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DECOMPOSITIONS OF *I*-CONTINUITY AND CONTINUITY

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ABSTRACT We introduce the notions of r_I-open sets and m_I-open sets and by using these sets we obtain decompositions of *I*-open sets. Furthermore, we introduce the notion of weakly *I*-locally-closed sets and obtain decompositions of open sets. Finally, we obtain decompositions of *I*-continuity and continuity.

KEY WORDS: Topological ideal, *I*-open sets, decomposition of *I*-continuity and decomposition of continuity.

1.INTRODUCTION

Throught the present paper, spaces always mean topological spaces on which no separation property is assumed unless explicitly stated. In a topological space (X,τ), the closure and the interior of any subset A of X will be denoted by $Cl(A)$ and $Int(A)$, respectively. An ideal is defined as a nonempty collection I of subsets of X satisfying the following two conditions: (1) If $A \in I$ and $B \subset A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then $(A \cup B) \in I$. Let (X,τ) be a topological space and I an ideal of subsets of X. An ideal topological space is a topological space (X,τ) with an ideal I on X and is denoted by (X,τ,I). For a subset $A \subset X$, $A^*(I) = \{ x \in X \mid (U \cap A) \notin I$ for each neighbourhood U of $x \}$ is called the local function of A with respect to I and τ [10]. X^* is

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often a proper subset of X. The hypothesis $X = X^*$ [5] is equivalent to the hypothesis $\tau \cap I = \{ \phi \}$ [11]. The ideal topological spaces which satisfy this hypothesis are called Hayashi-Samuels spaces. We simply write A^* instead of $A^*(I)$ in case there is no chance for confusion. For every ideal topological space (X,τ,I), there exists a topology $\tau^*(I)$, finer than τ, generated by $\beta(I,\tau) = \{ U \setminus \mathrm{I} \mid U \in \tau \text{ and } \mathrm{I} \in I \}$, but in general $\beta(I,\tau)$ is not always a topology [6]. It is well-known that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

2.PRELIMINARIES

Lemma 2.1. Let (X,τ,I) be an ideal topological space and A, B subsets of X. Then, the following properties hold:

a) If $A \subset B$, then $A^* \subset B^*$,

b) If $U \in \tau$, then $U \cap A^* \subset (U \cap A)^*$,

c) $A^* = Cl(A^*) \subset Cl(A)$,

d) $(A \cup B)^* = A^* \cup B^*$,

e) $(A \cap B)^* \subset A^* \cap B^*$ (Janković and Hamlett [5]).

First we shall recall some definitions used in the sequel.

Definition 2.1. A subset A of an ideal topological space (X,τ,I) is said to be

a) almost *I*-open [2] if $A \subset Cl(Int(A^*))$,

b) β-*I*-open [4] if $A \subset Cl(Int(Cl^*(A)))$,

c) *-perfect [5] if $A = A^*$,

d) τ^*-closed [5] if $A^* \subset A$,

e) pre-*I*-open [3] if $A \subset Int(Cl^*(A))$,

f) regular-*I*-closed [8] if $A = (Int(A))^*$,

g) *I*-open [7] if $A \subset Int(A^*)$,

h) *I*-locally-closed [3] if $A = U \cap V$, where U is open and V is *-perfect,

i) A_I-set [8] if $A = U \cap V$, where U is open and V is regular-*I*-closed.

3. R_I-OPEN AND M_I-OPEN SETS

Definition 3.1. A subset A of an ideal topological space (X, τ, I) is said to be

a) r_I-open if $Int(A) = Cl(Int(A^*))$,

b) m_I-open if $A = U \cap V$, where U is *I*-open and V is r_I-open.

We denote the family of all r_I-open (resp. m_I-open) sets of (X, τ, I) by $r_I(X, \tau)$ (resp. $m_I(X, \tau)$). In addition, we will use a symbol $IO(X, \tau)$ for the family of all *I*-open sets of (X, τ, I).

Proposition 3.1. For a subset of a Hayashi-Samuels space, the following properties hold:

a) Every *I*-open set is m_I-open,

b) Every r_I-open set is m_I-open.

Proof. Since $X \in IO(X, \tau) \cap r_I(X, \tau)$, the statements are obvious.

Remark 3.1. The converses of Proposition 3.1 need not be true as shown by the following examples.

Example 3.1. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, c\}, \{a, b, c\}\}$ and $I = \{\phi, \{c\}\}$. Since $X = X^*$, (X, τ, I) is a Hayashi-Samuels space.

