Commun. Fac. Sci. Univ. Ank. Series A1
V. 53.(No.2) pp. 23 -34 (2004)

BOUNDS FOR PRODUCT OF SINGULAR VALUES USING ROW (COLUMN) NORM AND DETERMINANT

A. D.GÜNGÖR[1] AND D. TAŞÇI[2]

[1] *Department of Mathematics, Art and Science Faculty, Selçuk University, KONYA*
[2] *Department of Mathematics, Art and Science Faculty, Gazi University, ANKARA*

(Received June 17, 2004; Revised Oct. 11, 2004; Accepted Dec. 13, 2004)

ABSTRACT

Let A be a $n \times n$ complex matrix with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ and let $1 \leq k < l \leq n$. Bounds for $\sigma_1 \ldots \sigma_k$, $\sigma_k \ldots \sigma_l$ and $\sigma_{n-k+1} \ldots \sigma_n$, involving k, n, $r_i(c_i)$, α_i and $\det A$ where $r_i(c_i)$ is the Euclidean norm of the *i*-th row (column) of A and α_i's are positive real numbers such that $\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2 = n$, are presented.

1. INTRODUCTION AND PRELIMINARIES

Let A be $n \times n$ an complex matrix . Let $\sigma_i(A)$'s be the singular values of A such that

$$\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_n(A).$$

It is well known that

$$\sigma_1^2(A) + \sigma_2^2(A) + \ldots + \sigma_n^2(A) = \|A\|_F^2 \tag{1.1}$$

and

$$\sigma_1(A)\sigma_2(A)\ldots\sigma_n(A) = |\det A| \tag{1.2}$$

where $\|A\|_F$ and $\det A$ denote the Frobenius norm of A and the determinant of A, respectively.

Wang and Zhang [2] have established some inequalities for the eigenvalues of the product of positive semidefinite Hermitian matrices. Merikoski and Virtanen [3] obtained bounds for eigenvalues using the trace and determinant. Rojo [4] defined matrix B.

In Section 2, we have obtained bounds for products of singular values using bounds for eigenvalues in [3] and (1.1) and (1.2) inequalities. In Section 3, we have found bounds for products of singular values using row (column) norm and determinant.

Firstly, we give some preliminaries related to our study.

We define

$$D = diag\left\{\frac{\alpha_1}{r_1(A)}, \frac{\alpha_2}{r_2(A)}, \ldots, \frac{\alpha_n}{r_n(A)}\right\}, \tag{1.3}$$

where $r_i(A)$ is the Euclidean norm of the i-th row of A and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive real numbers such that

$$\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2 = n. \tag{1.4}$$

Clearly, the Euclidean norm of the coefficient matrix $B=DA$ of the scaled system is equal to $\sqrt{n}$ and if $\alpha_1 = \alpha_2 = .. = \alpha_n = 1$ then each row of B is a unit vector in the Euclidean norm. Also, we can define $B=AD$,

$$D = diag\left\{\frac{\alpha_1}{c_1(A)}, \frac{\alpha_2}{c_2(A)}, \ldots, \frac{\alpha_n}{c_n(A)}\right\}, \tag{1.5}$$

where $c_i(A)$ is the Euclidean norm of the i-th column of A. Again, $\|B\|_F = \sqrt{n}$ and if $\alpha_1 = \alpha_2 = .. = \alpha_n = 1$ then each column of B is a unit vector in the Euclidean norm.

