathbf{N}} \\ \dot{\mathbf{B}} \end{pmatrix} = \begin{pmatrix} 0 & K & 0 \\ -K & 0 & \mathbf{T} \\ 0 & -\mathbf{T} & \mathbf{T} \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}. \tag{4.5}$$

As we see the Blaschke and Franet frames have one common axis $\mathbf{T} = \mathbf{A}_2$. Then a dual angle $\Psi = \psi + \varepsilon \psi^*$ specifies completely their relative position. If the dual plane $Sp\{\mathbf{A}_1, \mathbf{A}_3\}$ is rotated relative to the dual plane $Sp\{\mathbf{N}, \mathbf{B}\}$ by the dual angel Ψ , then we may write

$$\left. \begin{aligned} \mathbf{N} &= -\sin \Psi \mathbf{A}_1 + \cos \Psi \mathbf{A}_3 \\ \mathbf{B} &= \cos \Psi \mathbf{A}_1 + \sin \Psi \mathbf{A}_3 \end{aligned} \right\}. \tag{4.6}$$

Here

$$\sin \psi = \frac{1}{K} = R, \quad \Sigma = \cot \Psi, \quad (4.7)$$

$R = \rho + \varepsilon \rho^*$ is the dual radius of curvature of the curve $X = X(t)$. We use $T = \langle \dot{N}, \mathbf{B} \rangle$; this yields

$$T = -\frac{d\Psi}{dS}. \quad (4.8)$$

An Explicit Characterization of Dual Spherical Curve

Since T, N , and B is orthonormal basis in the dual space D^3 , then we can write that:

$$-X = F(S)T + G(S)N + H(S)B, \quad (4.9)$$

where F, G, H are differentiable dual functions of the dual arc-length S . Differentiating equation (4.9), with attention to (4.5), we get:

$$-T = (\dot{F} - GK)T + (\dot{G} + FK - HT)N + (\dot{H} + GT)B.$$

From this equation, we have successively that:

$$\dot{F} - GK + 1 = 0, \quad \dot{G} + FK - HT = 0, \quad \dot{H} + GT = 0. \quad (4.10)$$

From the fact that $\langle X, X \rangle = 1$, we get:

$$F^2 + G^2 + H^2 = 1. \quad (4.11)$$

Differentiating this equation, with account of (4.10), we get

$$F = 0, \quad G + \frac{1}{K} = 0, \quad H - \frac{1}{T} \frac{d}{dS} \left(\frac{1}{K} \right) = 0. \quad (4.12)$$

By (4.12), and in view of (4.9), the curve $X = X(S)$ is given in the form:

$$-X = RN + \frac{\dot{R}}{T} B. \quad (4.13)$$

Thus the dual spherical curves are those curves which satisfy the differential equation in the natural coordinates

$$R^2 + \left(\frac{\dot{R}}{T} \right)^2 = 1. \quad (4.14)$$

Substituting (4.12) into (4.10), gives:

$$G = R, \quad \dot{G} - HT = 0, \quad \dot{H} + GT = 0. \quad (4.15)$$

Equations (4.15) shows that:

$$RT + \frac{d}{dS} \left(\frac{\dot{R}}{T} \right) = 0. \quad (4.16)$$

By using (4.8), the last equation becomes simply:

$$R + \frac{d^2 R}{d\Psi^2} = 0. \quad (4.17)$$

Solutions are of the form

$$R(S) = \cos(\Psi(S) + \Psi_0), \quad \Psi(S) = -\int_0^S T(S)dS. \quad (4.18)$$

Note that these solutions depend on the torsion. Equation (4.18) is an explicit characterization of dual spherical curves (even when $T=0$ for some $S=0$).

According to (4.15 and (4.18), we have:

Theorem 4.1. The necessary and sufficient condition for the curve $X(S)$ to lie on a dual unit sphere, without having to assume its torsion is nowhere zero, may be given as follows:

- (i) K is nowhere pure dual (so that T is uniquely defined),
- (ii) There exists a C^1 -dual function $H=H(S)$, such that

$$H + \frac{dR}{d\Psi} = 0, \quad \frac{dH}{d\Psi} - R = 0, \quad H = \sin(\Psi + \Psi_0), \quad \Psi(S) = -\int_0^S T(S)dS.$$

Proof. Assume that $X(S)$ be a dual spherical curve. We can introduce a C^1 -dual function $H=H(S)$ defined by

$$H = -\langle X, B \rangle. \quad (4.19)$$

From (4.13), we have $H = \frac{\dot{R}}{T}$ or $H + \frac{dR}{d\Psi} = 0$ (in view of (4.8)) this is the first equation in the Theorem. Finally, differentiating (4.19) with respect to Ψ and making use of (4.13), we get $\frac{dH}{d\Psi} - R = 0$, which is second equation in the Theorem.

