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TRI-ADDITIVE MAPS AND PERMUTING TRI-DERIVATIONS

HASRET YAZARLI MEHMET ALI OZTÜRK AND YOUNG BAE JUN

ABSTRACT. In [7] ,Öztürk has proved some results concerning permuting tri-derivations on prime and semi-prime rings.We study permuting tri-additive maps with trace which is skew-commuting or skew-centralizing on s-unital rings and also we obtain a few results on trace of permuting tri-derivations in prime rings.

1. INTRODUCTION

The concept of a symmetric bi-derivation has been introduced by Maksa in [4] (see also [5]).In recent years,many mathematicans studied on commutativity of prime and semi-prime rings admitting suitably-constrained symmetric bi-derivations.

In [7] ,Öztürk introduced the notion of a permuting tri-derivations in rings and proved some results.The aim of this paper is to study some properties of a permuting tri-derivations of a s-unital rings and a prime rings.

2. PRELIMINARIES

Throughout this paper all rings R will be associative and the center (resp.extended centroid) of a ring will be denoted by Z (resp. C).

A mapping $f : R \to R$ is called commuting if $[x, f(x)] = 0$, for all $x \in R$. Similarly f is called skew-commuting (resp. skew-centralizing) on R if $xf(x) + f(x)x = 0$ (resp. $xf(x) + f(x)x \in Z$) for all $x \in R$.

A mapping $D(.,.) : R \times R \to R$ is called symmetric if $D(x,y) = D(y,x)$ for all $x, y \in R$. In follows, denote by $D(.,.)$ a symmetric mapping from $R \times R$ to R without otherwise specified. A mapping $d : R \to R$ is called the trace of $D(.,.)$ if $d(x) = D(x,x)$ for all $x \in R$. It is obvious that if $D(.,.)$ is bi-additive (i.e. additive in both arguments), then the trace d of $D(.,.)$ satisfies the identity $d(x+y) = d(x) + d(y) + 2D(x,y)$ for all $x, y \in R$. If $D(.,.)$ is bi-additive and satisfies the identity $D(xy,z) = D(x,z)y + zD(y,z)$ for all $x, y, z \in R$, we say that $D(.,.)$ is a symmetric bi-derivation.

A mapping $D(.,.,.) : R \times R \times R \to R$ is called tri-additive if

$$D(x+w,y,z) = D(x,y,z) + D(w,y,z)$$
$$D(x,y+w,z) = D(x,y,z) + D(x,w,z)$$
$$D(x,y,z+w) = D(x,y,z) + D(x,y,w)$$

for all $x, y, z, w \in R$. A tri-additive mapping $D(.,.,.) : R \times R \times R \to R$ is

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called permuting tri-additive if $D(x,y,z) = D(x,z,y) = D(z,x,y) = D(z,y,x) = D(y,z,x) = D(y,x,z)$ holds for all $x,y,z,w \in R$. A mapping $d: R\times R \to R$ defined by $d(x) = D(x,x,x)$ is called trace of $D(.,.,.)$, where $D(.,.,.): R\times R\times R \to R$ is a permuting tri-additive mapping.It is obvious that, if $D(.,.,.): R\times R\times R \to R$ is a permuting tri-additive mapping then the trace of $D(.,.,.)$ satisfies the relation $d(x+y) = d(x) + d(y) + 3D(x,x,y) + 3D(x,y,y)$ for all $x,y \in R$.

A permuting tri-additive mapping $D(.,.,.): R\times R\times R \to R$ is called permuting tri-derivation if $D(xw,y,z) = D(x,y,z)w + zD(x,y,w)$ are fulfilled for all $x,y,w,z \in R$. Then relation $D(x,yw,z) = D(x,y,z)w + yD(x,w,z)$ and $D(x,y,zw) = D(x,y,z)w + zD(x,y,w)$ are fulfilled for all $x,y,z,w \in R$. The mapping $d: R \to R$ defined by $d(x) = D(x,x,x)$ is an odd-function.

R is called a left s-unital (resp. s-unital) ring if for each $x \in R$ there holds $x \in Rx$ (resp.$x \in Rx \cap xR$). If R is a left s-unital (resp.s-unital) ring then for any finite subset F of R there exists an element e in R such that $ex = x$ (resp. $ex = xe = x$) for all $x \in F$ (see [9] ,*Theorem* 1 and [6], *Lemma* 1). Such an element e will be called a left pseudo-identity (resp. pseudo-identity) of F.

Throughout this paper e will be a left pseudo-identity of

$$E := \{x, d(x), d(e), D(x,e,e), D(x,x,e)\} \subset R$$

where x is an arbitrary element of R.