Theorem 1 [2] Let $A, B \in \mathbb{C}^{nxm}$ and let $1 \le i_1 < \ldots < i_k \le n$. Then

$$\prod_{t=1}^{k} \sigma_{i_t}(AB) \le \prod_{t=1}^{k} \sigma_{i_t}(A)\sigma_t(B) \tag{1.6}$$

and

$$\prod_{t=1}^{k} \sigma_t(AB) \ge \prod_{t=1}^{k} \sigma_{i_t}(A)\sigma_{n-i_t+1}(B). \tag{1.7}$$

Theorem 2 [1] Let $A \in \mathbb{C}^{nxm}$, $B \in \mathbb{C}^{nxm}$ and let $1 \le i_1 < \ldots < i_k \le n$. Then

$$\prod_{t=1}^{k} \sigma_{i_t}(AB) \ge \prod_{t=1}^{k} \sigma_{i_t}(A)\sigma_{n-t+1}(B). \tag{1.8}$$

2. BOUNDS FOR PRODUCT OF SINGULAR VALUES USING NORM AND DETERMINANT

We will give bounds that we obtained for singular values as a result of some bounds for eigenvalues which is obtained in [3] where A be a square matrix with singular values

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0.$$

Corollary 3 Let $1 \leq k \leq l \leq n$. Then

$$(2.1) \qquad \left(\frac{k-1}{\|A\|_F^2}\right)^{(k-1)/2} |\det A|^{1/n} \leq \sigma_k \ldots \sigma_l \leq \left(\frac{\|A\|_F^2}{l}\right)^{l/2} \left(\frac{1}{|\det A|}\right)^{(k-1)/n}.$$

Corollary 4 Let $1 \leq k \leq n-2$. Then

(2.2)

$$\sigma_1 \ldots \sigma_k \leq \left[\frac{1}{(\det A)^2}\left(\frac{1}{n-k}\left(\frac{\|A\|_F^2}{k+1}\right)^{k+1}\right)^{n-k}\right]^{1/2(n-k-1)}.$$

Let $2 \leq k \leq n-1$. Then

$$(2.2) \qquad \sigma_{n-k+1} \ldots \sigma_n \geq \left[k(\det A)^2 \left(\frac{n-k+1}{\|A\|_E^2}\right)^{n-k+1}\right]^{k/2(k-1)}.$$

Corollary 5 Let $1 \leq k \leq l \leq n-2$. Then

(2.4)

$$\left\{|\det A|^{2\left(1-\frac{k}{n}\right)l+2}\left[(n-k+1)\left(\frac{k}{\|A\|_E^2}\right)^k\right]^{n-k+1}\right\}^{1/2(n-k)} \leq \sigma_k \ldots \sigma_l$$

$$\leq \left\{ \left(\frac{1}{|\det A|} \right)^{2[(n-l)k+l-k+1]/n} \left[\frac{1}{n-l} \left(\frac{\|A\|_E^2}{l+1} \right)^{l+1} \right]^{n-l} \right\}^{1/2(n-l-1)}$$

and

$$\left[(n-k+1)(\det A)^2 \left(\frac{k}{\|A\|_E^2} \right)^k \right]^{(l-k+1)/2(n-k)} \leq \sigma_k \ldots \sigma_l \tag{2.5}$$

$$\leq \left\{ \frac{1}{(\det A)^2} \left[\frac{1}{n-l} \left(\frac{\|A\|_E^2}{l+1} \right)^{l+1} \right]^{n-l} \right\}^{(l-k+1)/2l(n-l-1)}$$

3. BOUNDS FOR PRODUCT OF SINGULAR VALUES USING ROW (COLUMN) NORM AND DETERMINANT

Since the matrices PA, AP and A have the same singular values for any permutation matrix P, we assume for this section, without loss of generality, that the rows and columns of A are such that

$$r_1(A) \leq r_2(A) \leq \ldots \leq r_n(A) \tag{3.1}$$

$$c_1(A) \leq c_2(A) \leq \ldots \leq c_n(A) \tag{3.2}$$

and let

$$0 < \alpha_n \leq \ldots \leq \alpha_2 \leq \alpha_1 \tag{3.3}$$

for α_i 's in (1.4).