Conversely, let $X(S)$ satisfy the conditions (i), and (ii) and consider the curve defined by

$$M(S) = X(S) + R(S)N + H(S)B(S). \quad (4.20)$$

And the dual vector function $Y(S)$ is defined by

$$\langle Y, Y \rangle = \langle M - X, M - X \rangle = R^2 + H^2. \quad (4.21)$$

Differentiating equations (4.20) and (4.21), with account of (4.4), (4.8), we get that

$$\frac{dM}{d\Psi} = 0, \quad \frac{d}{d\Psi} \langle Y, Y \rangle = 0.$$

These show that M and Y are constants vectors, therefore the curve $X=X(S)$ is a dual spherical curve.

Evolute Dual Curve

In the Euclidean space E^3 , the vector equation of the evolutes $e = e(s)$ of a general space curve $x=x(s)$ has the form

$$e(s) = x(s) + \rho(n(s) + a b(s)), \quad (4.22)$$

where ρ is the radius of curvature of $x(s)$, and $a = a(s)$ is scalar function given by

$$a = \cot\left(\int_0^s \tau ds + c\right), \quad (4.23)$$

where c is arbitrary constant, while \mathbf{n} , and \mathbf{b} are, respectively, the usual principal normal and binormal unit vectors of the curve $\mathbf{x} = \mathbf{x}(s)$.

In analogy to the work in [4] of plane kinematics, we give the dual spherical evolute of the dual curve $\mathbf{X}=\mathbf{X}(S)$. According to (4.23) the dual spherical evolute may be given by:

$$\mathbf{E}(S) = \mathbf{X}(S) + \frac{1}{K}(\mathbf{N}(S) + A \mathbf{B}(S)), \quad (4.24)$$

where $A = a + \varepsilon a^*$ is scalar dual function of the dual arc length of the dual curve $\mathbf{X}=\mathbf{X}(S)$. From equation (4.6), we have that:

$$\mathbf{X} = \mathbf{A}_1 = -\sin \Psi \mathbf{N} + \cos \Psi \mathbf{B}. \quad (4.25)$$

By making use this equation in (4.24), with attention to (4.7), we get:

$$\mathbf{E}(S) = (\cos \Psi + A \sin \Psi) \mathbf{B}. \quad (4.26)$$

Since \mathbf{E} is dual unit vector, then

$$\mathbf{E}(S) = \mathbf{B}(S), \quad (4.27)$$

and

$$\cos \Psi + A \sin \Psi = 1. \quad (4.28)$$

If we calculate the real and dual parts of this equation, we get:

$$a = (\rho + \cot \int_0^s \tau ds), \quad a^* = \Psi^* (1 + a \cot \int_0^s \tau ds). \quad (4.29)$$

It is noted that ρ in (4.29) is not constant from point to point on the dual spherical evolute as it was for evolutes of general space curves. However, the dual spherical evolutes are, in complete analogy with planar evolutes, remains on the dual unit sphere on which the curves lies.

As we can see from equation (4.27), the dual unit vector $\mathbf{B} = \mathbf{E}$ is the instantaneous screw axis of the Blaschke frame. It means that:

$$\frac{d}{ds} \mathbf{A}_i = \sqrt{1 + \sum^2} \mathbf{B} \times \mathbf{A}_i, \quad (i = 1, 2, 3), \quad (4.30)$$

Hence, the dual spherical evolutes defines the locus of the axis of curvature of the motion of the Blaschke frame. And as \mathbf{B} does not have a component in the \mathbf{A}_2 – direction. Then the Blaschke frame as it moves along the striction curve on the ruled surface $\mathbf{X} = \mathbf{X}(S)$, the axis of curvature generates a right helicoid in E^3 . Thus, in general the ruled surface generated by the instantaneous screw axis of the Blaschke frame is a right helicoid whose striction line coincides continuously with \mathbf{A}_2 .

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