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ON HARMONIC CURVATURES OF CURVES IN LORENTZIAN N-SPACE

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ABSTRACT. In this study we consider the harmonic curvatures of a Frenet curve in Lorentzian space $\mathbb{L}^n$. We give a charecterization of the r^{th}–curvature center $C_r(t)$ of a Frenet curve of $\mathbb{L}^n$ with respect to its harmonic curvatures H_j, $1 \le j \le r$.

1. Introduction

Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be nonzero vectors in n-dimensional real vector space $\mathbb{R}^n$. For $X, Y \in \mathbb{R}^n$

$$\langle X, Y\rangle = -x_1y_1 + \sum_{i=2}^{n} x_iy_i \tag{1.1}$$

is called *Lorentzian inner product.* The couple $\{\mathbb{R}^n, \langle,\rangle\}$ is called *Lorentzian space* and denoted by $\mathbb{L}^n$ [2]. The vector X of $\mathbb{L}^n$ is called (see [7])

i) time-like if $\langle X, X\rangle < 0$,

ii) space-like if $\langle X, X\rangle > 0$ or $X = 0$,

iii) null or null vector if $\langle X, X\rangle = 0$, $X \neq 0$.

In [4] the first author consider the curvature center of the curves on a hypersurface M^n in $\mathbb{E}^{n+1}$ which was partially contained some results from [5].

In [6] the same author consider the curvature center of the curves a hypersurface in a n-dimensional Lorentzian space $\mathbb{L}^n$. She has shown that the locus of centers of spheres that has $\alpha(t)$ as the r-multiple contact point with α are on the $(n-r)$-curvature hyperplane $D_{(n-r)}(t)$.

In this study we consider the harmonic curvatures of a Frenet curve in Lorentzian space $\mathbb{L}^n$. We give a characterization of the r^{th}–curvature center $C_r(t)$ of a Frenet curve of $\mathbb{L}^n$ with respect to its harmonic curvatures H_j, $1 \le j \le r$. We also consider the general helices in $\mathbb{L}^n$.

2. Basic Definitions

The Frenet curvature and Frenet equations of curve $\mathbb{L}^n$ can be defined as follows.

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Definition 2.1. Let $\alpha : I \longrightarrow \mathbb{L}^n$be a curve in $\mathbb{L}^n$ and $k_1, k_2..., k_{(n-1)}$ the Frenet curvatures of α. Then for the unit tangent vector $V_1 = \alpha'(t)$ over M the i^{th} *e-curvature function* m_i is defined by (see [5])

$$m_i = \begin{cases} 0 & , \quad i = 1 \\ (\varepsilon_1 \varepsilon_2)/k_1 & , \quad i = 2 \\ \left[\frac{d}{dt}(m_{i-1}) + \varepsilon_{i-2} k_{i-2} m_{i-2}\right] (\varepsilon_i)/k_{i-1} & , \quad 2 < i \leq n \end{cases} \tag{2.1}$$

where $\varepsilon_i = \langle V_i, V_i \rangle = \pm 1$.

Let $\alpha : I \longrightarrow \mathbb{L}^n$ be a unit speed non-null curve in $\mathbb{L}^n$. The curve α is called *Frenet curve of osculating order* d if its higher order derivatives $\alpha'(t), \alpha''(t), \dots, \alpha^d(t)$ are linearly independent and $\alpha'(t), \alpha''(t), \dots, \alpha^d(t), \alpha^{d+1}(t)$ are no longer linearly independent for all $t \in I$. For each Frenet curve of order d one can associate an orthonormal $d-$frame $V_1, V_2, ..., V_d$ along α (such that $\alpha'(t) = V_1$) called the Frenet frame and $d-1$ functions $k_1, k_2, ..., k_{(d-1)} : I \longrightarrow \mathbb{E}$ called the *Frenet curvatures*, such that the Frenet formulas are defined in the usual way;

$$\begin{aligned} V_1' &= \nabla_{v_1} \alpha' = \varepsilon_2 k_1 V_2, \\ V_2' &= \nabla_{v_1} V_2 = -\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_3, \\ &\vdots \\ V_i' &= \nabla_{v_1} V_i = -\varepsilon_{(i-1)} k_{(i-1)} V_{(i-1)} + \varepsilon_{(i+1)} k_i V_{(i+1)}, \\ V_{i+1}' &= \nabla_{v_1} V_{i+1} = -\varepsilon_i k_i V_i \end{aligned} \tag{2.2}$$

where ∇ is the Levi-Civita connection of $\mathbb{L}^n$.

