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# THE NEW EXPRESSIONS OF GAUSSIAN CURVATURE AT SPACELIKE CONGRUENCES

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**ABSTRACT**

We obtained two fundamental formula in [1] for the geometry of spacelike congruences by means of dual hyperbolic and central angles. Beside we stated and proved the Manheim's, Liouville's and Hamilton's formulae.

In this study we give the new expressions of the Gaussian curvature for the spacelike congruences by means of fundamental formulae.

**Key Words:** Study's Mapping, Spacelike Ruled Surface, Timelike Ruled Surface, Blaschke Vectors, Dual Hyperbolic Angle, Spacelike Congruences.

## 1. INTRODUCTION

We defined in [1] the Blaschke vectors of Blaschke trihedron of timelike and spacelike ruled surfaces at their striction points, similar to the Darboux vectors of spacelike and timelike curves on a timelike (or spacelike) surface at Minkowski 3-space $R_1^3$ with signature (+,+,-) (See, [2] and [3] ). Using these vectors, we stated the important two theorems for the geometry of spacelike congruences as follows:

**Theorem 1.** Let $\widetilde{b}_i$ $(i = 0,1,2,3)$ be Blaschke vectors of the ruled surfaces $(\widetilde{c}_i)$ passing through the common spacelike line $\widetilde{a}_0$ of spacelike congruence $\widetilde{a}_0(u,v)$. If $\theta$ is dual hyperbolic angle between the normals $\widetilde{a}_{22}$ and $\widetilde{a}_{32}$ of the parameter ruled surfaces, then the Blaschke vector of spacelike ruled surface $(\widetilde{c}_1)$ is given by

$$\widetilde{b}_1 = \widetilde{b}_2 \cosh\theta + \widetilde{b}_3 \sinh\theta + \widetilde{a}_0 \frac{d\theta}{d\widetilde{s}_1}, \tag{1}$$

where $d\widetilde{s}_1$ is the dual arc element of $(\widetilde{c}_1)$.

**Proof.** See, [1].

**Theorem2.** In Theorem1, if we choose the timelike ruled surface $(\widetilde{c}_0)$ instead of $(\widetilde{c}_1)$ then the Blaschke vector of $(\widetilde{c}_0)$ is given by

$$\widetilde{b}_0 = \widetilde{b}_2 \sinh\theta + \widetilde{b}_3 \cosh\theta + \widetilde{a}_0 \frac{d\theta}{d\widetilde{s}_0}, \tag{2}$$

where $\theta$ is dual central angle between the normals of ruled surfaces $(\widetilde{c}_0)$ and $(\widetilde{c}_3)$, and $d\widetilde{s}_0$ is the arc distance of $(\widetilde{c}_0)$.
**Proof.** See, [1].

## 2. THE NEW EXPRESSIONS OF THE GAUSSIAN CURVATURE FOR SPACELIKE CONGRUENCES

**Proposition 1.** Suppose that the parameter ruled surface $(\widetilde{c}_2)$ and $(\widetilde{c}_3)$ passing through a spacelike line $\widetilde{a}_0$ of a spacelike congruence are perpendicular. Let $(\widetilde{c}_1)$ be an arbitrary timelike ruled surface. Let $\widetilde{b}_i$ $(i = 0,1,2,3)$ be the Blaschke vectors of the Blaschke trihedrons of this ruled surfaces. Then the mixed product $(\widetilde{a}_0, \widetilde{b}_1, \widetilde{b}_0)$ is invariant and the Gaussian curvature is given by

$$K = -(\widetilde{a}_0, \widetilde{b}_1, \widetilde{b}_0) = -(\widetilde{a}_0, \widetilde{b}_2, \widetilde{b}_3) \tag{3}$$

**Proof.** Let's multiply the equalities (1) and (2) in the vectoral mean :

$$\widetilde{b}_1 \wedge \widetilde{b}_0 = \left( \widetilde{b}_1 + \widetilde{a}_0 \frac{d\theta}{d\widetilde{s}_1} \right) \wedge \left( \widetilde{b}_0 + \widetilde{a}_0 \frac{d\theta}{d\widetilde{s}_0} \right)$$

$$\widetilde{b}_1 \wedge \widetilde{b}_0 = \widetilde{b}_1 \wedge \widetilde{b}_0 + \widetilde{b}_1 \wedge \widetilde{a}_0 \frac{d\theta}{d\widetilde{s}_0} + \frac{d\theta}{d\widetilde{s}_1} \widetilde{a}_0 \wedge \left( \widetilde{b}_0 + \widetilde{a}_0 \frac{d\theta}{d\widetilde{s}_0} \right).$$

