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OBTAINMENT OF TRANSITION PROBABILITIES $P_{ij}(s,t)$ FROM THE KOLMOGOROV EQUATION UNDER THE SPECIAL CASE FELLER-ARLEY PROCESS

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ABSTRACT

This paper obtains Feller- Arley non-homogeneous birth and death process by using transition probabilities and hence Kolmogorov's forward equation. For the purpose, we first derive the relevant probability generating function (PGF) $\Phi(X,t)$ for the process from the transition probabilities using Kolmogorov's forward equation [3]. Consequently, we obtain the desired general solution for non-homogeneous Feller-Arley process transition probabilities from the probability generating function in question. Moreover this model can be used in two-stage models for carcinogenesis and data analysis.

KEYWORDS Transition probabilities, Feller-Arley process, birth and death process, Kolmogorov's forward equation, instantaneous, transition rates and moment generating function.

1. INTRODUCTION

Let $X(t)$, $t>0$ be a stochastic process with parameter $t>0$, state space $S=\{0,1,2,...\}$. For example, $X(t)$ can be the number of accidents during $[0,t]$ or the number of bacteria in a plate dish at time t, ($t>0$) starting with N_0 bacteria at time $t=0$ [1,4]. Note that for given t, $X(t)$ is obviously a random variable with the state space $S=\{0,1,2,...\}$. The probability $\Pr(X(t)=j|X(s)=i)$, $t>s$ is called the transition probability from $X(s)=i$ to $X(t)=j$. The purpose of this paper is to examine the aspects of Feller-Arley process as possible applications for carcinogenesis, data analysis.

Definition 1.

If $\Pr\left(X(t)=j \middle| X(t_1)=i_1, X(t_2)=i_2,\ldots,X(t_k)=i_k\right)=$
$\Pr\left(X(t)=j \middle| X(t_k)=i_k\right)$

for all $t_1 < t_2 < \ldots < t_k < t$ and for all $i_1, i_2, \ldots, i_k$, then $X(t)$, $t \geq 0$ will be a Markov process [2].

Definition 2.

A markov process $X(t)$, $t > 0$ with the state space $S = \{0,1,2,\ldots\}$ is a birth and death process with birth rate $b_j(t)$ and death rate $d_j(t)$ [4], if

(i) $\Pr\left(X(t+\Delta t)=j+1 \middle| X(t)=j\right)=b_j(t)\Delta t+o(\Delta t)$

(ii) $\Pr\left(X(t+\Delta t)=j-1 \middle| X(t)=j\right)=d_j(t)\Delta t+o(\Delta t)$

(iii) $\Pr\left(X(t+\Delta t)=j \middle| X(t)=i\right)=o(\Delta t)$ if $|j-i| \geq 2$.

Simply conditions (i), (ii) and (iii) imply

$$\Pr\left(X(t+\Delta t)=j+1 \middle| X(t)=i\right)=1-\left(b_j(t)+d_j(t)\right)\Delta t+o(\Delta t).$$

If $b_j(t)=b_j$ and $d_j(t)=d_j$ independently of t, the process $X(t)$, $t>0$ is a homogeneous birth-death process. Otherwise, it is called non-homogeneous birth-death process. If $d_j(t)=0$, $X(t)$, $t>0$ becomes a pure birth process. If $b_j(t)=0$, $X(t)$, $t>0$ is a pure death process. More over if $d_j(t)=0$ and $b_j(t)=\lambda$, $X(t)$ will be a homogeneous Poisson process. If $d_j(t)=0$ and $b_j(t)=\lambda(t)$, $X(t)$ is a non homogeneous Poisson process.

2. INSTANTANEOUS TRANSITION RATES

Note that

$$\Pr\left(X(t)=j \middle| X(t)=i\right)=\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise}\end{cases}.$$

Writing $\Pr\left(X(t)=j \middle| X(s)=i\right)=p_{ij}(s,t)$, $s=t$ implies that $p_{ij}(s,s)=\delta_{ij}$.