Remark 2.1. Let R be a ring and $D(.,.,.)$ be a permuting tri-derivation of R. In this case, for any fixed $a \in R$ and for all $x,y \in R$, a mapping $D_1(.,.): R\times R \to R$ defined by $D_1(x,y) = D(a,x,y)$ and a mapping $d_2: R \to R$ defined by $d_2(x) = D(a,a,x)$ is a symmetric bi-derivation (in this meaning, permuting 2-derivation is a symmetric bi-derivation) and is a derivation, respectively.

3. Tri-additive Maps with Skew-Commuting and Skew-Centralizing Trace

Theorem 3.1. *Let R be a $2,3-torsion\ free$ left s-unital ring . Let $D(.,.,.)$ be a permuting tri-additive mapping of R with the trace d. If d is skew-commuting on R, then $D = 0$.*

Proof. Let e be pseudo-identity of $E \subset R$ and x be an arbitrary element of R. Using the fact that d is skew-commuting on R, we get □

$$xd(x) + d(x)x = 0, \quad x \in R \tag{3.1}$$

Writing e for x in (3.1)

$$ed(e) + d(e)e = d(e) + d(e)e = 0 \tag{3.2}$$

and right multiplying by e gives $2d(e)e = 0 = d(e)e$. Therefore by (3.2), $d(e) = 0$. Now writing x for $x+e$ in (3.1) and using the fact that D is permuting tri-additive, we get,

$$3xD(x,x,e) + 3xD(x,e,e) + d(x) + 3D(x,x,e) + 3D(x,e,e) + d(x)e$$

$$+3D(x,x,e)x + 3D(x,x,e)e + 3D(x,e,e)x + 3D(x,e,e)e = 0, \quad x \in R \tag{3.3}$$

Replacing $-x$ for x in (3.3) and subtracting (3.3) with the results and using the fact that R is $2,3-torsion\ free$ and d is an odd function, we get,

$$xD(x,e,e)+D(x,x,e)+D(x,x,e)e+D(x,e,e)e=0,\quad x\in R \tag{3.4}$$

Now writing x for $x+e$ in (3.4). Using the fact that D is permuting tri-additive, we get,

$$xD(x,e,e)+3D(x,e,e)+2D(x,x,e)e+3D(x,e,e)e=0,\quad x\in R \tag{3.5}$$

Replacing $-x$ by x in (3.5) and subtracting (3.5) with the result and using the fact that R is $2,3-torsion\ free$, we get,

$$D(x,e,e)+D(x,e,e)e=0,\quad x\in R \tag{3.6}$$

Right multiplication of (3.6) by e gives $2D(x,e,e)=D(x,e,e)$, and so, by (3.6), we get $D(x,e,e)=0$, $x\in R$. Hence, we get

$d(x+e)=d(x)+d(e)+3D(x,x,e)+3D(x,e,e)\quad =d(x)+3D(x,x,e)\ ,\ \ x\in R$

Using the fact that d is skeww-commuting on R, we have $(x+e)d(x+e)+d\,(x+e)\,(x+e)=0$. The last relation now becomes

$$d(x)+3xD(x,x,e)+3D(x,x,e)+d(x)e+3D(x,x,e)x+3D(x,x,e)e\ =0,\quad x\in R \tag{3.7}$$

Replacing $-x$ for x in (3.7) and subtracting (3.7) with the results and using the fact that R is $2,3-torsion\ free$, we get,

$$D(x,x,e)+D(x,x,e)e=0,\ x\in R \tag{3.8}$$

Right multiplication of (3.8) by e gives $2D(x,x,e)e=0=D(x,x,e)$ and so, by (3.8), we get, $D(x,x,e)=0$, $x\in R$. Therefore, by (3.7)

$$d(x)+d(x)e=0,\quad x\in R \tag{3.9}$$

Right multiplication of (3.9) by e gives $2d(x)e=0=d(x)$ and hence the relation (3.9) implies $d(x)=0$, $x\in R$. Thus $D=0$.

Theorem 3.2. *Let R be a $2,3-torsion\ free$ left s-unital ring. Let $D(.,.,.)$ be a permuting tri-additive mapping of R with the trace d. If d is skew-centralizing on R, then d is commuting on R.*

Proof. Let e be pseudo-identity of $E\subset R$ and x be an arbitrary elelment of R. Using the fact that d is skew-centralizing on R, we get,

$$xd(x)+d(x)x\in Z,\ x\in R \tag{3.10}$$

Writing e for x in (3.10), we get,

$$ed(e)+d(e)e=d(e)+d(e)e\in Z \tag{3.11}$$

Commuting with e gives $d(e)=d(e)e$; and by (3.11) $2d(e)\in Z$, thus $d(e)\in Z$. Writing x for $x+e$ in (3.10) and using the fact that D is permuting tri-addittive, we get,

$$2xd(e)+3xD(x,x,e)+3xD(x,e,e)+d(x)+3D(x,x,e)+3D(x,e,e)+$$

$$d(x)e+3D(x,x,e)x+3D(x,x,e)e+3D(x,e,e)x+3D(x,e,e)e\in Z \quad x\in R \tag{3.12}$$