1) Set $A = \{c\}$. Then A is m_I-open, but not *I*-open. Since $B^* = \{b, d\}$, $Int(B^*) = \phi$ and we have $Cl(Int(B^*)) = \phi$ and besides since $Int(B) = \phi$, we obtain
$Int(B^*) = Cl(Int(B^*))$. This shows that $B \in r_I(X, \tau)$. Additionally, since $C^* = \{a, b, c, d\} = X$ and $C = \{a, c\} \subset X = Int(C^*)$, we have that C is an *I*-open set. Consequently, we have that $A = B \cap C = \{c\}$ is m_I-open. On the other hand, since $A^* = \phi$ and $A = \{c\} \not\subset \phi = Int(A^*)$, we have that A is not *I*-open.

2) Set $A=\{a,c\}$. Then A is m_I-open, but not r_I-open. Since $A^*=\{a,b,c,d\}=X$ and $A=\{a,c\}\subset Int(A^*)$, we have that A is *I*-open and hence m_I-open by using Proposition 3.1.a). On the other hand, since $Cl(Int(A^*))=X$ and A is an open set, we have $Cl(Int(A^*))=X\neq\{a,c\}=Int(A)$. This shows that A is not r_I-open.

Remark 3.2. Almost *I*-openness and r_I-openness are independent of each other as following examples show.

Example 3.2. Let $(\Re,\tau)$ be the real numbers with the usual topology τ and F the ideal of all finite subsets of $\Re$. Let Q be the set of all rational numbers. Then Q is almost *I*-open, but it is not r_I-open. For $Q\subset\Re$, since $Q^*=\Re$ and $Int(Q^*)=\Re$, we have $Q\subset\Re=Cl(Int(Q^*))$. This shows that Q is almost *I*-open. On the other hand, since $Int(Q)=\phi$, we have $Cl(Int(Q^*))=\Re\neq\phi=Int(Q)$. This shows that Q is not r_I-open.

Example 3. 3. Let (X,τ,I) be an ideal topological space such that Let $X=\{a,b,c,d\}$, $\tau=\{\phi,X,\{a,c\},\{a,c,d\}\}$ and $I=\{\phi,\{c\},\{d\},\{c,d\}\}$. Set $A=\{b,d\}$. Then A is r_I-open, but it is not almost *I*-open. For $A=\{b,d\}$, since $A^*=\{b\}$ and $Int(A^*)=\phi$, we have $Cl(Int(A^*))=\phi$. Hence $Cl(Int(A^*))=\phi=Int(A)$ and we obtain that A is r_I-open. On the other hand, we have $A=\{b,d\}\not\subset\phi=Cl(Int(A^*))$. This shows that A is not almost *I*-open.

Proposition 3.2. For a subset A of a Hayashi-Samuels space (X,τ,I), the following properties are equivalent:

a) A is *I*-open,

b) A is an almost *I*-open and m_I-open set.

Proof. a) $\Rightarrow$ b) : Let A be an *I*-open set. Then A is almost *I*-open by Definition 2.1. On the other hand, since $A=A\cap X$, where $A\in IO(X,\tau)$ and $X\in r_I(X,\tau)$, A is m_I-open.

b) $\Rightarrow$ a) : Let A be an almost *I*-open and m_I-open set. Then, $A = U \cap V$ for some *I*-open set U and r_I-open set V. So,

$$\begin{aligned} A &\subset Cl(Int(A^*)) \\ &= Cl(Int((U \cap V)^*)) \\ &\subset Cl(Int(U^*) \cap Int(V^*)) \\ &\subset Cl(Int(U^*)) \cap Cl(Int(V^*)) \\ &= Cl(Int(U^*)) \cap Int(V) \qquad (1), \end{aligned}$$

where $V \in r_I(X,\tau)$ and $U \in IO(X,\tau)$. Since $A = U \cap V$ and $A \subset U$, we obtain that

$$\begin{aligned} A &= A \cap U \\ &\subset U \cap (Cl(Int(U^*)) \cap Int(V)) \\ &= (U \cap Cl(Int(U^*))) \cap Int(V) \\ &= U \cap Int(V) \\ &\subset Int(U^*) \cap Int(V) \\ &= Int(U^* \cap Int(V)) \\ &\subset Int((U \cap Int(V))^*) \\ &\subset Int((U \cap V)^*) \\ &= Int(A^*) \end{aligned}$$

by using Lemma 1 and (1). Consequently, $A \subset Int(A^*)$ hence A is *I*-open.

4.WEAKLY *I*-LOCALLY-CLOSED SETS

Definition 4.1. A subset A of an ideal topological space (X,τ,I) is said to be weakly *I*-locally-closed if $A = U \cap V$, where U is open and V is τ^*-closed.