A non-null curve $\alpha : I \longrightarrow \mathbb{L}^n$ is called a *W-curve* (or *helix*) *of rank d*, if α is a Frenet curve of osculating order d and the Frenet curvatures k_i , $1 \leq i \leq d-1$ are non-zero constants.

Definition 2.2. Let α be a non-null curve of osculating order d. The functions $H_j : I \longrightarrow \mathbb{L}^n$, $1 \leq j \leq d-2$, defined by

$$\begin{aligned} H_0 &= 0, \ H_1 = \frac{k_1}{k_2} \\ H_j &= \{\nabla_{v_1}(H_{j-1}) + \varepsilon_{j-2} H_{j-2} k_j\} \frac{\varepsilon_j}{k_{j+1}}, 2 \leq j \leq d-2 \end{aligned} \tag{2.3}$$

are called the *harmonic curvatures of* α, where $k_1, k_2, ..., k_{(d-1)}$ are not necessarily constant [1].

Proposition 1. [8] *Let α be a non-null curve of osculating order d then*

$$k_r(t) = \frac{\varepsilon_{r-2} \left(\sum\limits_{i=1}^{r-2} H_i^2 \right)'}{2H_{r-2} H_{r-1}}, 2 < r \leq d-2 \tag{2.4}$$

where $(H_i)'$ stands for differentiation with respect to parameter t.

3. General Helices in $\mathbb{L}^n$

In the present section we will consider general helices in $\mathbb{L}^n$.

Definition 3.1. Let α be a non-null curve of osculating order d. Then α is called a *general helix of rank* $(d-2)$ if *(see* [1]*)*

$$\sum_{i=1}^{d-2} H_i^2 = c, \tag{3.1}$$

holds, where c is a real constant.

We have the following result.

Theorem 3.2. *For the non-null curve of* $\mathbb{L}^n$, *if the* j^{th} *harmonic curvature* H_j $(j \neq 1)$ *vanishes identically, i.e.* $H_j = 0$, *then* α *is a general helix of rank* $j-1$.

Proof :Let α be a non-null curve of $\mathbb{L}^n$ then by (2.3) the harmonic curvatures of α become

$$H_j = \left((H_{j-1})' + \varepsilon_{j-2} H_{j-2} k_j\right) \frac{\varepsilon_j}{k_{j+1}}.$$

Suppose, $H_j = 0$ then we have

$$(H_{j-1})' + \varepsilon_{j-2} H_{j-2} k_j = 0. \tag{3.2}$$

So, substituting (2.4) into (3.2) we may get

$$H_{j-1} H_{j-1}^{'} + H_1 H_1^{'} + H_2 H_2^{'} + \cdots + H_{j-2} H_{j-2}^{'} = 0. \tag{3.3}$$

Hence, by virtue of (3.3) an easy calculation gives

$$\sum_{i=1}^{j-1} H_i^2 = c$$

where c =const. This completes the proof of the theorem. □

Corollary 1. *If* $H_1 = 0$ *then* α *is a straight line.*

Corollary 2. *If* H_1 *is constant then* α *is a general helix of rank* 1.

Corollary 3. *If* α *is a general helix of rank* 2 *then* $H_2^{'} + \varepsilon_1 H_1 k_3 = 0$.

4. Curvature Centers of a Frenet Curve

In the present part we give a characterization of the curvature centers $C_r(t)$ of a Frenet curve with respect to its harmonic curvatures H_j.