Again writing x for $x+e$ in (3.12) , since D is permuting tri-additive, we get,

$$15xd(e)+3xD(x,x,e)+9xD(x,e,e)+9D(x,x,e)+21D(x,e,e)+d(x)$$

$$+d(x)e+9D(x,x,e)e+21D(x,e,e)e+3D(x,x,e)x+9D(x,e,e)x\in Z,\ x\in R \tag{3.13}$$

Replacing $-x$ by x in (3.13) and subtracting (3.13) with results and using the fact that d is an odd function and R is $2,3-torsion\ free$, we get,

$$xD(x,e,e)+D(x,x,e)+D(x,e,e)x\in Z,\ x\in R \tag{3.14}$$

We use (3.14) in (3.12)

$$2xd(e)+3xD(x,x,e)+d(x)+3D(x,x,e)+3D(x,e,e)+d(x)e+$$

$$3D(x,x,e)x+3D(x,e,e)e\in Z,\ x\in R \tag{3.15}$$

Replacing $-x$ by x in (3.15) and subtracting (3.15) with results, and using the fact that R is $2,3-torsion\ free$ and d is an odd function, we get $D(x,x,e)\in Z$, $x\in R$. We use (3.14) in (3.13)

$$15xd(e)+3xD(x,x,e)+21D(x,e,e)+d(x)+d(x)e+21D(x,e,e)e$$

$$+3D(x,x,e)x\in Z,\ x\in R \tag{3.16}$$

Commuting with e gives;

$$21\left[D(x,x,e),e\right]+\left[d(x),e\right]=0, x\in R \tag{3.17}$$

Writing x for $x+e$ in (3.17). Using the fact that D is permuting tri-additive and R is $3-torsion\ free$, we get,

$$\left[D(x,x,e),e\right]=0,\ x\in R \tag{3.18}$$

We use last relation in (3.17), $[d(x), e] = 0$ and so, by (3.17), we get $d(x) = d(x)e$, $x \in R$. By (3.15), we get,

$$2xd(e) + 6xD(x, x, e) + 2d(x) + 6D(x, e, e) \in Z, \ x \in R \qquad (3.19)$$

Writing x for $x + e$ in (3.19) and using the fact that D is permuting tri-additive and R is $3 - torsion\ free$, we get,

$$2xD(x, e, e) + xd(e) + 3D(x, e, e) \in Z, x \in R \qquad (3.20)$$

Replacing $-x$ by x in (3.20)and subtracting (3.20) with result and using the fact that d is an odd function and R is $2 - torsion\ free$, we get,

$$xd(x) + 3D(x, e, e) \in Z, \ x \in R \qquad (3.21)$$

We use last relation in (3.19)

$$6xD(x, x, e) + 2d(x) \in Z, \ x \in R \qquad (3.22)$$

Commuting with e gives $[d(x), x] = 0$, $x \in R$, since R is $2 - torsion\ free$. Thus d is commuting on R □

For $n \geq 2$, a mapping $f : R \to R$ is called n-skew commuting (resp. n-skew centralizing) on R if $x^n f(x) + f(x)x^n = 0$ (resp. $x^n f(x) + f(x)x^n \in Z$) *for all* $x \in R$.

Now we extend the result n-skew-commuting mappings (*Theorem* 1) to n-skew commuting ones.

Theorem 3.3. *Let $n \geq 2$, let R be an $n! - torsion\ free$ left s-unital ring with $charR \neq 3$. Let $D(.,.,.)$ be a permuting tri-additive mapping of R with the trace d. If d is n-skew commuting on R., then $D = 0$.*

Proof. Let e be pseudo-idendity of $E \subset R$ and x be an arbitrary element of R.. Suppose that

$$x^n d(x) + d(x)x^n = 0, \ x \in R \qquad (3.23)$$

Using similar approach as in the proof of *Theorem* 1, we get $d(e) = 0$. Replacing $x + e$ by x in (3.23), we get

$$(x + e)^n d(x + e) + d(x + e)(x + e)^n = 0, \ \ x \in R \qquad (3.24)$$

This can be written in the from $\sum_{i=1}^{n} s_i(x, e)d(x + e) + d(x + e)s_i(x, e) = 0$, where $s_i(x, e)$ is the sum of terms involving i factors of e in the exponsion of $(x + e)^n$. Replacing e by $2e$, $3e, ...,$ ne in turn, and expressing the resulting system of n homogeneous equations, we see that the coefficient matrix of the system is a *van der Monde matrix*

$$\begin{pmatrix} 1 & 1 & . & . & . & 1 \\ 2 & 2^2 & . & . & . & 2^n \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ n & n^2 & . & . & . & n^n \end{pmatrix}$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than n, and since R is $n!-torsion\ free$, it follows immediately that $s_i(x,e)d(x+e)+d(x+e)s_i(x,e)=0 \quad for\ all\ i=1,2,...,n$