Also $\Pr\left(X(t+\Delta t)=j-1 \middle| X(t)=j\right)=d_j(t)\Delta t+o(\Delta t)\Leftrightarrow$

$$\lim_{\Delta t\to 0}\frac{p_{j,j-1}(t,t+\Delta t)-p_{j,j-1}(t,t)}{\Delta t}=d_j(t)+\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}=d_j(t)$$

where Δt is some time-increment and $o(\cdot)$ stands for the order of magnitude.

$$\Pr\left(X(t+\Delta t)=j+1 \middle| X(t)=j\right)=b_j(t)\Delta t+o(\Delta t)$$

$$\Leftrightarrow \lim_{\Delta t\to 0}\frac{p_{j,j+1}(t,t+\Delta t)-p_{j,j+1}(t,t)}{\Delta t}=b_j(t)+\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}=b_j(t)$$

$$\Pr\left(X(t+\Delta t)=j \middle| X(t)=i\right)=o(\Delta t) \text{ if } |j-i|\geq 2$$

$$\Leftrightarrow \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(p_{i,j}(t,t+\Delta t)-p_{i,j}(t,t)\right)=0$$

Given the instantaneous rate, we shall try to answer the question "how to find $p_{ij}(s,t)$ $t>s$". One way is to use Kolmogorov's forward or backward equation.

Theorem 1. The transition probability $p_{ij}(s,t)$ satisfy the following two equations, If $b_{-1}(t)=p_{-1,k}(s,t)=p_{j,-1}(s,t)=0$, then,

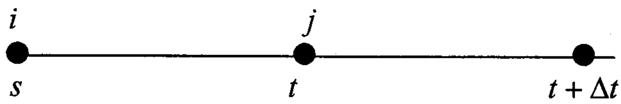
$$\frac{\partial}{\partial t}p_{ij}(s,t)=p_{i,j-1}(s,t)b_{j-1}(t)+p_{i,j+1}(s,t)d_{j+1}(t)-\left[b_j(t)+d_j(t)\right]p_{ij}(s,t) \qquad (1)$$

$$\frac{\partial}{\partial s}p_{ij}(s,t)=p_{i+1,j}(s,t)b_i(s)+p_{i-1,j}(s,t)d_i(s)-\left[b_i(s)+d_i(s)\right]p_{ij}(s,t) \qquad (2)$$

which are called *Kolmogorov forward* and *Kolmogorov backward equations* respectively, with

$$p_{ij}(s,s)=p_{ij}(t,t)=\delta_{ij},\quad \delta_{ij}=\begin{cases}1 & if\ i=j\\ 0 & otherwise\end{cases}.$$

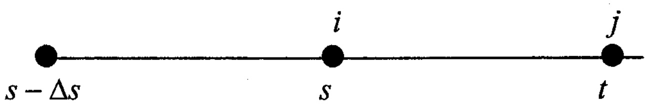
Proof (Kolmogorov Forward equation). Consider the time interval based on Definition 2



$$p_{ij}(s,t+\Delta t) = p_{i,j-1}(s,t)b_{j-1}(t)\Delta t + p_{i,j+1}(s,t)d_{j+1}(t)\Delta t$$
$$+ [1-(b_j(t)+d_j(t))\Delta t]p_{ij}(s,t) + o(\Delta t)$$
$$p_{ij}(s,t+\Delta t) - p_{ij}(s,t) = p_{i,j-1}(s,t)b_{j-1}(t)\Delta t + p_{i,j+1}(s,t)d_{j+1}(t)\Delta t$$
$$-[b_j(t)+d_j(t)]\Delta t p_{ij}(s,t) + o(\Delta t)$$
$$\frac{p_{ij}(s,t+\Delta t) - p_{ij}(s,t)}{\Delta t} = p_{i,j-1}(s,t)b_{j-1}(t) + p_{i,j+1}(s,t)d_{j+1}(t)$$
$$-(b_j(t)+d_j(t))p_{ij}(s,t) + \frac{o(\Delta t)}{\Delta t}$$
$$\lim_{\Delta t\to 0}\frac{p_{ij}(s,t+\Delta t) - p_{ij}(s,t)}{\Delta t} = \frac{\partial p_{ij}(s,t)}{\partial t}$$
$$= p_{i,j-1}(s,t)b_{j-1}(t) + p_{i,j+1}(s,t)d_{j+1}(t)$$
$$-[b_j(t)+d_j(t)]p_{ij}(s,t)$$

with $p_{ij}(s,s) = \delta_{ij}$.