Definition 4.1. Let $\alpha : I \longrightarrow \mathbb{R}_1^n = \mathbb{L}^n$ be a non-null curve. If $m_1, ..., m_n$ denote the j^{th} e-curvature functions of α and $\{V_1, ..., V_n\}$ the Frenet frame field of α then

$$C_r(t) = \left(\alpha + \sum_{j=2}^{r} a_j m_j V_j\right)(t), \quad a_j = \pm 1 \tag{4.1}$$

is called r^{th} $(a_1, ..., a_r)$*-curvature center of* α at the point $\alpha(t)$ [6].

By the use of (2.1) and (2.2) with (4.1) we obtain the following result.

Theorem 4.2. *Let α be a non-null curve of $\mathbb{L}^n$. Then the 2^{nd} and 3^{th}-curvature centers are given by*

$$C_2(t) = \alpha(t) + a_2\left(\frac{\varepsilon_1\varepsilon_2}{H_1 k_2}\right)V_2(t) \tag{4.2}$$

and

$$C_3(t) = \alpha(t) + a_2\left(\frac{\varepsilon_1\varepsilon_2}{H_1 k_2}\right)V_2(t) + a_3\left(\frac{\varepsilon_1\varepsilon_2\varepsilon_3}{H_1^2 k_2^3}\right)(-H_1^{'}k_2 + H_1 k_2^{'})V_3(t) \tag{4.3}$$

respectively, where $a_j = \pm 1$ and

$$m_2 = \left(\frac{\varepsilon_1\varepsilon_2}{H_1 k_2}\right), \quad m_3 = \left(\frac{\varepsilon_1\varepsilon_2\varepsilon_3}{H_1^2 k_2^3}\right)(-H_1^{'}k_2 + H_1 k_2^{'}). \tag{4.4}$$

Proposition 2. *Let α be a Frenet curve of osculating order 3 in $\mathbb{L}^3$. If the 2^{nd} curvature center of α lies in osculating plane of α, i.e. $C_2(t) = \lambda_1 V_1 + \lambda_3 V_3$ for some functions λ_i then*

$$\begin{aligned} \lambda_1^{'} &= 1 - \varepsilon_2 a_2, \\ \lambda_1 k_1 - \lambda_3 k_3 &= \left(\frac{a_2\varepsilon_1}{k_1}\right)^{'} \\ \lambda_3^{'} &= \frac{a_2\varepsilon_1\varepsilon_2\varepsilon_3}{H_1} \end{aligned} \tag{4.5}$$

where $k_1 \neq 0$ and $k_2 \neq 0$.

Proof : Differentiating $C_2(t) = \lambda_1(t)V_1 + \lambda_3(t)V_3$ with respect to parameter t and using (2.2) we get

$$C_2^{'}(t) = \lambda_1^{'}V_1 + \lambda_1\varepsilon_2 k_1 V_2 + \lambda_3^{'}V_3 + \lambda_3(-\varepsilon_2 k_2 V_2) \tag{4.6}$$

Similarly, differentiating (4.2) we may obtain

$$C_2^{'} = \alpha^{'}(t) + \left(\frac{a_2\varepsilon_1\varepsilon_2}{k_1}\right)^{'}V_2 + \left(\frac{a_2\varepsilon_1\varepsilon_2}{k_1}\right)(-\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_3) \tag{4.7}$$

Hence, comparing (4.6) with (4.7) we get the result. □

As a consequence of Proposition 11 we have the following result.

Corollary 4. *Let α be a helix of osculating order 3 in $\mathbb{L}^3$ such that the 2^{nd} curvature center of α lies in osculating plane of α then*

$$\begin{aligned} \lambda_1 &= 2t, \quad a_2\varepsilon_2 = -1 \\ \lambda_3 &= \lambda_1 H_1, \\ H_1^2 &= \frac{a_2\varepsilon_1\varepsilon_2\varepsilon_3}{2}, \quad \varepsilon_1\varepsilon_2 = -1 \end{aligned} \tag{4.8}$$

where $a_2\varepsilon_2 = \varepsilon_1\varepsilon_2 = -1$.

