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A NOTE ON THE BOUNDEDNESS OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY AND ADVANCED ARGUMENTS

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ABSTRACT

In this note, we discuss the boundedness of a class of systems, if unique global solutions exist.

KEYWORDS Advanced and delay arguments, boundedness, Lyapunow function

1. INTRODUCTION

The study of boundedness of functional differential equations with delays has been investigated by many authors (see for example [1-7]). In this study, we consider functional differential equations having both advanced and delay arguments of the form

$$\begin{aligned} x_1'(t) &= -a_1(t)x_1(t) + a_2(t)x_2(t) + b_1(t)x_1(t-h)u(x_2(t)) \\ &\quad - kb_1(t+h)u(x_2(t+h))x_1(t+h) \\ x_2'(t) &= -a_3(t)x_2(t) + a_2(t)x_1(t) - b_2(t)x_2(t+h)z(x_1(t+h)) \\ &\quad + \ell b_2(t-h)z(x_1(t))x_2(t-h) \end{aligned} \tag{1}$$

where $a_1, a_2, a_3, b_1, b_2, u, z : IR^+ \to R^+$, are continuous functions and $k > 1,\ 0 < \ell < 1,\ h > 0$ are real numbers.

We assume that

$$a_1(t)a_3(t) - a_2^2(t) \geq 0 \tag{2}$$

for all $t \geq 0$.

In this article, we have discussed the boundedness of all positive solutions of (1).

2. BOUNDEDNESS

Theorem. Suppose there exist $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that

$$0 < \alpha_1 \le u(s) \le \beta_1,$$
$$0 < \alpha_2 \le z(s) \le \beta_2$$

for $s \in IR$.

If (2) holds and $\alpha_1 > \frac{1}{2k}$, $\beta_1 < \frac{1}{2}$, $\alpha_2 > \frac{1}{2}$, $\beta_2 < \frac{1}{2\ell}$ then every solution $x_1(t), x_2(t)$ of (1) are bounded.

Proof. Let $x_1(t), x_2(t)$ denote a solution of (1). Let us define

$$V(t, x_1(t), x_2(t)) = x_1^2(t) + x_2^2(t) + \int_{t-h}^{t} b_1(s_1+h)x_1(s_1)x_1(s_1+h)ds_1 + \int_{t}^{t+h} b_2(s_2-h)x_2(s_2)x_2(s_2-h)ds_2 . \tag{3}$$

It is clear that

$$x_1^2(t) + x_2^2(t) \le V(t, x_1(t), x_2(t)) \tag{4}$$

and $V(t,0,0) = 0$. Hence V is positive definite.

Now let us compute the derivative of V.

$$\begin{aligned}
V'(t, x_1(t), x_2(t)) \le\ & -2a_1(t)x_1^2(t) + 4a_2(t)x_1(t)x_2(t) - 2a_3(t)x_2^2(t) \\
& + b_1(t)x_1(t-h)\left[2u(x_2(t))x_1(t) - x_1(t)\right] \\
& + b_1(t)x_1(t+h)\left[-2ku(x_2(t+h))x_1(t) - x_1(t)\right] \\
& + b_2(t-h)x_2(t-h)\left[2\ell z(x_1(t))x_2(t) - x_2(t)\right] \\
& + b_2(t)x_2(t+h)\left[-2z(x_1(t+h))x_2(t) + x_2(t)\right] \\
\le\ & -2a_1(t)x_1^2(t) + 4a_2(t)x_1(t)x_2(t) - 2a_3(t)x_2^2(t) \\
& + b_1(t)x_1(t-h)(2\beta_1 - 1)x_1(t) \\
& + b_1(t+h)x_1(t+h)(1 - 2k\alpha_1)x_1(t) \\
& + b_2(t-h)x_2(t-h)(2\ell\beta_2 - 1)x_2(t) \\
& + b_2(t)x_2(t+h)(1 - 2\alpha_2)x_2(t).
\end{aligned}$$

If $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfy the assumptions given in the statement of the Theorem, we find

$$V'(t, x_1(t), x_2(t)) \leq -2(x_1(t) \quad x_2(t))\begin{pmatrix} a_1(t) & -a_2(t) \\ -a_2(t) & a_3(t) \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}. \tag{5}$$

Let $\lambda_1(t)$, $\lambda_2(t)$ denote eigenvalues of the matrix

$$\begin{pmatrix} a_1(t) & -a_2(t) \\ -a_2(t) & a_3(t) \end{pmatrix}.$$

Since

$$\lambda_1(t) + \lambda_2(t) = a_1(t) + a_3(t) \geq 0$$

and by (2)

$$\lambda_1(t)\lambda_2(t) = a_1(t)a_3(t) - a_2^2(t) \geq 0$$

we see that

$$\lambda_1(t) \geq 0, \ \ \lambda_2(t) \geq 0.$$

Let $\lambda(t) = \min\{\lambda_1(t), \lambda_2(t)\}$. It follows from (5) that

$$V'(t, x_1(t), x_2(t)) \leq -2C\lambda(t)(x_1^2(t) + x_2^2(t)) \tag{6}$$

for some $C>0$.

Therefore V is decreasing. Integrating (6) from $0 \leq t_0$ to t, we get

$$V(t) \leq V(t_0). \tag{7}$$

From (4) and (7) it follows that

$$x_1^2(t) + x_2^2(t) \leq V(t_0)$$

for all $t_0 \geq 0$.

This means that solution of $x_1(t), x_2(t)$ is bounded.

Remark: Since all solutions are bounded, local existence and uniqueness of solutions result in global existence and uniqueness of solutions of (1) under the conditions of the Theorem.

Another interesting problem is to obtain stability results concerning trivial solution of (1). Further results in this direction will be reported in due courses.

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ÖZET: Bu çalışmada, global çözüme sahip lineer olmayan bir sistemin pozitif çözümlerinin sınırlılığı tartışılmıştır.

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