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ON SOME NEW DIFFERENCE SEQUENCE SPACES

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ABSTRACT

In this paper we investigate some new difference sequence spaces which naturaly emerge from the concept of almost convergence. Strongly almost summable sequences have been discussed by Nanda [6]. The object of this paper is to introduce the spaces of strongly almost Δ-summable sequences which happen to be complete paranormed spaces under certain conditions. Some topological results and inclusion relations of such sequences have been discussed.

KEYWORDS Strongly almost sequences,strongly almost summable.

1. INTRODUCTION

Let S be the set of all sequences of real or complex terms and ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences $x=(x_k)$, respectively, normed by $\|x\| = \sup_k |x_k|$. The zero sequence *(0 , 0 , 0 , - - -)* is denoted by θ. Let D be the shift operator on S, that is $Dx = \{x_k\}_{k=1}^{\infty}$, $D^2x = \{x_k\}_{k=2}^{\infty}$ and so on. It may be recalled that the Banach limit L is a non-negative linear functional on ℓ_∞ such that L is invariant under the shift operator (that is, $L(Dx)=L(x)$ for all $x \in \ell_\infty$) and $L(e)=1$, where $e=(1,1,1,...)$, [1]. A sequence $x \in \ell_\infty$ is called almost convergent Lorentz [12] if all Banach limits of x coincide.

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Let $\hat{c}$ denote the space of almost convergent sequences. It is proved by Lorentz [12] that

$$\hat{c}= \{x: \ \lim_{m\to\infty} t_{m,n}(x) \textit{ exists uniformly in } n\}$$

where

$$t_{m,n}(x)=\frac{1}{m+1}\sum_{i=0}^{m} D^i x_n \ , (D^o=1).$$

Several authors including Lorentz [12], King [11] and Nanda [6, 8] have studied almost convergent sequences. Just as convergence gives rise to absolute and strong convergence, it was quite natural to expect that almost convergence must give rise to the concepts of absolute almost and strong almost convergence. Absolutely almost and strongly almost convergent sequences have been introduced and discussed in a natural way by Das, Kuttner and Nanda [9] and Maddox [4].

The summability methods of real or complex sequences by infinite matrices are of three types ordinary, absolute and strong, [5]. In the same vein, it is expected that
the concept of almost convergence must give rise to three types of summability methods-
almost, absolutely almost and strongly almost. The almost summable sequences have been discussed by King [11],Schaefer [13] and some others. More recently Das, Kuttner and Nanda [9] and Nanda [10] have introduced the concept of absolute almost summability along with the concept of absolute almost convergence. The spaces of strongly almost summable sequences were defined and studied by Nanda [6,7].

The purpose of this paper is to introduce the space of strongly almost Δ-summable sequences which happen to be complete paranormed spaces under certain conditions and also obtain some topological results and inclusion relations.

Let $A=(a_{nk})$ be an infinite matrix of complex numbers and $x=(x_k)$ be a sequence of complex numbers. The sequence $(A(x))_n$ defined by $A(x)_n=\sum_{k=1}^{\infty} a_{nk}x_k$ for all $n\in N$ and is called the A-transform of x whenever this series converges for each n.

Throughout $A = (a_{nk})$ denotes an infinite matrix of nonnegative real numbers and let $p=(p_k)$ be a sequence of real numbers such that $p_k > 0$ for all k and $\sup_k p_k = H < \infty$. The following inequality will be used throughout

Let $p=(p_k)\in \ell_\infty$, then for sequences (a_k) and (b_k) of complex numbers we have

$$|a_k + b_k|^{p_k} \leq K(|a_k|^{p_k} + |b_k|^{p_k}) \tag{1}$$

where $K = max\,(1,2^{H-1})$ (see for instance Maddox [2]).

2. DEFINITIONS AND PRELIMINARIES

Now we define the following new difference sequences spaces :

$$[A_\Delta, p]_0 = \{x = (x_k) \in w: \ \lim_{m\to\infty} t_{m,n}(\Delta x) = 0, \quad uniformly\ in\ n\},$$

$$[A_\Delta, p] = \left\{\begin{matrix} x = (x_k) \in w: \ \lim_{m\to\infty} t_{m,n}(\Delta x - le) = 0,\ for\ some\ l, \\ uniformly\ in\ n \end{matrix}\right\},$$

$$[A_\Delta, p]_\infty = \{x = (x_k) \in w: \ \sup_{m,n} t_{m,n}(\Delta x) < \infty\},$$

where

$$t_{m,n}(\Delta x) = \frac{1}{m+1}\sum_{i=0}^{m} A_{n+i,k}(\Delta x) = \sum_k a(n,k,m)|\Delta x_k|^{p_k},$$

$$a(n,k,m) = \frac{1}{m+1}\sum_{i=0}^{m} a_{n+i,k} \text{ and}$$

$$\Delta x = (\Delta x_k) = (x_k - x_{k+1}) \text{ for all } k \in N.$$

The sets, $[A_\Delta, p]_0$, $[A_\Delta, p]$ *and* $[A_\Delta, p]_\infty$ will respectively be called the spaces of strongly almost Δ-summable to zero, strongly almost Δ-summable and strongly almost Δ-bounded sequences.