If we multiply the two side of the last equation by $\widetilde{a}_0$ from left we obtain

$$\left\langle \widetilde{a}_0, \widetilde{b}_1 \wedge \widetilde{b}_0 \right\rangle = \left\langle \widetilde{a}_0, \widetilde{b}_1 \wedge \widetilde{b}_0 \right\rangle. \tag{4}$$

From [1] we know that

$$\widetilde{b}_1 = \widetilde{b}_2 \cosh\theta + \widetilde{b}_3 \sinh\theta,$$
$$\widetilde{b}_0 = \widetilde{b}_2 \sinh\theta + \widetilde{b}_3 \cosh\theta.$$

Thus we have

$$\begin{aligned} \widetilde{b}_1 \wedge \widetilde{b}_0 &= (\widetilde{b}_2 \cosh\theta + \widetilde{b}_3 \sinh\theta) \wedge (\widetilde{b}_2 \sinh\theta + \widetilde{b}_3 \cosh\theta) \\ &= \widetilde{b}_2 \wedge \widetilde{b}_3 \cosh^2\theta - \widetilde{b}_2 \wedge \widetilde{b}_3 \sinh^2\theta) \\ &= \widetilde{b}_2 \wedge \widetilde{b}_3. \end{aligned}$$

This shows that the equation is satisfied.

Now, let's show that this mixed product is equal to the Gaussian curvature K:

$$\begin{aligned}\langle \tilde{a}_0, \tilde{b}_1 \wedge \tilde{b}_0 \rangle &= \langle \tilde{a}_{02}, \tilde{b}_1 \wedge \tilde{b}_0 \rangle \\ &= -\langle \tilde{a}_{12}, \tilde{b}_1 \rangle \langle \tilde{a}_{02}, \tilde{b}_0 \rangle + \langle \tilde{a}_{02}, \tilde{b}_1 \rangle \langle \tilde{a}_{12}, \tilde{b}_0 \rangle \\ &= -\tilde{p}_1 \tilde{p}_0,\end{aligned}$$

$$\begin{aligned}\langle \tilde{a}_0, \tilde{b}_2 \wedge \tilde{b}_3 \rangle &= \langle \tilde{a}_{22} \wedge \tilde{a}_{32}, \tilde{b}_2 \wedge \tilde{b}_3 \rangle \\ &= -\langle \tilde{a}_{22}, \tilde{b}_2 \rangle \langle \tilde{a}_{32}, \tilde{b}_3 \rangle + \langle \tilde{a}_{22}, \tilde{b}_3 \rangle \langle \tilde{a}_{32}, \tilde{b}_2 \rangle \\ &= -\tilde{p}_3 \tilde{p}_2,\end{aligned}$$

where the dual number $p_i$ $(i=0,1,2,3)$ is the dual curvature of ruled surface $(\tilde{c}_i)(i=0,1,2,3)$.
It is clear that the equations (5) and (6) are equal the Gaussian curvature.

**Proposition2.** The Gaussian curvature is also given by

$$K = \frac{D(\tilde{a}_{22}, \tilde{a}_{32})}{D(\tilde{s}_2, \tilde{s}_3)} = -\frac{1}{\sqrt{\tilde{e}\,\tilde{g}}}\left[\frac{\partial}{\partial u}\left(\frac{(\sqrt{\tilde{g}})_u}{\sqrt{\tilde{e}}}\right) + \frac{\partial}{\partial v}\left(\frac{(\sqrt{\tilde{e}})_v}{\sqrt{\tilde{g}}}\right)\right]. \tag{7}$$

**Proof.** Considering the equation (3) we have

$$\begin{aligned}-K &= \langle \tilde{b}_2 \wedge \tilde{b}_3, \tilde{a}_0 \rangle \\ &= \langle \tilde{b}_2 \wedge \tilde{b}_3, \tilde{a}_{22} \wedge \tilde{a}_{32} \rangle \\ &= -\langle \tilde{b}_2 \wedge \tilde{a}_{22}, \tilde{b}_3 \wedge \tilde{a}_{32} \rangle + \langle \tilde{b}_2 \wedge \tilde{a}_{32}, \tilde{b}_3 \wedge \tilde{a}_{22} \rangle \\ &= -\left\langle \frac{\partial \tilde{a}_{22}}{\partial \tilde{s}_2}, \frac{\partial \tilde{a}_{32}}{\partial \tilde{s}_3} \right\rangle + \left\langle \frac{\partial \tilde{a}_{32}}{\partial \tilde{s}_2}, \frac{\partial \tilde{a}_{22}}{\partial \tilde{s}_3} \right\rangle \\ &= \frac{D(\tilde{a}_{22}, \tilde{a}_{32})}{D(\tilde{s}_2, \tilde{s}_3)}.\end{aligned}$$