We denote the family of all weakly *I*-locally-closed (resp. τ^*-closed) sets of (X,τ,I) by $w_I LC(X,\tau)$ (resp. $\tau_C^*(X,\tau)$). We recall that in [9] was used notation of st-*I*-locally closed set instead of notation of weakly *I*-locally-closed set.

Proposition 4.1. In an ideal topological space (X,τ,I), $\tau_C^*(X,\tau) \subset w_I LC(X,\tau)$ and $\tau \subset w_I LC(X,\tau)$ ([9]).

Proof. Since $X \in \left(\tau \cap \tau_C^*(X,\tau)\right)$, the statements are obvious.

Remark 4.1. The converses of Proposition 4.1 need not be true as shown by the following examples ([9]).

Example 4.1. Let (X,τ,I) be an ideal topological space such that $X=\{a,b,c,d\}$, $\tau=\{\phi,X,\{a,c\},\{d\},\{a,c,d\}\}$ and $I=\{\phi,\{c\},\{d\},\{c,d\}\}$.

1) Set $A=\{b,d\}$. Then A is a weakly I-locally-closed set which is not open. For $A=\{b,d\}$, since $A^*=\{b\}$, we have $A^*=\{b\}\subset\{b,d\}=A$. This shows that $A\in\tau_C^*(X,\tau)$ and hence we have $A\in w_I LC(X,\tau)$ by using Proposition 4.1. On the other hand, since $A\notin\tau$, A is not open.

2) Set $A=\{a,c\}$. Then A is an open set which is not τ^*-closed. For $A=\{a,c\}$, it is obvious that A is open. Therefore, we have $A\in w_I LC(X,\tau)$ by using Proposition 4.1. On the other hand, for $A=\{a,c\}$, since $A^*=\{a,b,c\}\neq\{a,c\}=A$, we have that A is not τ^*-closed.

Proposition 4.2. Every I-locally-closed set is weakly I-locally-closed ([9]).

Proof. Let A be an I-locally-closed set of an ideal topological space (X,τ,I). Then we have $A=U\cap V$, where $U\in\tau$ and V is *-perfect. Therefore, since every *-perfect set is τ^*-closed, $V\in\tau_C^*(X,\tau)$. This shows that A is weakly I-locally-closed.

According to Proposition 4.2 and Proposition 5.a) of [8], we have the following diagram.

A_I-set $\Rightarrow$ I-locally-closed set $\Rightarrow$ weakly I-locally-closed set

Diagram I

Remark 4.2. The converse of Proposition 4.2. need not be true as shown by the following example.

Example 4.2. Let (X,τ,I) be the same ideal topological space as Example 4.1., that is, $X=\{a,b,c,d\}$, $\tau=\{\phi,X,\{a,c\},\{d\},\{a,c,d\}\}$ and $I=\{\phi,\{c\},\{d\},\{c,d\}\}$. Set $A=\{b,d\}$. Then A is a weakly I-locally-closed set which is not *I*-locally-closed. For $A=\{b,d\}$, we showed that A is a weakly *I*-locally-closed set in Example 4.1. On the other hand, for $A=\{b,d\}$, since $A^*=\{b\}\neq\{b,d\}=A$, we have A is not *-perfect and hence A is not *I*-locally-closed ([9]).

Remark 4.3. Pre-*I*-openness and weakly *I*-locally-closedness are independent of each other as the following examples show ([9]).

Example 4.3. Let (X,τ,I) be the same ideal topological space as Example 4.1., that is, $X=\{a,b,c,d\}$, $\tau=\{\phi,X,\{a,c\},\{d\},\{a,c,d\}\}$ and $I=\{\phi,\{c\},\{d\},\{c,d\}\}$. Set $A=\{b,d\}$. Then A is a weakly *I*-locally-closed set which is not pre-*I*-open. Since we have already shown that $A\in w_I LC(X,\tau)$ in Example 4.1., we only show that A is not pre-*I*-open. Since both $A^*=\{b\}$ and $Cl^*(A)=A\cup A^*=A=\{b,d\}$, we have $A=\{b,d\}\not\subset\{d\}=Int(\{b,d\})=Int(Cl^*(A))$. This shows that A is not pre-*I*-open.(This example is given in [9].)

Example 4.4. Let $(\Re,\tau)$ be the real numbers with the usual topology τ and F the ideal of all finite subsets of $\Re$. Let Q be the set of all rationals. Then Q is pre-*I*-open [3], but it is not weakly *I*-locally-closed. For $Q\subset\Re$, since $Q^*=\Re$, $Q\notin\tau$ and $Q^*=\Re\not\subset Q$, we have that Q is not weakly *I*-locally-closed.