When $\Delta^0 x = x$ we obtain the spaces $[A,p]_0$, $[A,p]$ *and* $[A,p]_\infty$ respectively , which were defined by Nanda [6]. The sets $[A,p]_0$, $[A,p]$ *and* $[A,p]_\infty$ are respectively called the spaces of *strongly almost summable to zero, strongly almost summable* and *strongly bounded sequences.*

A sequence space E is said to be *solid* if $(\alpha_k x_k) \in E$, when ever $(x_k) \in E$ and for all sequences α_k of scalars with $|\alpha_k| \leq 1$ for alll $k \in N$.

A sequence space E is said to be *symmetric* if $(x_{\pi(n)}) \in E$, whenever $(x_n) \in E$, where $\pi(n)$ is a permutation of N.

For $0 < r \leq 1$ a non-void subset U of a linear space is said to be absolutely r-convex if $x,y \in U$ and $|\lambda|^r + |\mu|^r \leq 1$ together imply that $\lambda x + \mu y \in U$. It is clear that if U is absolutely r-convex, then it is absolutely t-convex for $t < r$. A linear topological space X is said to be r-convex if every neighbourhood of $\theta \in X$ contains an absolutely r-convex neighbourhood of $\theta \in X$. The r-convexity for $r > 1$ is of little interest, since X is r-convex for $r > 1$ if and only if X is the only neighbourhood of $\theta \in X$, [3]. A subset B of X is said to be bounded if for each neighbourhood U of $\theta \in X$ there exists an integer $J > 1$ such that $B \subseteq JU$. X is called locally bounded if there is a bounded neighbourhood of zero.

The following results will be used for establishing the results of this article .

Lemma 1 (Nanda [6], Proposition 2). $[A,p] \subset [A,p]_\infty$ if

$$\| A \| = \sup_{m,n} \sum_k a(n,k,m) < \infty \ . \tag{2}$$

Lemma 2 (Nanda [6], Theorem 2). Let $0 < p_k \leq 1$. Then $[A,p]_0$ *and* $[A,p]_\infty$ are locally bounded if $\inf p_k > 0$. If (2) holds, then $[A,p]$ has the same property.

Lemma 3 (Nanda [6], Theorem 3). Let $0 \prec p_k \leq 1$. Then $[A,p]_0$ *and* $[A,p]_\infty$ are r-convex for all r, where $0 < r < \liminf p_k$. Moreover, if $p_k = p \leq 1$ for all k, then they are p-convex. If (2) holds, then $[A,p]$ has the same property.

3. THE MAIN RESULTS

The proof of the following result is a routine work in view of the inequality (1).

Theorem 1. If $p \in \ell_\infty$, then $[A_\Delta, p]_0, [A_\Delta, p]$ *and* $[A_\Delta, p]_\infty$ are linear spaces over the field C of complex numbers .

The following result follows from Lemma 1 .

Theorem 2. $[A_\Delta, p] \subset [A_\Delta, p]_\infty$ if (2) holds .

Theorem 3. Let $p \in l_\infty$, then $[A_\Delta, p]_0$ and $[A_\Delta, p]_\infty$ $(\inf p_k > 0)$ are complete linear topological spaces paranormed (not necessarily totally) by g defined by

$$g(x) = |x_1| + \sup_{m,n}[t_{m,n}(\Delta x)]^{1/M},$$

where M=max$(1, H = \sup p_k)$. $[A_\Delta, p]$ is paranormed by g if (2) holds. Further $[A_\Delta, p]$ *is complete if*

$$\sum_k a(n,k,m) \to 0 \quad \textit{uniformly in n.}$$

Proof. Clearly $g(\theta) = 0$, $g(-x) = g(x)$ and $g(x+y) \le g(x) + g(y)$. We now show that the scalar multiplication is continuous. We have

$$g(\lambda x) \le \max(|\lambda|, \sup_k |\lambda|^{p_k/M}) g(x) \tag{3}$$

Therefore $x \to \theta \Rightarrow \lambda x \to \theta$ (for fixed λ) by (3). Now let $\lambda \to 0$ and $x \in [A_\Delta, p]_0$ be fixed. Without loss of generality , let $|\lambda| < 1$. Then the inequality (3) takes the form

$$g(\lambda x) \le \sup_k |\lambda|^{p_k/M} g(x) .$$

Now the proof that $\lambda x \to \theta$ becomes a routine work .