Thus we can write

$$\frac{\partial \tilde{a}_{22}}{\partial \tilde{s}_2} = \frac{\partial \tilde{a}_{22}}{\partial u}\frac{du}{d\tilde{s}_2} = \frac{\partial \tilde{a}_{22}}{\partial u}\frac{1}{\sqrt{\tilde{e}}}$$

$$\frac{\partial \tilde{a}_{32}}{\partial \tilde{s}_2} = \frac{\partial \tilde{a}_{32}}{\partial u}\frac{du}{d\tilde{s}_2} = \frac{\partial \tilde{a}_{32}}{\partial u}\frac{1}{\sqrt{\tilde{e}}}.$$

$$\frac{\partial \tilde{a}_{22}}{\partial \tilde{s}_3}=\frac{\partial \tilde{a}_{22}}{\partial v}\frac{dv}{d\tilde{s}_3}=\frac{\partial \tilde{a}_{22}}{\partial v}\frac{1}{\sqrt{\tilde{g}}}$$

$$\frac{\partial \tilde{a}_{32}}{\partial \tilde{s}_3}=\frac{\partial \tilde{a}_{32}}{\partial v}\frac{dv}{d\tilde{s}_3}=\frac{\partial \tilde{a}_{32}}{\partial v}\frac{1}{\sqrt{\tilde{g}}}.$$

Using these equalities, desired formula is obtained.

**Corollary 1.** There exists the relation

$$(\tilde{b}_1,\tilde{b}_2,\tilde{b}_3) = -K\frac{d\theta}{d\tilde{s}_1},$$

among the Blaschke vector $\tilde{b}_i$, $i=1,2,3$.

**Proof.** The mixed product of the vectors $\tilde{b}_i$, $i=1,2,3$ is

$$(\tilde{b}_1,\tilde{b}_2,\tilde{b}_3)=\left\langle \tilde{b}_1,\tilde{b}_2\wedge\tilde{b}_3\right\rangle$$

$$=\left\langle \tilde{b}_2\cosh\theta+\tilde{b}_3\sinh\theta+\tilde{a}_0\frac{d\theta}{d\tilde{s}_1},\tilde{b}_2\wedge\tilde{b}_3\right\rangle$$

$$=\left\langle \tilde{a}_0,\tilde{b}_2\wedge\tilde{b}_3\right\rangle\frac{d\theta}{d\tilde{s}_1}$$

$$=-K\frac{d\theta}{d\tilde{s}_1}$$

This shows that the Darboux vectors is on the same plane if $\theta$ = constant or $K = 0$.

**Corollary 2.** The Gaussian curvature is also given by equation

$$K=-\left(\tilde{a}_0,\frac{\partial\tilde{a}_0}{\partial\tilde{s}_2},\frac{\partial\tilde{a}_0}{\partial\tilde{s}_3}\right)$$

by means of spacelike unit vector $\tilde{a}_0$ and its derivatives.

**Proof.** By [1] we know that

$$\frac{\partial\tilde{a}_0}{\partial\tilde{s}_2}=\tilde{b}_1\wedge\tilde{a}_0,$$

$$\frac{\partial\tilde{a}_0}{\partial\tilde{s}_3}=\tilde{b}_2\wedge\tilde{a}_0.$$

Then we have

$$\frac{\partial\tilde{a}_0}{\partial\tilde{s}_2}\wedge\frac{\partial\tilde{a}_0}{\partial\tilde{s}_3}=(\tilde{b}_1\wedge\tilde{a}_0)\wedge(\tilde{b}_2\wedge\tilde{a}_0)$$

$$=-<\tilde{b}_1,\tilde{b}_2\wedge\tilde{a}_0>\tilde{a}_0+<\tilde{a}_0,\tilde{b}_2\wedge\tilde{a}_0>\tilde{b}_1$$

$$=-(\tilde{b}_1,\tilde{b}_2,\tilde{a}_0)\,\tilde{a}_0$$

$$=-K\,\tilde{a}_0.$$

Multiplying the two sides of last equation by $\tilde{a}_0$, the value of $\kappa$ is found.

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