Proposition 3. *Let α be a Frenet curve of osculating order 3 in $\mathbb{L}^3$. If the 2^{nd} curvature center of α lies in normal plane of α, i.e. $C_2(t) = \lambda_2 V_2 + \lambda_3 V_3$ for some functions λ_i then*

$$\begin{cases} -\lambda_2\varepsilon_1 k_1 = 1 - a_2\varepsilon_2, \\ \lambda_2^{'} - \lambda_3\varepsilon_2 k_2 = a_2\varepsilon_1\varepsilon_2\left(\dfrac{1}{k_1}\right)^{'}, \\ \lambda_2\varepsilon_3 k_2 + \lambda_3^{'} = a_2\varepsilon_1\varepsilon_2\varepsilon_3\left(\dfrac{1}{H_1}\right). \end{cases} \tag{4.9}$$

where $a_2\varepsilon_2 \neq 1, k_1 \neq 0$ and $k_2 \neq 0$.

Proof : Differentiating $C_2(t) = \lambda_2(t)V_2 + \lambda_3(t)V_3$ with respect to t and using (2.2) we may get

$$C_2' = \lambda_1' V_2 + \lambda_1(-\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_3) + \lambda_3' V_3 + \lambda_3(-\varepsilon_2 k_2 V_2). \tag{4.10}$$

Similarly, comparing (4.7) with (4.10) we get the result. □

As a consequence of Proposition 11 we have the following result.

Corollary 5. *There is no helix of osculating order 3 in $\mathbb{L}^3$ whose curvature center lies in normal plane itself.*

*Proof :*Let α be a helix of osculating order 3 in $\mathbb{L}^3$. Then by (4.10) we get

$$\begin{aligned} \lambda_2 &= \frac{a_2\varepsilon_2 - 1}{\varepsilon_1 k_1}, \\ \lambda_3 &= 0. \end{aligned}$$

So $C_2(t)$ can not be written of the form $C_2(t) = \lambda_2(t)V_2 + \lambda_3(t)V_3$. □

Proposition 4. *There is no Frenet curve α in $\mathbb{L}^3$ of osculating order 3 which its 2^{nd} curvature center lies in the tangent plane of α.*

Proof : Let the 2^{nd} curvature center of α lies in tangent plane of α then $C_2(t) = \lambda_1 V_1 + \lambda_2 V_2$. So, differentiating $C_2(t)$ with respect to t and using (2.2) we may get

$$C_2' = \lambda_1' V_1 + \lambda_1(\varepsilon_2 k_1 V_2) + \lambda_2' V_2 + \lambda_2(-\varepsilon_1 k_1 V_1 + \varepsilon_3 k_2 V_3). \tag{4.11}$$

Furthermore, comparing (4.7) with (4.11) we may obtain

$$\begin{cases} \lambda_1' - \lambda_2\varepsilon_1 k_1 = 1 - a_2 m_2 \varepsilon_1 k_1, \\ \lambda_1\varepsilon_2 k_1 + \lambda_2' = (a_2 m_2)', \\ \lambda_2 = a_2 m_2. \end{cases} \tag{4.12}$$

But, from 2^{nd} and 3^{rd} equations we may get $\lambda_1 = 0$. Substituting $\lambda_1 = 0$ into (4.12) one can show that the resultant system of differential equation is not consistent.

ÖZET: Bu çalışmada, L^n Lorentz uzayındaki Frenet eğrilerinin harmonik eğrilikleri H_j, $(1 \leq j \leq r)$ ele alınmıştır. Bununla beraber L^n deki Frenet eğrilerinin $r-$inci eğrilik merkezleri $C_r(t)$ lerin bir karakterizasyonu verilmiştir. Eğrilerin helis olması durum ayrıca incelenmiştir.

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