In particular,

$s_n(x,e)d(x+e)+d(x+e)s_n(x,e)=ed(x+e)+d(x+e)e=d(x+e)+d(x+e)e$, $x\in R$ and as in the proof of *Theorem* 1 we get $d(x+e)=0$, $x\in R$. On the other hand, $d(x+e)=d(x)+d(e)+3D(x,x,e)+3D(x,e,e)$ and $d(e)=0$ and hence we get

$$d(x)+3D(x,x,e)+3D(x,e,e)=0,\ x\in R \tag{3.25}$$

Replacing $-x$ by x in (3.25) and subtracting (3.25) with the result, we get $D(x,x,e)=0$, $x\in R$. By (3.25),

$$d(x)+3D(x,e,e)=0, x\in R \tag{3.26}$$

Writing x for $x+e$ in (2.26), and using the fact that D is permuting tri-additive and R is $3-torsion\ free$, we get $D(x,e,e)=0$, $x\in R$. By (3.25), $d(x)=0$, $x\in R$. Thus $D=0$. □

4. Permuting Tri-derivations in Prime Rings

Lemma 4.1. *Let R be prime ring with $charR\neq 2,3$. Let $D(.,.,.)$ be permuting tri-derivation of R with the trace d. If*

$$ad(x)=0,\ x\in R \tag{4.1}$$

where a is a fixed element of R, then $a=0$ or $D=0$.

Proof. Let $x,y\in R$. Writing x for $x+y$ in (4.1), we get

$$aD(x,x,y)+aD(x,y,y)=0 \tag{4.2}$$

Replacing $-x$ by x in (4.2) and subtracting with result, we get,

$$aD(x,y,y)=0 \tag{4.3}$$

Writing x for xy in (4.3). Since D is permuting tri-derivation, we get, $axd(y)=0$ $for\ all\ x\in R$. Since R is a prime ring, we get, $a=0$ or $d(y)=0\ for\ all\ x\in R$. Thus $a=0$ or $D=0$. □

Lemma 4.2. *Let R be a prime ring with $charR \neq 2,3$ and let d_1 and d_2 be traces of permuting tri-derivations $D_1(.,.,.)$ and $D_2(.,.,.)$, respectively. If the identity*

$$d_1(x)d_2(y) = d_2(x)d_1(y) \qquad \text{for all } x, y \in R \tag{4.4}$$

holds and $d_1 \neq 0$, then there exist $\lambda \in C$ such that $d_2(x) = \lambda d_1(x)$.

Proof. Let $x, y, z \in R$. Writing y for $y+z$ in (4.4), we get

$$d_1(x)D_2(y,y,z) + d_1(x)D_2(y,z,z) = d_2(x)D_1(y,y,z) + d_2(x)D_1(y,z,z) \tag{4.5}$$

since $D_1$and D_2 are permuting tri-derivations and $charR \neq 3$.

Again writing y for $y+z$ in (4.5), we get,

$$d_1(x)D_2(y,z,z) = d_2(x)D_1(y,z,z) \tag{4.6}$$

since D_1 and D_2 are permuting tri-derivations and $charR \neq 2$.

Writing y for yz in (4.6), we get,

$$d_1(x)yd_2(z) = d_2(x)yd_1(z) \tag{4.7}$$

Replacing z by x in (4.7), we get,

$$d_1(x)yd_2(x) = d_2(x)yd_1(x) \tag{4.8}$$

Thus if $d_1(x) \neq 0$, then by (4.8) and [1, *Corollary to Lemma* 1.3.2] we have $d_2(x) = \lambda(x)d_1(x)$ for some $\lambda(x) \in C$. Hence if $d_1(x) \neq 0$ and $d_1(z) \neq 0$, then $(\lambda(z) - \lambda(x))d_1(x)yd_1(z) = 0$ by (4.7). Since R is prime, it follows from *Lemma* 4 that $\lambda(x) = \lambda(z)$. This shows that there exist $\lambda \in C$ such that $d_2(x) = \lambda d_1(x)$ under the condition $d_1(x) \neq 0$. On the other hand, assume that $d_1(x) = 0$. Since $d_1 \neq 0$ and R is prime, it follows from (4.7) that $d_2(x) = 0$ as well. Thus $d_2(x) = \lambda d_1(x)$. This completes the proof. □

Theorem 4.3. *Let R a prime ring with $charR \neq 2,3$ and let $d_1(\neq 0)$, d_2, d_3 and $d_4(\neq 0)$ be trace of permuting tri-derivations $D_1(.,.,.)$, $D_2(.,.,.)$, $D_3(.,.,.)$ and $D_4(.,.,.)$ respectively.If the identity*

$$d_1(x)d_2(y) = d_3(x)d_4(y) \tag{4.9}$$

holds for all $x, y \in R.$, then there exists $\lambda \in C$ such that $d_2(x) = \lambda d_4(x)$ and $d_3(x) = \lambda d_1(x)$.