Proposition 4.3. For a subset A of an ideal topological space (X,τ,I), the following properties are equivalent:

a) A is open,

b) A is pre-*I*-open and weakly *I*-locally-closed.

Proof. a) $\Rightarrow$ b) : The proof is immediately obtained by using Theorem 2.2 of [3] and Proposition 4.1.

b) $\Rightarrow$ a) : Let A be pre-*I*-open set and $A\in w_I LC(X,\tau)$. Then, we have $A\subset Int(Cl^*(A))$ and $A=U\cap V$, where $U\in\tau$ and $V\in\tau_C^*(X,\tau)$, respectively. Therefore, we have

$$A\subset Int(Cl^*(A))$$

$$= Int(Cl^*(U \cap V))$$
$$\subset Int(Cl^*(U) \cap Cl^*(V))$$
$$= Int(Cl^*(U)) \cap Int(Cl^*(V))$$
$$= Int(Cl^*(U)) \cap Int(V).$$

(2)

Since $A = U \cap V$ and $A \subset U$, we have

$$A = A \cap U$$
$$\subset (Int(Cl^*(U)) \cap Int(V)) \cap U \subset$$
$$= (U \cap Int(Cl^*(U))) \cap Int(V)$$
$$= Int(U \cap Cl^*(U)) \cap Int(V)$$
$$= Int(U \cap V)$$
$$= Int(A)$$

by using (2) and hence $A \in \tau$.

5. DECOMPOSITIONS OF *I*-CONTINUITY AND CONTINUITY

Definition 5.1. A function $f:(X,\tau,I) \to (Y,\varphi)$ is said to be m_I-continuous (resp. $w_I LC$-continuous, almost *I*-continuous[2], *I*-continuous[1], pre-*I*-continuous[3]) if for every $V \in \varphi$, $f^{-1}(V)$ is an m_I-open set (resp. weakly *I*-locally-closed, almost *I*-open, *I*-open, pre-*I*-open) set. We recall that in [9] was used notion of st-*I*-LC-continuous instead of notion of $w_I LC$-continuous.

Theorem 5.1. Let (X,τ,I) be a Hayashi-Samuels space. Then, for a function $f:(X,\tau,I) \to (Y,\varphi)$, the following statements are equivalent:

a) f is *I*-continuous,

b) f is almost *I*-continuous and m_I-continuous.

Proof. This follows from Proposition 3.2.

Theorem 5.2. For a function $f:(X,\tau,I) \to (Y,\varphi)$, the following statements are equivalent:

a) *f* is continuous,

b) *f* is pre-*I*-continuous and $w_I LC$-continuous.

Proof. This follows from Proposition 4.3.

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ÖZET r_I-açık küme ve m_I-açık küme kavramlarını tanımladık ve bu kümeleri kullanarak *I*-açık kümelerin ayrışımlarını elde ettik. Ayrıca, weakly *I*-lokal-kapalı küme kavramını tanımladık ve açık kümelerin ayrışımlarını elde ettik. Son olarak, *I*-sürekli ve sürekli fonksiyonların ayrışımlarını elde ettik.

REFERENCES

[1] M. E. Abd El-Monsef, E. F. Lashien and A. A. Nasef, On *I*-open sets and *I*-continuous functions, Kyungpook Math. J. , 32(1) (1992), 21-30.

[2] M. E. Abd El-Monsef, R. A. Mahmoud and A. A. Nasef, Almost *I*-openness and almost *I*-continuity, J. Egypt. Math. Soc. 7(1999), 191-200.

[3] J. Dontchev, On pre-*I*-open sets and a decomposition of *I*-continuity, Banyan Math. J., 2(1996).

[4] E. Hatir and T. Noiri, On decompositions of continuity via idealization, Acta Math. Hungar., 96(4) (2002), 341-349.

[5] E Hayashi, Topologies defined by local properties, Math. Ann., 156 (1964), 205- 215.

[6] D. Janković and T. R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly, 97 (1990), 295-310.

[7] D. Janković and T. R. Hamlett, Compatible extensions of ideals, Boll. Un. Mat. Ital. (7) 6-B (1992), 453-465.

[8] A. Keskin, S. Yuksel and T. Noiri, Idealization of decomposition theorem, Acta Math. Hungar. 102(4) (2004), 269-277.

[9] A. Keskin, New Decompositions of continuity in ideal topological spaces, Ph. D. Dissertation, Univ. of Selcuk at Konya, 2003.

[10] K. Kuratowski, Topology, Vol. I, Academic Press, New York, 1966.

[11] P. Samuels, A topology formed from a given topology and ideal, J. London Math. Soc.(2), 10(1975), 409-416.