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TWO THEOREMS ON CONTINUOUSLY DIFFERENTIABILITY OF THE FINITE- DIMENSIONAL REPRESENTATIONS OF THE GROUPS $\mathbb{C}_0^1, R^n$

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ABSTRACT

This article will focus on, and little extend, the part of the earlier study [5] that concerns the representations of the groups $\mathbb{C}_0^1$, R^n, using the method given for continuously differentiability of $f(\alpha)$ [4].

1. INTRODUCTION

Historical Background

The first extensive investigation of the group representations was carried out in a series articles by Frobenius, who developed much of the theory of the representations of finite groups, and in particular, of the symetric groups. On the other hand, since 1940's an intensive study of foundations of the theory of group representations has been in progress. For example, in 1965, Stein constructed, in a fairly simple manner representations in $Gl(2n, \mathbb{C})$ which where not contained in the list of Gelfand and Naimark [1,3] and in 1998.

Author constructed, [6] two new representations of the topological group K_2 and then in the recent paper [5], he has obtained representations of the groups $\mathbb{C}_0^1$, R^n that we will work. The purpose of this paper is to obtain two results on continuously differentiability of all the one-dimensional representations $\varphi(z) = |z|^k e^{im \arg z \det(\beta)}$ of the group $\mathbb{C}_0^1$ and the representations $f \to f(\alpha_1, \alpha_2, ..., \alpha_n)$ of the group R^n, where k is a complex constant, m is an integer and $\alpha_1,\alpha_2,...,\alpha_n$ are standart basis vectors of the vector space R^n.

2. THE SOME REQUIRED CONCEPTS ABOUT SUBJECT

Let us note briefly some highlights of the mentioned papers. The usual terms and notations not described in this paper can be found in any standart book [such as 2 and 7]. In this section we shall outline the ideas of algebraic foundations of the representation theory and basic concept of some compact topological groups. Let $f=f(g)$ be a complex-valued function defined on a topological group G. The function f is called continuous on the topological space G.

Next, let X be a finite-dimensional complex vector space of dimensional n and let $e_1,e_2,...,e_n$ be a fixed basis in X. Let f be a vector–valued function with values in X defined on the topological group G and for $g\in G$, let $(f_1(g),f_2(g),...,f_n(g))$ be the components of the point $f(g)$ in the basis $e_1,e_2,...,e_n$. The function f is called continuous on the group if the complex-valued functions $f_1(g),f_2(g),...,f_n(g)$ are continuous on G. This definition is independent of the choice of a basis $e_1,e_2,...,e_n$. For each $g\in G$, the components $f_j'(g)$ of element $f(g)$ in another basis $e_1',e_2',...,e_n'$ are linear combinations with constant coefficients of the components $f_j'(g)$. If the functions f_j are continuous, so are functions f_j'.

Finally, let $A=A(g)$ be an operator-valued function on G whose values are linear operators on X. The function A is said to be continuous on G if $g\to A(g)x$ is a continuous vector function on G for every $x\in X$. Then we say that the function $A(g)$ is continuous if and only if its matrix elements $a_{jl}(g)$, in any basis $e_1,e_2,...,e_n$ in X, are continuous complex-valued functions on G. Let G be a topological group and let X be a finite-dimensional complex linear space different from $\{0\}$. A representation of G in X is any mapping that carries every $g\in G$ into linear operator $T(g)$ on X in such a way that;

1) $T(e)=1$, where 1 is the identity operator in X.
2) $T(g_1g_2)=T(g_1)T(g_2)$.
3) $T(g)$ is a continuous operator-valued function on G.

Thus we add to the usual definition the condition of continuity (to emphasize this, representations of a topological group are often called continuous) of the operator valued-function $T(g)$. This requirement will play a significant role in the sequel. Representations that not necessarily satisfying condition (3) will now be called algebraic representations. By a continuous representations of G we shall mean a representation (π, H) of G, where H is a Hilbert space and the map $(g,v) \to \pi(g)v$, $G \times H \to H$ is continuous. For a continuous representation (π, H) of G, a vector $v \in H$ is called K -finite if the span of all $\pi(k)v$, $k \in K$, is finite dimensional, where K is a compact subgroup. Some very important classes of groups have large compact subgroups, for example, reductive groups over local fields.

3. ON CONTINUOUSLY DIFFERENTIABILITY

We shall first introduce some basic groups and their continuously differentiable representations. Every one-dimensional representation of the group R^1 can be considered a continuous complex-valued function $\alpha \to f(\alpha)$ [5], $\alpha \in R^1$, that satisfies conditions,

$$f(0) = 1, f(\alpha_1 + \alpha_2) = f(\alpha_1)f(\alpha_2) \tag{2.1}$$

and is continuous.

Let $\omega(\alpha)$ be a continuously differentiable function on R^1 equal to zero outside of some neighborhood of the point $\alpha_0 \in R^1$ for which

$$c = \int_{-\infty}^{\infty} f(\alpha)\omega(\alpha)d\alpha \neq 0$$

by virtue of (2.1) such a function $\omega(\alpha)$ exists. Multiply both sides of (2.1) by $\omega(\alpha_2)$ and integrate both sides of the resulting equality from $-\infty$ to $+\infty$ with respect to α_2.