Proof. Let $x, y, z \in R$. Writing y for $y+z$ in (4.9), we get, $d_1(x)D_2(y,y,z) + d_1(x)D_2(y,z,z) = d_3(x)D_4(y,y,z) + d_3(x)D_4(y,z,z)$, since D_2 and D_4 are permuting tri-derivations and $charR \neq 3$.

Again writing y for $y+z$ in last relation, we get,

$$d_1(x)D_2(y,z,z) = d_3(x)D_4(y,z,z) \tag{4.10}$$

sinceD_2 and D_4 are permuting tri-derivations and $charR \neq 2$.

Writing y for yz in (4.10), we get,

$$d_1(x)yd_2(z) = d_3(x)yd_4(z) \tag{4.11}$$

Replacing y by $yd_4(w)$ in (4.11), we get, $d_1(x)yd_4(w)d_2(z) = d_3(x)yd_4(w)d_4(z) = d_1(x)yd_2(w)d_4(z)$, so that $d_1(x)y(d_4(w)d_2(z) - d_2(w)d_4(z) = 0$. Since $d_1 \neq 0$ and R is prime, it follows that $d_4(w)d_2(z) = d_2(w)d_4(z)$. Applying *Lemma* 4, there exist $\lambda \in C$ such that $d_2(z) = \lambda d_4(z)$, which implies (4.11) that $(\lambda d_1(x) - d_3(x))yd_4(z) = 0$ so that $d_3(x) = \lambda d_1(x)$. This completes the proof. □

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ÖZET: Öztürk [7] de asal ve yarı-asal halkalar üzerinde permuting tri-türevlerle ilgili bazı sonuçlar ispatladı. Biz s-unital halkalar üzerinde çarpık-kommuting veya çarpık-merkezleyen izli tri-toplamsal dönüşümleri çalıştık ve ayrıca asal halkalarda permuting tri-türevlerin izleriyle ilgili bazı sonuçlar elde ettik.

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Current address: Hasret Yazarli. Department of Mathematics, Faculty of Art and Sciences-Cumhuriyet University 58140 Sivas-TURKEY

E-mail address: `h.yazarli@mynet.com`.

Current address: Mehmet Ali Oztürk. Department of Mathematics Faculty of Art and Sciences Cumhuriyet University 58140 Sivas-TURKEY

E-mail address: `trmaozturk@cumhuriyet.edu.tr`

Current address: Young Bae Jun. Department of Mathematics Education Gyeongsang National University Chinju 660-701, KOREA

E-mail address: `ybjun@nongae.gsnu.ac.kr`

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MEROMORPHIC CLOSE-TO-CONVEX FUNCTIONS WITH ALTERNATING COEFFICIENTS

M. K. AOUF, F. M. AL-OBOUDI AND M. M. HADAIN

ABSTRACT. Coefficient inequalities and distortion theorems are obtained for certain subclass of meromorphically close -to- convex functions with alternating coefficients. Further class preserving integral operators are obtained.

1. INTRODUCTION

Let $\sum$ denotes the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \qquad (1.1)$$

which are regular in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. Define

$$\begin{aligned} D^0 f(z) &= f(z); \\ D^1 f(z) &= \frac{1}{z} + 3a_1 z + 4a_2 z^2 + \cdots \\ &= \frac{\left(z^2 f(z)\right)'}{z}, \\ D^2 f(z) &= D\left(D^1 f(z)\right); \end{aligned}$$

and for $n = 1, 2, 3, ...$

$$\begin{aligned} D^n f(z) &= D\left(D^{n-1} f(z)\right) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} (k+2)^n a_k z^k \\ &= \frac{\left(z^2 D^{n-1} f(z)\right)'}{z}. \end{aligned}$$