To prove the backward equation: Again consider the time interval



thus

$$p_{ij}(s-\Delta s,t) = p_{i+1,j}(s,t)b_i(s-\Delta s)\Delta s + p_{i-1,j}(s,t)d_i(s-\Delta s)\Delta s \\ +[1-(b_i(s-\Delta s)+d_i(s-\Delta s))\Delta s]p_{ij}(s,t)+o(\Delta s)$$

therefore we can obtain

$$\frac{p_{ij}(s-\Delta s,t)-p_{ij}(s,t)}{\Delta s} = p_{i+1,j}(s,t)b_i(s-\Delta s) + p_{i-1,j}(s,t)d_i(s-\Delta s) \\ -(b_i(s-\Delta s)+d_i(s-\Delta s))p_{ij}(s,t)+\frac{o(\Delta s)}{\Delta s}$$

Assume that $b_i(s)$ and $d_i(s)$ are continuous function of s [5]. Then letting $\Delta s \to 0$. We have the following equation

$$\lim_{\Delta s\to 0}\frac{p_{ij}(s-\Delta s,t)-p_{ij}(s,t)}{\Delta s} = \frac{\partial p_{ij}(s,t)}{\partial s} \\ = p_{i+1,j}(s,t)b_i(s)+p_{i-1,j}(s,t)d_i(s)-(b_i(s) \\ +d_i(s))p_{ij}(s,t), \qquad p_{ij}(s,s)=\delta_{ij}.$$

Definition 3.
If $b_i(t)=ib(t)$ and $d_i(t)=id(t)$, then $X(t)$, $t>0$ is called *non-homogeneous Feller-Arley Process.*

If the state space is $S=\{0,1,2,\ldots,m\}$ and $b_i(t)=i(1-\frac{i}{m})b(t)$ and $d_i(t)=i(1-\frac{i}{m})d(t)$, then $X(t)$ is called non-homogeneous Logistic Process.

If $b_i(t) = ib(t)$ and $d_i(t) = id(t)$ and $b(t) - d(t) = \beta e^{-\delta t}$, $\beta > 0$, $\delta > 0$ $X(t)$ is a Gompertz process.

3. THE TRANSITION PROBABILITIES $p_{ij}(s,t)$ FOR SOME SPECIAL CASES

We now illustrate how to obtain transition probabilities from the Kolmogorov's forward equation. For simplicity, let $i = 1$ $s = t_0$ and write $p_{ij}(s,t) = p_{1j}(t_0,t) = p_j(t_0,t)$.

The Kolmogorov's forward equation (1) becomes

$$\frac{\partial}{\partial t} p_{ij}(t_0,t) = p_{j-1}(t_0,t)b_{j-1}(t) + p_{j+1}(t_0,t)d_{j+1}(t) - \left[b_j(t) + d_j(t)\right]p_j(t_0,t) \quad (3)$$

with $p_j(t_0,t) = \delta_{ij}$.

3.1. Feller-Arley Process

Now assume $b_j(t) = jb(t)$ and $d_j(t) = jd(t)$ in Equation 3. We have already illustrated how to obtain $p_{ij}(t_0,t)$. For the special case (Feller-Arley Process) [5], define $\Phi(X,t) = \sum_{j=0}^{\infty} X^j p_j(t_0,t)$. $\Phi(X,t)$ is probability generating function (PGF) of $p_{ij}(t_0,t)$ in the usual sense that $p_j(t_0,t) = \frac{1}{j!}\left(\frac{\partial^j}{\partial X^j}\Phi(X,t)\bigg|_{X=0}\right)$.

By multiplying both sides of the Kolmogorov's forward equation (1) by X^j and summing over j from 0 to ∞ and noting that $b_{-1}(t) = 0$, we have

$$\Phi(X,t) = \sum_{j=0}^{\infty} X^j p_j(t_0,t)$$

$$\sum_{j=0}^{\infty} X^j \frac{d}{dt} p_j(t_0,t) = \frac{d}{dt}\sum_{j=0}^{\infty} X^j p_j(t_0,t) = \frac{d}{dt}\Phi(X,t).$$

The left-hand side yields

$$\sum_{j=0}^{\infty} X^j [p_{j-1}(t_0,t)b_{j-1}(t) + p_{j+1}(t_0,t)b_{j+1}(t) - (b_j(t) + d_j(t))p_j(t_0,t)]$$

$$= b(t)\sum_{j=0}^{\infty} X^j (j-1) p_{j-1}(t_0,t) + d(t)\sum_{j=0}^{\infty} X^j (j+1) p_j(t_0,t) - (b(t)$$

$$+ d(t))\sum_{j=0}^{\infty} jX^j p_j(t_0,t)$$

$$= b(t)X^2 \sum_{j=0}^{\infty} (j-1) X^{j-2} p_{j-1}(t_0,t) + d(t)\sum_{j=0}^{\infty} (j+1) X^j p_j(t_0,t)$$

$$-(b(t)+d(t))X\sum_{j=0}^{\infty} X^j p_j(t_0,t)$$

$$= [b(t)X^2 + d(t) - X(b(t)+d(t))]\sum_{j=0}^{\infty} jX^{j-1} p_{j-1}(t_0,t)$$

$$= [b(t)X^2 + d(t) - X(b(t)+d(t))]\frac{\partial}{\partial X}\Phi(X,t)$$

and the right-hand side becomes

$$\frac{\partial}{\partial t}\Phi(X,t) = \left[b(t)X^2 + d(t) - X(b(t)+d(t))\right]\frac{\partial}{\partial X}\Phi(X,t) \tag{4}$$