For $[A_\Delta, p]_\infty$, let $\inf p_k = h > 0$. Then clearly $x \to \theta \Rightarrow \lambda x \to \theta$ (for fixed λ). Now let $\lambda \to 0$ and $x \in [A_\Delta, p]_\infty$ be fixed. Without loss of generality, let $|\lambda| < 1$. Then $\lambda x \to \theta$ follows from the following inequality.

$$g^M(\lambda x) \le |\lambda|^h g^M(x) \tag{4}$$

Consider the case $[A_\Delta, p]$. If (2) holds, then g(x) exists for each $x \in [A_\Delta, p]$. Then the contunuity of the scalar multiplication becomes a routine work in view of the inequality (4).

Let $\{x^i\}$ be a Cauchy sequence in $[A_\Delta, p]_0$. Then there exists a sequence $x = (x_k) \in \ell_\infty$ such that $g(x^i - x) \to 0$ $(i \to \infty)$. Thus for a given $\varepsilon > 0$, there exists n_0 such that $g(x^i - x) < \varepsilon$ for all $i > n_0$. Hence $x^i - x \in [A_\Delta, p]_0$. Since $[A_\Delta, p]_0$ it follows that $x = x^i - (x^i - x) \in [A_\Delta, p]_0$. The completness of $[A_\Delta, p]_\infty$ can similarly be obtained.

We now consider $[A_\Delta, p]$. If (3) holds and $\{x^i\}$ is a Cauchy sequence in $[A_\Delta, p]$. Then there exists $x = (x_k) \in \ell_\infty$ such that $g(x^i - x) \to 0$ $(i \to \infty)$. Further as

$$\sum_k a(n,k,m) \to 0 \quad \textit{uniformly in } n,$$

it is clear that $[A_\Delta, p] = [A_\Delta, p]_0$.

This completes the proof.

Remark. Note that the strong summability field of the matrix method A= (a_{nk}) is not a *K*-space (a K-space of sequences for which the coordinate linear functionals are continuous) if the matrix A= (a_{nk}) contains zero columns.

Theorem 4 . The spaces $[A_\Delta, p]_0$, $[A_\Delta, p]$ *and* $[A_\Delta, p]_\infty$ are not solid in general.

Proof. It follows from the following example.

Example1. Consider the matrix $A = (a_{nk})$ defined by $a_{nk} = 1$ for all $n,k \in N$. Let $p_k = 1$ for all $k \in N$. Consider the sequence $x = (x_k)$ defined by $x_k = 1$ for all $k \in N$. Then $x \in [A_\Delta, p]_0$. Consider the sequence (α_k) of scalars defined by $\alpha_k = (-1)^k$ for all $k \in N$. Then it is clear that $(\alpha_k x_k) \notin [A_\Delta, p]_0$. Hence $[A_\Delta, p]_0$ is not solid. Similarly it can be shown that the two other spaces are not solid.

Theorem 5. The spaces $[A_\Delta, p]_0, [A_\Delta, p]$ *and* $[A_\Delta, p]_\infty$ are not symmetric in general.

Proof. This is clear from the following example.

Example 2. Consider the matrix $A = (a_{nk})$ defined by $a_{nk} = k^{-1}$ for all $n=k \in N$ and $a_{nk} = 0$, otherwise. Let $p_k = 1$ for all $k \in N$. Consider the sequence $x = (x_k)$ defined by $x_k = 1$ for all $k = j^2$, $j \in N$ and $x_k = 0$ otherwise. Then $x \in [A_\Delta, p]_0$. Now consider the rearrangement (y_k) of (x_k) defined as $y_k = 1$ for k odd and $y_k = 0$ for k even. Then $[A_\Delta, p]_0$. Hence $[A_\Delta, p]_0$ is not symmetric. The other cases can similarly be shown.

The proof of the following result are routine works in view of Lemma 2 and Lemma 3.

Theorem 6 (*a*). Let $0 \prec p_k \leq 1$. Then $[A_\Delta, p]_0$ and $[A_\Delta, p]_\infty$ are r-convex for all r, where $0 < r < \liminf p_k$. Moreover, if $p_k = p \leq 1$ for all k, then they are p-convex. If (2) holds, then $[A_\Delta, p]$ has the same property.