Let $K_n(\alpha, \beta, \gamma)$ denote the class of functions $f(z)$ in $\sum$ satisfying the condition

$$\left| \frac{z^2 (D^n f(z))' + 1}{(2\gamma - 1) z^2 (D^n f(z))' + (2\alpha\gamma - 1)} \right| < \beta \qquad n \in N_o = \{0, 1, 2, \cdots\} \qquad (1.2)$$

for some $\alpha\,(0 \leq \alpha < 1), \beta\,(0 < \beta \leq 1), \gamma\,(\frac{1}{2} \leq \gamma \leq 1)$, and for all $z \in U^*$.We note that $K_0(\alpha, \beta, \gamma) = \sum(\alpha, \beta, \gamma)$ (Cho, Lee and Owa [3]).Let σ_A be the subclass of

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$\sum$ which consisting of functions of the form

$$\begin{aligned} f(z) &= \frac{1}{z} + a_1 z - a_2 z^2 + a_3 z^3 - \cdots, \qquad a_k \geq 0 \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k, \qquad a_k \geq 0 \end{aligned} \tag{1.3}$$

and let $\sigma^*_{A,n}(\alpha,\beta,\gamma) = K_n(\alpha,\beta,\gamma) \cap \sigma_A$.

In this paper, coefficient inequalities and distortion theorems for the class $\sigma^*_{A,n}(\alpha,\beta,\gamma)$ are determine. Techniques used are similar to these of Silverman [4], Uralegaddi and Ganigi [5], Aouf and Darwish [1] and Aouf and Hossen [2]. Finally, the class preserving integral operators of the form

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\, dt \qquad (c > 0) \tag{1.4}$$

is considered.

2. Coefficient inequalities

Theorem 2.1. *Let* $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k$. *If*

$$\sum_{k=1}^{\infty} (k+2)^n k (1 + 2\beta\gamma - \beta) |a_k| \leq 2\beta\gamma (1-\alpha), \tag{2.1}$$

then $f(z) \in \sigma^*_{A,n}(\alpha,\beta,\gamma)$.

Proof Suppose (2.1) holds for all and missible values of α, β, γ and n. It suffices to show that

$$\left| \frac{z^2 (D^n f(z))' + 1}{(2\gamma - 1) z^2 (D^n f(z))' + (2\alpha\gamma - 1)} \right| < \beta \tag{2.2}$$

for $|z| < 1$. We have

$$\begin{aligned} &\left| \frac{z^2 (D^n f(z))' + 1}{(2\gamma - 1) z^2 (D^n f(z))' + (2\alpha\gamma - 1)} \right| \\ = &\left| \frac{\sum_{k=1}^{\infty} k (k+2)^n a_k z^{k+1}}{2\gamma (1-\alpha) - \sum_{k=1}^{\infty} (2\gamma - 1) k (k+2)^n a_k z^{k+1}} \right| \\ \leq &\frac{\sum_{k=1}^{\infty} k (k+2)^n |a_k|}{2\gamma (1-\alpha) - \sum_{k=1}^{\infty} (2\gamma - 1) k (k+2) |a_k|}. \end{aligned}$$

The last expression is bounded above by β,provided

$$\sum_{k=1}^{\infty} k (k+2)^n |a_k| \leq \beta \left\{ 2\gamma (1-\alpha) - \sum_{k=1}^{\infty} (2\gamma - 1) k (k+2)^n |a_k| \right\}$$

which is equivalent to

$$\sum_{k=1}^{\infty} k(k+2)^n (1+2\beta\gamma-\beta)|a_k| \le 2\beta\gamma(1-\alpha) \tag{2.3}$$

which is true by hypothesis.

For functions in $\sigma^*_{A,n}(\alpha,\beta,\gamma)$ the converse of the above theorem is also true.

Theorem 2.2. *A function $f(z)$ in σ_A is in $\sigma^*_{A,n}(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{k=1}^{\infty} k(k+2)^n (1+2\beta\gamma-\gamma) a_k \le 2\beta\gamma(1-\alpha)\cdot$$

Proof. In view of Theorem 1 it suffices to show that the only if part. Let us assume that $f(z) = \frac{1}{z} + \sum\limits_{k=1}^{\infty} (-1)^{k-1} a_k z^k$ $(a_k \ge 0)$ is in $\sigma^*_{A,n}(\alpha,\beta,\gamma)$. Then

$$\left|\frac{z^2(D^n f(z))' + 1}{(2\gamma-1)z^2(D^n f(z))' + (2\alpha\gamma-1)}\right|$$

$$= \left|\frac{\sum\limits_{k=1}^{\infty}(-1)^{k-1} k(k+2)^n a_k z^{k+1}}{2\gamma(1-\alpha) - \sum\limits_{k=1}^{\infty}(-1)^{k-1}(2\gamma-1)k(k+2)^n a_k z^{k+1}}\right| < \beta\cdot$$

for all $z \in U^*$. Using the fact that $\operatorname{Re} z \le |z|$ *for all z, it follows that*

$$\operatorname{Re}\left\{\frac{\sum\limits_{k=1}^{\infty}(-1)^{k-1} k(k+2)^n a_k z^{k+1}}{2\gamma(1-\alpha) - \sum\limits_{k=1}^{\infty}(-1)^{k-1}(2\gamma-1)k(k+2)^n a_k z^{k+1}}\right\} < \beta \, (z \in U^*)\cdot \tag{2.4}$$