Theorem 6 Let $2 \leq k \leq n-1$. Then

$$\sigma_1 \ldots \sigma_k \geq \prod_{i=n-k+1}^{n} \frac{r_i}{\alpha_i} \left\{ k \left(\prod_{i=1}^{n} \frac{\alpha_i}{r_i} |\det A| \right)^2 \left(\frac{n-k+1}{n} \right)^{n-k+1} \right\}^{k/2(k-1)} \tag{3.4}$$

where r_i 's and α_i 's be as in (3.1) and (3.3), respectively.

Proof. We write inequality (2.3) for matrix $B=DA$ which is defined in Section 1. Taking $i_1 = n-k+1$ and $i_k = n$ in (1.6) and applying $B = DA$, we obtain (3.4).

Theorem 7 Let $1 < k \le n-2$. Then
(3.5)

$$\sigma_1 \ldots \sigma_k \le \prod_{i=n-k+1}^{n} \frac{\min(r_i, c_i)}{\alpha_i} \left\{ \left(\frac{\prod_{i=1}^{n} \frac{\min(r_i, c_i)}{\alpha_i}}{|\det A|} \right)^2 \left[\frac{1}{n-k} \left(\frac{n}{k+1} \right)^{k+1} \right]^{n-k} \right\}^{1/2(n-k-1)} \tag{3.5}$$

where $r_i(c_i)$'s and α_i's be as in (3.1) $((3.2))$ and (3.3), respectively.

Proof. Write (2.2) for matrix $B=DA$ which is defined in Section 1. Taking $i_1 = n-k+1$ and $i_k = n$ in (1.7) and applying matrix $B=DA$. Then taking $i_1 = 1$ and $i_k = k$ in (1.8) and applying matrix $B=AD$ which is defined in Section 1, we have (3.5).

Theorem 8 Let $1 < k \le n-2$. Then

$$\sigma_{n-k+1} \ldots \sigma_n \le \prod_{i=1}^{k} \frac{\min(r_i, c_i)}{\alpha_i} \left\{ \left(\frac{\prod_{i=1}^{n} \frac{\min(r_i, c_i)}{\alpha_i}}{|\det A|} \right)^2 \left[\frac{1}{n-k} \left(\frac{n}{k+1} \right)^{k+1} \right]^{n-k} \right\}^{1/2(n-k-1)} \tag{3.6}$$

where $r_i(c_i)$'s and α_i's be as in (3.1) $((3.2))$ and (3.3), respectively.

Proof. First, we write (2.2) according to matrix B such that $\|B\|_F = \sqrt{n}$. Taking $i_1 = 1$, $i_k = k$ and $i_1 = n-k+1$, $i_k = n$ in (1.7) and applying matrices $B=DA$ and $B = AD$ which are defined in Section 1, respectively, we get (3.6).

Theorem 9 Let $1 \le k < l \le n$. Then

$$\prod_{i=1}^{l-k+1} \frac{c_i}{\alpha_i} \left\{ (n-k+1) \left(\prod_{i=1}^{n} \frac{\alpha_i}{c_i} |\det A| \right)^2 \left(\frac{k}{n} \right)^k \right\}^{(l-k+1)/2(n-k)} \le \sigma_k \ldots \sigma_l \tag{3.7}$$

$$\le \prod_{i=n-l+k}^{n} \frac{c_i}{\alpha_i} \left\{ \left(\frac{\prod_{i=1}^{n} \frac{c_i}{\alpha_i}}{|\det A|} \right)^2 \left[\frac{1}{n-l} \left(\frac{n}{l+1} \right)^{l+1} \right]^{n-l} \right\}^{(l-k+1)/2l(n-l-1)}$$

where c_i's and α_i's be as in (3.2) and (3.3), respectively.

Proof. We write inequality (2.1) for matrix $B = AD$ which is defined in Section 1. Taking $i_1 = k$, $i_k = l$ in (1.6) and applying matrix $B = AD$, we obtain (3.7).