(b). Let $0 < p_k \leq 1$. Then $[A_\Delta, p]_0$ and $[A_\Delta, p]_\infty$ are locally bounded if $\inf p_k > 0$. If (2) holds, then $[A_\Delta, p]$ has the same property.

Theorem 7(a). Let $0 < p_k \leq q_k \leq 1$ for all k, then $[A_\Delta, q]_\infty$ is a closed subspace of $[A_\Delta, p]_\infty$.

(b). Suppose that $\|A\| \prec \infty$, $0 < p_k \leq q_k$ and $\dfrac{q_k}{p_k}$ is bounded for all k, then

$$[A_\Delta, q] \subset [A_\Delta, p].$$

(c). Let $0 < \inf p_k \leq p_k \leq 1$, then $[A_\Delta, p] \subset [A_\Delta]$.

(d). Let $1 \le p_k \le \sup p_k < \infty$, then $[A_\Delta] \subset [A_\Delta, p]$,

where

$$[A_\Delta] = \left\{ x : \sum_k a(n,k,m) |\Delta x_k - l| \to 0, \quad uniformly\ in\ n \right\}.$$

Proof (*a*). Let $x \in [A_\Delta, q]_\infty$. Then there exists a constant $B > 1$ such that

$$\sum_k a(n,k,m) |\Delta x_k|^{q_k} \le B \qquad (\text{ for every } m,n\,)$$

and so

$$\sum_k a(n,k,m) |\Delta x_k|^{p_k} \le B \qquad (\text{ for every } m,n\,),$$

thus $x \in [A_\Delta, p]_\infty$. To show that $[A_\Delta, q]_\infty$ is closed, suppose that $x^i \in [A_\Delta, q]_\infty$ and $x^i \to x \in [A_\Delta, p]_\infty$. Then for every $0 < \varepsilon < 1$, there exists n_0 such that for all m and n

$$\sum_k a(n,k,m) |\Delta(x_k^i - x_k)|^{p_k} < \varepsilon \qquad (\text{ for every } i > n_0).$$

Now

$$\sum_k a(n,k,m) |\Delta(x_k^i - x_k)|^{q_k} < \sum_k a(n,k,m) |\Delta(x_k^i - x_k)|^{p_k} < \varepsilon \qquad (\text{ for every } i > n_0)$$

This completes the proof.

(*b*) Write $w_k = |\Delta x_k - l|^{q_k}$ and $\dfrac{q_k}{p_k} = \lambda_k$ for all k. So that $0 < \lambda \le \lambda_k \le 1$ (λ constant). Let $x \in [A_\Delta, q]$. Then

$$\sum_k a(n,k,m) w_k \to 0 \quad uniformly\ in\ n.$$

Define

$$u_k = \begin{cases} w_k, & w_k \geq 1 \\ 0, & w_k < 1 \end{cases}$$

and

$$v_k = \begin{cases} 0, & w_k \geq 1 \\ w_k, & w_k < 1 \end{cases}$$

so that $w_k = u_k + v_k$, $w_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows $u_k^{\lambda_k} \leq u_k + w_k$, $v_k^{\lambda_k} \leq v_k^{\lambda}$. We have the inequality (see Maddox [2])

$$\sum_k a(n,k,m) w_k^{\lambda_k} \leq \sum_k a(n,k,m) w_k + \left(\sum_k a(n,k,m) v_k \right)^{\lambda} \|A\|^{1-\lambda}$$

Therefore $x \in [A_\Delta, p]$ and this completes the proof.

(*c*). The proof follows from 7(b).

(*d*). The proof follows from 7(b).

We have the following result which follows from the above result.

Corollary. Suppose that $\|A\| \prec \infty$.

(a). If $0 < \inf p_k \leq p_k \leq 1$, then $[A_\Delta, p] = [A_\Delta]$.

(b). If $1 \leq p_k \leq \sup p_k < \infty$, then $[A_\Delta] = [A_\Delta, p]$.

ÖZET

Bu çalışmada hemen hemen yakınsaklık kavramından doğal olarak ortaya çıkan bazı yeni fark dizi uzayları tanımlanmıştır. Kuvvetli hemen hemen toplanabilir dizi uzayları Nanda [6] tarafından tartışılmıştır. Bu makalenin amacı, bazı şartlar altında tam olabilen kuvvetli hemen hemen Δ-toplanabilir diziler uzayını tanımlayıp, bu tür dizilerin bazı topolojik sonuçlarını ve kapsam bağıntılarını tartışmaktır.

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