Now choose values of z on the real axis so that $z^2(D^n f(z))'$ is real. Upon clearing the denominator in (2.3) and letting $z \to -1$ through real values, we obtain

$$\sum_{k=1}^{\infty} k(k+2)^n a_k \le \beta\left\{2\gamma(1-\alpha) - \sum_{k=1}^{\infty}(2\gamma-1)k(k+2)^n a_k\right\}$$

which is equivalent to

$$\sum_{k=1}^{\infty} k(k+2)^n (1+2\beta\gamma-\beta) a_k \le 2\beta\gamma(1-\alpha)\cdot$$

This completes the proof of Theorem 2.

Corollary 1. *Let the function $f(z)$ defined by (1.3) be in the class $\sigma^*_{A,n}(\alpha,\beta,\gamma)$, then*

$$a_k \le \frac{2\beta\gamma(1-\alpha)}{k(k+2)^n(1+2\beta\gamma-\beta)} \qquad (k \ge 1).$$

Equality holds for the functions of the from

$$f_k(z) = \frac{1}{z} + (-1)^{k-1}\frac{2\beta\gamma(1-\alpha)}{k(k+2)^n(1+2\beta\gamma-\beta)} z^k.$$

3. Distortion theorems

Theorem 3.1. *Let the function $f(z)$ defined by (1.3) in the class $\sigma^*_{A,n}(\alpha,\beta,\gamma)$, then for $0<|z|=r<1$,*

$$\frac{1}{r}-\frac{2\beta\gamma(1-\alpha)}{3^n(1+2\beta\gamma-\beta)}r\le|f(z)|\le\frac{1}{r}+\frac{2\beta\gamma(1-\alpha)}{3^n(1+2\beta\gamma-\beta)}r \tag{3.1}$$

with equality for the function

$$f(z)=\frac{1}{z}+\frac{2\beta\gamma(1-\alpha)}{3^n(1+2\beta\gamma-\beta)}z \quad at\ z=r,ir. \tag{3.2}$$

Proof. *Suppose $f(z)$ is in the class $\sigma^*_{A,n}(\alpha,\beta,\gamma)$. In view of Theorem 2, we have*

$$3^n(1+2\beta\gamma-\beta)\sum_{k=1}^{\infty}a_k\le\sum_{k=1}^{\infty}k(k+2)^n(1+2\beta\gamma-\beta)a_k\le2\beta\gamma(1-\alpha)$$

which evidently yields

$$\sum_{k=1}^{\infty}a_k\le\frac{2\beta\gamma(1-\alpha)}{3^n(1+2\beta\gamma-\beta)}. \tag{3.3}$$

Consequently, we obtain

$$|f(z)|\le\frac{1}{r}+\sum_{k=1}^{\infty}a_kr^k\le\frac{1}{r}+r\sum_{k=1}^{\infty}a_k\le\frac{1}{r}+\frac{2\beta\gamma(1-\alpha)}{3^n(1+2\beta\gamma-\beta)}r,$$

by (3.3). This gives the right-hand inequality of (3.1).Also

$$|f(z)|\ge\frac{1}{r}-\sum_{k=1}^{\infty}a_kr^k\ge\frac{1}{r}-r\sum_{k=1}^{\infty}a_k\ge\frac{1}{r}-\frac{2\beta\gamma(1-\alpha)}{3^n(1+2\beta\gamma-\beta)}r,$$

by (3.3),which gives the left-hand side of (3.1). It can be easily seen that the function $f(z)$ defined by (3.2) is extremal for the theorem.

Theorem 3.2. *Let the function $f(z)$ defined by (1.3) be in the class $\sigma^*_{A,n}(\alpha,\beta,\gamma)$, then for $0<|z|=r<1$,*

$$\frac{1}{r^2}-\frac{2\beta\gamma(1-\alpha)}{3^n(1+2\beta\gamma-\beta)}\le|f'(z)|\le\frac{1}{r^2}+\frac{2\beta\gamma(1-\alpha)}{3^n(1+2\beta\gamma-\beta)}. \tag{3.4}$$

The result is sharp, the extremal function being of the form (3.2).