Theorem 10 Let $1 \le k < l \le n$. Then

$$\sigma_{n-l+1} \ldots \sigma_{n-k+1} \le \tag{3.8}$$

$$\prod_{i=k}^{l} \frac{r_i}{\alpha_i} \left\{ \left(\frac{\prod_{i=1}^{n} \frac{r_i}{\alpha_i}}{\det A} \right)^2 \left[\frac{1}{n-l+k-1} \left(\frac{n}{l-k+2} \right)^{l-k+2} \right]^{n-l+k-1} \right\}^{1/2(n-l+k-2)}$$

where r_i's and α_i's be as in (3.1) and (3.3), respectively.

Proof. Firstly, we write inequality (2.3) for matrix $B = DA$ which is defined in Section 1. Then taking $i_1 = k$, $i_k = l$ in (1.7) and applying matrix $B = DA$, we have (3.8).

Theorem 11 Let $1 \le k < l \le n-2$. Then

(3.9) $\sigma_1 \ldots \sigma_{l-k+1} \ge$

$$\prod_{i=k}^{l} \frac{r_i}{\alpha_i} \left\{ \left(\prod_{i=1}^{n} \frac{\alpha_i}{r_i} |\det A| \right)^{2\left(1-k/n\right)l+2} \left[(n-k+1) \left(\frac{k}{n} \right)^k \right]^{n-k+1} \right\}^{1/2(n-k)}$$

where r_i's and α_i's be as in (3.1) and (3.3), respectively.

Proof. Write inequality (2.4) for matrix $B = DA$ which is defined in Section 1. Taking $i_1 = k$, $i_k = l$ in (1.6) and applying matrix $B = DA$, we get (3.9)

Theorem 12 Let $1 \le k < l \le n-2$. Then

$$(3.10) \quad \sigma_{n-l+k} \ldots \sigma_n \le \prod_{i=k}^{l} \frac{r_i}{\alpha_i} \left\{ \left(\frac{\prod_{i=1}^{n} \frac{r_i}{\alpha_i}}{|\det A|} \right)^2 \left[\frac{1}{n-l} \left(\frac{n}{l+1} \right)^{l+1} \right]^{n-l} \right\}^{(l-k+1)/2l(n-l-1)}$$

where r_i's and α_i's be as in (3.1) and (3.3), respectively.

Proof. To show (3.10), we write (2.5) for matrix $B = DA$ which is defined in Section 1. Taking $i_1 = k$, $i_k = l$ in (1.8) and applying matrix $B = DA$, we get (3.10).

Teorem 13 Let $1 \le k < l \le n-2$. Then

(3.11) $\sigma_k ... \sigma_l \le$

$$\prod_{i=n-l+1}^{n-k+1} \frac{c_i}{\alpha_i} \left\{ \left(\frac{\prod_{i=1}^{n} \frac{c_i}{\alpha_i}}{|\det A|} \right)^2 \left[\frac{1}{n-l+k-1} \left(\frac{n}{l-k+2} \right)^{l-k+2} \right]^{n-l+k-1} \right\}^{1/2(n-l+k-2)}$$

where c_i's and α_i's be as in (3.2) and (3.3), respectively.

Proof. To show (3.11), write inequality (2.2) for matrix $B = AD$ which is defined in Section 1. Taking $i_1 = k$, $i_k = l$ in (1.7) and applying matrix $B = AD$, we get (3.11).

Theorem 14 Let $2 \le k \le n-1$. Then

(3.12) $$\sigma_{n-k+1} ... \sigma_n \ge \prod_{i=1}^{k} \frac{c_i}{\alpha_i} \left\{ k \left(\prod_{i=1}^{n} \frac{\alpha_i}{c_i} |\det A| \right)^2 \left(\frac{n-k+1}{n} \right)^{n-k+1} \right\}^{k/2(k-1)}$$

where c_i's and α_i's be as in (3.2) and (3.3), respectively.