We obtain [3]

$$\int_{-\infty}^{\infty} f(\alpha_1 + \alpha_2)\omega(\alpha_2)d\alpha_2 = f(\alpha_1)\int_{-\infty}^{\infty} f(\alpha_2)\omega(\alpha_2)d\alpha_2 \tag{2.2}$$

and so

$$f(\alpha_1) = \frac{1}{c}\int_{-\infty}^{\infty} f(\alpha_1 + \alpha_2)\omega(\alpha_2)d\alpha_2 = \frac{1}{c}\int_{-\infty}^{\infty} f(\alpha)\omega(\alpha - \alpha_1)d\alpha \tag{2.3}$$

But the right side of (2.3) is a continuously differentiable function of α_1, Since $\omega(\alpha - \alpha_1)$ has this property and the integral is over a finite interval. That is, $f(\alpha_1)$ is continuously differentiable on R^1 [5]..

The function $f(\alpha)$ satisfies the differentiable equation

$$df/d\alpha = kf,$$

where k is a certain constant.

Differentiating both sides of (2.1) by α_1 and setting $\alpha_1 = 0$ and setting $\alpha_2 = \alpha$, we obtain

$$f'(\alpha) = f'(0)f(\alpha) = kf(\alpha), \text{ where } k = f'(0).$$

Any solution of the equation (2.4) that satisfies $f(0) = 1$ has the form $f(\alpha) = e^{k\alpha}$.

Thus all one-dimensional representations $\alpha \to f(\alpha)$ of the group R^1 are described by the formula

$$f(\alpha) = e^{k\alpha} \tag{2.5}$$

where k is a complex constant.

The unitary one-dimensional representations $f(\alpha)$ of the group R^1 are now easily identified the property of beeing unitary in the one-dimensional case means that $|f(\alpha)| = 1$. For $f(\alpha) = e^{k\alpha}$ this holds if and only if k is purely imaginary, $k = i\tau$, for some $\tau \in R^1$. Thus all unitary one-dimensional representations $\alpha \to f(\alpha)$ of the group R^1 have the form.

$$f(\alpha) = e^{i\tau\alpha}$$

where τ is a real constant. Conversely, for every $\tau \in R^1$ the formula $f(\alpha) = e^{i\tau\alpha}$ defines a one-dimensional unitary representation of the group R^1.

Note that continuity of a representation $\alpha \to f(\alpha)$ implies differentiability of the function $f(\alpha)$.

4. THEOREMS AND THEIR PROOFS ON THE CONTINUOUSLY DIFFERENTIABILITY

On Continuously Differentiability of $\varphi(z)$.

Every rotation of a circle is described by an angle α of the rotation, where rotations by α and $\alpha + 2\pi$ are identical. Therefore, in a one-dimensional representation $\alpha \to f(\alpha)$ of the group Γ^1 the function $f(\alpha)$ must, besides the conditions in equations (2.1), also satisfy the condition $f(\alpha + 2\pi) = f(\alpha)$. From equation (2.1), we see that $f(\alpha) = e^{k\alpha}$, and from $f(\alpha + 2\pi) = f(\alpha)$ that $k = im$, where m is an integer. Thus all one-dimensional representations $\alpha \to f(\alpha)$ of the group Γ^1 have the form $f(\alpha) = e^{im\alpha}$.

The group $\mathbb{C}_0^1$ ($\mathbb{C}_0^1 = \mathbb{C} - \{0\},.$) is isomorphic to the direct product $R_0^+ \times \Gamma^1$ and the mapping $z \to (r,\alpha)$ for $|z| = r$, $\alpha = |z|$, is an isomorphism of the group $\mathbb{C}_0^1$ onto $R_0^+ \times \Gamma^1$. Combining $f(\alpha) = e^{im\alpha}$ and $\varphi(\beta) = \beta^k = e^{k\ln\beta}$, then all one-dimensional representations $z \to \varphi(z)$ [5] of the group $\mathbb{C}_0^1$ have the form

$$\varphi(z) = |z|^k \, e^{im\arg z \det(\beta)} \tag{3.1}$$

where k is a complex constant and m is an integer.

Theorem 3.1. The representation $\varphi(z)$ in equation (3.1) is differentiable and its derivative is continuous throughout $\mathbb{C}_0^1$.