Proof. From Theorem 2, we have

$$3^n(1+2\beta\gamma-\beta)\sum_{k=1}^{\infty}ka_k\le\sum_{k=1}^{\infty}k(k+2)^n(1+2\beta\gamma-\beta)a_k\le2\beta\gamma(1-\alpha)$$

which evidently yields

$$\sum_{k=1}^{\infty}ka_k\le\frac{2\beta\gamma(1-\alpha)}{3^n(1+2\beta\gamma-\beta)}. \tag{3.5}$$

Consequently, we obtain

$$|f'(z)|\le\frac{1}{r^2}+\sum_{k=1}^{\infty}ka_kr^{k-1}\le\frac{1}{r^2}+\sum_{k=1}^{\infty}ka_k\le\frac{1}{r^2}+\frac{2\beta\gamma(1-\alpha)}{3^n(1+2\beta\gamma-\beta)}.$$

Also

$$|f'(z)| \geq \frac{1}{r^2} - \sum_{k=1}^{\infty} k a_k r^{k-1} \geq \frac{1}{r^2} - \sum_{k=1}^{\infty} k a_k \geq \frac{1}{r^2} - \frac{2\beta\gamma(1-\alpha)}{3^n(1+2\beta\gamma-\beta)}.$$

This completes the proof of Theorem 4.

Putting $n = 0$ in Theorem 4, we get:

Corollary 2. *Let the function $f(z)$ defined by (1.3) be in the class $\sigma^*_{A,n}(\alpha,\beta,\gamma) = \sigma^*_{A,0}(\alpha,\beta,\gamma)$, then for $0 < |z| = r < 1$,*

$$\frac{1}{r^2} - \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)}.$$

The result is sharp.

Putting $n = 0$ and $\beta = \gamma = 1$ in Theorem 6, we get:

Corollary 3. *Let the function $f(z)$ defined by (1.3) be in the class $\sigma^*_{A,0}(\alpha,1,1) = \sigma^*_A(\alpha)$, then for $0 < |z| = r < 1$,*

$$\frac{1}{r^2} - (1-\alpha) \leq |f'(z)| \leq \frac{1}{r^2} + (1-\alpha).$$

This result is sharp.

4. Class preserving integral operators

In this section we consider the claa preserving integral operators of the form (1.4).

Theorem 4.1. *Let the function $f(z)$ defined by (1.3) be in the class $\sigma^*_{A,n}(\alpha,\beta,\gamma)$, then*

$$F(z) = cz^{-c-1}\int_0^z t^c f(t)\,dt = \frac{1}{z} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{c}{c+k+1}a_k z^k, \quad c > 0$$

belongs to the class $\sigma^*_{A,n}(\lambda(\alpha,c),\beta,\gamma)$, where

$$\lambda(\alpha,c) = 1 - \frac{c(1-\alpha)}{(c+2)}. \tag{4.1}$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{3^n(1+2\beta\gamma-\beta)}z.$$

Proof. Suppose $f(z) \in \sigma^*_{A,n}(\alpha,\beta,\gamma)$,then

$$\sum_{k=1}^{\infty} k(k+2)^n(1+2\beta\gamma-\beta)a_k \leq 2\beta\gamma(1-\alpha).$$

In view of Theorem 2, we shall find the largest value of λ for which

$$\sum_{k=1}^{\infty} \frac{k\,(k+2)^n\,(1+2\beta\gamma-\beta)}{2\beta\gamma\,(1-\lambda)} \cdot \frac{c}{c+k+1} a_k \leq 1.$$

It suffices to find the range of λ for which

$$\frac{ck\,(k+2)^n\,(1+2\beta\gamma-\beta)}{2\beta\gamma\,(1-\lambda)\,(c+k+1)} \leq \frac{k\,(k+2)^n\,(1+2\beta\gamma-\beta)}{2\beta\gamma\,(1-\alpha)}.$$

Solving the above inequality for λ we obtain

$$\lambda \leq 1 - \frac{c\,(1-\alpha)}{(c+k+1)}.$$

Since

$$A\,(k) = 1 - \frac{c\,(1-\alpha)}{(c+k+1)}, \tag{4.2}$$

is an increasing function of $k\,(k \geq 1)$, letting $k=1$ in (4.2), we obtain

$$\lambda = A\,(1) = 1 - \frac{c\,(1-\alpha)}{(c+2)}$$

and the theorem follows at once.

ÖZET: Bu çalışmada, alterne katsayılı konvekse - yakın meromorfik fonksiyonların belirli bir altsınıfı için katsayı eşitsizlikleri ve bükülme teoremleri elde edilmiştir. Ayrıca sınıf koruyan integraller incelenmiştir.

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Current address: M. K. Aouf. Dept. of Mathematics, Girls collage of education, Science Section, Jeddah, SAUDI ARABIA, M. AL-Oboudi. Dept. of Mathematics, Girls collage of education, Science Section, Riyadh, SAUDI ARABIA, M. M. Hadain. Dept. of Mathematics, Girls collage of education, Science Section, Abha, SAUDI ARABIA