Proof. . Write inequality (2.3) for matrix $B = AD$ which is defined in Section 1. Taking $i_1 = n-k+1$, $i_k = n$ in (1.6) and applying matrix $B = AD$, we have inequality (3.12).

Example 15 Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 2 \end{bmatrix}$. Singular values of matrix A are $\sigma_1 = 6.673$, $\sigma_2 = 1.949$, $\sigma_3 = 1.181$ and $\sigma_4 = 0.521$. In the following, the best bounds are underlined.

l / *k*	*1*	*2*	*3*	*4*
1	σ_1	$\sigma_1\sigma_2$	$\sigma_1\sigma_2\sigma_3$	$\sigma_1\sigma_2\sigma_3\sigma_4$
2		σ_2	$\sigma_2\sigma_3$	$\sigma_2\sigma_3\sigma_4$
3			σ_3	$\sigma_3\sigma_4$
4				σ_4

For product of singular values, we have the following:

1.THE VALUES OF PRODUCT OF SİNGULAR VALUES

l / *k*	*1*	*2*	*3*	*4*
1	6.673	13.008	15.362	8
2		1.949	2.302	1.199
3			1.181	0.615
4				0.521

2. LOWER BOUNDS FOR PRODUCT OF SINGULAR VALUES:

(3.4)

l \ *k*	*1*	*2*	*3*	*4*
1		0.02	0.415	2.063

(3.7)

l \ *k*	*1*	*2*	*3*	*4*
1		0.223	0.267	2.017
2			0.032	0.038
3				0.002

(3.9)

l \ *k*	*1*	*2*	*3*	*4*
1		0.198	0.415	2.063

(3.12)

l \ *k*	*1*	*2*	*3*	*4*
1				2.017
2				0.038
3				0.002

3. UPPER BOUNDS FOR PRODUCT OF SINGULAR VALUES:

(3.4)

k \ l	1	2
1		4415.874

(3.6)

k \ l	1	2	3	4
1				
2				
3				358.542

(3.7)

k \ l	1	2
1		5277.946

(3.8)

k \ l	1	2	3	4
1				
2				
3				27990.74

(3.10)

k \ l	1	2	3	4
1				
2				
3				404.69

(3.11)

l / k	1	2
1		390281.06

The bounds for individual singular values are found from the diagonals of these tables.

We have seen that some bounds for $\alpha_1 = \alpha_2 = .. = \alpha_n = 1$ are better than those for α_i's $(i = 1,...,n)$ such that $\alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2 = n$ while some bounds are worse.

ÖZET

A, singüler değerleri $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n$ olan $n \times n$ tipinde bir kompleks matris ve $1 \leq k < l \leq n$ olsun. Bu çalışmada; $r_i(c_i)$, A nın *i*-inci satır Euclidean normu (sütun Euclidean normu) ve α_i ler, $\alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2 = n$ olacak şekildeki pozitif reel sayılar olmak üzere $\sigma_1 ... \sigma_k$, $\sigma_k ... \sigma_l$ ve $\sigma_{n-k+1} ... \sigma_n$ için k, n, $r_i(c_i)$, α_i ve $\det A$ içeren sınırlar elde edildi.

REFERENCES

[1] Marshall A. W., Olkin I. , Inequalities, Theory of Majorization and its Applications, Academic, 1979, New York.

[2] Wang B., Zhang F., Some Inequalities for the Eigenvalues of the Product of Positive Semidefinite Hermitian Matrices, Linear Algebra and its Applications 160 (1992), 113-118.

[3] Merikoski J. K., Virtanen A. , Bounds for Eigenvalues Using the Trace and Determinant, Linear Algebra and its Applications 264 (1997), 101-108.

[4] Rojo O., Further bounds for the smallest singular values and the spectral condition numbers, Computers and Mathematics with Applications, Vol. 38, No. 7-8 (1999), 215-228.

e-mail addresses: ayse2dilek@yahoo.com
dtasci@gazi.edu.tr