Proof. The representation $\omega = \omega(z)$ is a continuously differentiable function on continous $\mathbb{C}_0^1$ in some neighborhood of the point $z_0 \in \mathbb{C}_0^1$ for which

$$c = \int_{-\infty}^{\infty} \varphi(z)\omega(z)dz \neq 0$$

by virtue of $\varphi(z) \neq 0$, for $z_0 \in \mathbb{C}_0^1$ such a function $\omega(z)$ exists. Multiply both sides of $\varphi(z_1z_2) = \varphi(z_1)\varphi(z_2)$ by $\omega(z_2)$ and integrate both sides of the resulting equality from $-\infty$ to $+\infty$ with respect to z_2 and setting $z_1z_2 = z$, we obtain

$$\int_{-\infty}^{\infty} \varphi(z_1z_2)\,\omega(z_2)\,dz_2 = \varphi(z_1)\int_{-\infty}^{\infty} \varphi(z_2)\,\omega(z_2)dz_2$$

and so

$$\begin{aligned} \omega(z_1) &= \frac{1}{c}\int_{-\infty}^{\infty} \varphi(z_1z_2)\,\omega(z_2)\,dz_2 \\ &= \frac{1}{c}\int \varphi(z)\omega(z/z_1)dz. \end{aligned} \tag{3.2}$$

But the right side of (3.2) is continuously differentiable function of z/z_1 , since $\omega(z/z_1)$ has this property and the integral is over a finite interval. That is to say, $\varphi(z_1)$ is continuously differentiable. Briefly, the representation $\varphi(z)$ is continuously differentiable on $\mathbb{C}_0^1$. Since $\omega = \omega(z)$ can be taken infinitely differentiable, $\varphi(z)$ is infinitely differentiable.

Theorem 3.2. Let $\alpha_1,\alpha_2,...,\alpha_n$ be standard basis vectors of the vector space R^n. Then the representation $f \to f(\alpha_1,\alpha_2,...,\alpha_n)$ is differentiable and its derivative is continuous throughout R^n.

Proof. Let $\omega=\omega(\alpha)=(\alpha_1,\alpha_2,...,\alpha_n)$ be a continuously differentiable function on R^n equal to zero outside of some neighborhood of the point $\alpha_1,\alpha_2,...,\alpha_n \in R^n$ for which

$$c=\int_{-\infty}^{\infty} f(\alpha_1,\alpha_2,...,\alpha_n)\,\omega(\alpha_1,\alpha_2,...,\alpha_n)d\alpha_i \neq 0 \quad (i=1,2,...).$$

By virtue of $f(\alpha_1,\alpha_2,...,\alpha_n)\neq 0$ such a function of $\omega=\omega(\alpha_1,\alpha_2,...,\alpha_n)$ exists. Multiply the representation of $f=f(\alpha_1,\alpha_2,...,\alpha_n)$ by $\omega=\omega(\alpha_2,...,\alpha_n)$ and integrate both sides of the resulting equality from $-\infty$ to $+\infty$ with respect to $\alpha_i(i=1,2,....)$. We obtain

$$\int_{-\infty}^{\infty} f(\alpha_1,\alpha_2,...,\alpha_n)\,\omega(\alpha_1,\alpha_2,...,\alpha_n)\,d\alpha_i = f(\alpha_1)\int_{-\infty}^{\infty} f(\alpha_2,...,\alpha_n)\,\omega(\alpha_2,...,\alpha_n)\,d\alpha_i \qquad (3.3)$$

and so

$$f(\alpha_1)=\frac{1}{c}\int_{-\infty}^{\infty} f(\alpha_2,...,\alpha_n)\,\omega(\alpha_2,...,\alpha_n)\,d\alpha_i$$

$$f(\alpha_1)=\frac{1}{c}\int_{-\infty}^{\infty} f(\alpha)\,\omega(\alpha/\alpha_1)\,d\alpha. \qquad (3.4)$$

But the right side of (3.4) is a continuously differentiable function of α_1, Since $\omega(\alpha/\alpha_1)$ has this property and the integral is over a finite interval, $f(\alpha_1)$ is continuously differentiable.

Similarly, if we do similar procedures for $f(\alpha_2)$, we obtain differentiability on R^n.

$$f(\alpha_2)=\frac{1}{c}\int_{-\infty}^{\infty} f(\alpha_1,\alpha_2,...,\alpha_n)\,\omega(\alpha_1,\alpha_3,...,\alpha_n)\,d\alpha_i \qquad i=1,3,4,...$$

$$f(\alpha_2)=\frac{1}{c}\int_{-\infty}^{\infty} f(\alpha)\,\omega(\alpha/\alpha_2)\,d\alpha .$$

Briefly $f(\alpha_2)$ is continuously differentiable on R^n. If we continue in this way get

$$f(\alpha_n)=\frac{1}{c}\int_{-\infty}^{\infty} f(\alpha_1,\alpha_2,...,\alpha_{n-1})\,\omega(\alpha_1,\alpha_2,...,\alpha_{n-1})\,d\alpha_i, \quad i=1,2,3,...,n-1$$

and we have

$$f(\alpha_n)=\frac{1}{c}\int_{-\infty}^{\infty} f(\alpha)\,\omega(\alpha/\alpha_n)\,d\alpha_i\ , \qquad i=1,2,3,\ldots,n-1$$

Then $f(\alpha_n)$ is continuously differentiable on R^n. Since functions $f(\alpha_1), f(\alpha_2), f(\alpha_3),\ldots, f(\alpha_n)$ are continuously differentiabile, then representation $f \to f(\alpha_1,\alpha_2,\ldots,\alpha_n)$ are continuously differentiable in R^n.

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