The initial condition is $\Phi(X,t_0) = \sum_{j=0}^{\infty} X^j p_j(t_0,t)$ if $t_0 = t$,

thus $\Phi(X,t_0) = \sum_{j=0}^{\infty} X^j p_j(t_0,t_0) = \sum_{j=0}^{\infty} X^j \delta_{1j} = X$.

Now we shall illustrate how to solve the above first order partial differential equation to obtain $\Phi(X,t)$ and hence $p_j(t_0,t)$. We may use the equation *(4)*, that is:

$$\frac{\partial}{\partial t}\Phi(X,t) = \left[X^2 b(t) + d(t) - X(b(t)+d(t))\right]\frac{\partial}{\partial X}\Phi(X,t)$$

with the initial condition $\Phi(X,t_0) = X$. After simplification of *(4)*, we simply get

$$\frac{dX}{dt} + \left[X^2 b(t) + d(t) - X(b(t)+d(t))\right] = 0$$

or

$$\frac{dX}{dt}+(X-1)[(X-1)b(t)+\varepsilon(t)]=0 \tag{5}$$

where $\varepsilon(t)=b(t)-d(t)$. Let $z=\frac{1}{X-1}$. Dividing the equation *(5)* by $(X-1)^2$, it becomes

$$\frac{1}{(X-1)^2}\frac{dX}{dt}+b(t)+\frac{1}{(X-1)}\varepsilon(t)=0 \Rightarrow -\frac{d}{dt}\left(\frac{1}{X-1}\right)+b(t)+\frac{1}{(X-1)}\varepsilon(t)$$

$$\Rightarrow \frac{d}{dt}z=b(t)+z\varepsilon(t) \text{ or } \frac{d}{dt}z-z\varepsilon(t)=b(t) \tag{6}$$

If we multiply both sides of the equation (6) by $\int_{t_0}^{t}\varepsilon(t)dt$, that is

$$\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\}\left(\frac{d}{dt}z-z\varepsilon(t)\right)=\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\}b(t)$$

$$\Rightarrow \frac{d}{dt}\left[z\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\}\right]=b(t)\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\} \tag{7}$$

Thus the solution of (7) is simply

$\Phi(X,t)=\Psi(U(X,t))$ where

$$\Psi(X,t)=\frac{1}{X-1}\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\}-\int_{t_0}^{t}b(t)\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\}dt = \text{Constant}$$

The solution is

$$\Phi(X,t)=\Psi\left(\frac{1}{X-1}\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\}-\int_{t_0}^{t}b(t)\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\}dt\right)$$

if $t = t_0$, then $\Phi(X,t_0) = X = \Psi\left(\frac{1}{X-1}\right)$ so that $\Psi(z) = \frac{1+z}{z}$

$$\Phi(X,t) = \Psi\left(\frac{1}{X-1}\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\} - \int_{t_0}^{t}b(t)\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\}dt\right)$$

$$= \frac{1+\frac{1}{X-1}\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\} - \int_{t_0}^{t}b(t)\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\}dt}{\frac{1}{X-1}\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\} - \int_{t_0}^{t}b(t)\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\}dt}$$

Finally we obtain general solution

$$\Phi(X,t) = 1 + \frac{X-1}{\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\} - \int_{t_0}^{t}b(t)\exp\left\{-\int_{t_0}^{t}\varepsilon(t)dt\right\}dt}$$

with the initial condition $\Phi(X,t_0) = X$.

One may can use this result for the two-stage model for carcinogenesis and hazard rate of cancer tumor cells at time $t > 0$.

ÖZET

Bu çalışmada, Kolmogorov ileri ve geçiş olasılıkları yardımıyla homojen olmayan Feller-Arley doğum-ölüm yöntemi elde edilmiştir. Bunun için olasılık üreten fonksiyonu $\Phi(X,t)$ Kolmogorov ileri denkleminden bulunmuştur. Sonuç olarak homojen olmayan Feller-Arley yöntemine olasılık üreten fonksiyonu ile ulaşılmıştır. Ayrıca bu yöntem iki-aşamalı carcinogenesis ve veri analizi modelleri için kullanılabilir.

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