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SOME RESULTS ON NEAR-RINGS WITH GENERALIZED DERIVATIONS

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ABSTRACT. Let N be a prime right near-ring with multiplicative center Z, $f : R \to R$ a generalized derivation associated with derivation d. The following results are proved: (i) If $f^2(N) = 0$ then $f = 0$. (ii) If $f(N) \subset Z$ then N is commutative ring. (iii) $f(xy) = f(x)f(y)$ or $f(xy) = f(y)f(x)$ for all $x, y \in N$ then $d = 0$.

1. INTRODUCTION

Throughout this paper, N stands for a right near-ring with multiplicative center Z. An additive map $d : N \to N$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$-or equivalently(cf.[7]) that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in [2]. The notion of generalized derivation of a prime ring was introduced by M. Bresar and B. Hvala in [4] and [6]. Some recent results concerning commutativity in prime near-rings with derivation have been generalized in several ways. Many authors have investigated these theorems for generalized derivation. It is my purpose to extend some comparable results on near-rings with generalized derivation.

According to [2], a near ring N is said to be prime if $xNy = \{0\}$ for $x, y \in N$ implies $x = 0$ or $y = 0$. For $x, y \in N$ the symbol (x, y) will denote the additive-group commutator $x+y-x-y$, while the symbol $[x, y]$ will denote the commutator $xy-yx$. Let S be a nonempty subset of N and d be a derivation of N. If $d(xy) = d(x)d(y)$ or $d(xy) = d(y)d(x)$ for all $x, y \in S$, then d is said to act as a homomorphism or anti-homomorphism on S, respectively.

In [3], Bell and Kappe proved that if d is a derivation of a semi-prime ring R which is either an endomorphism or anti-endomorphism, then $d = 0$. Argaç extended that above conclusion holds for near-rings in [1].

Two results are obtained in this paper: The first result states that if f is a generalized derivation of N such that $f^2 = 0$ then $f = 0$. The second result proves that f is generalized derivation of a prime near-ring N which is either a homomorphism or an anti-homomorphism on N, then $d = 0$. As for terminologies used here without mention, we refer to G. Pilz [8].

We shall give a description of generalized derivation associated with d by motivated [5, Definition 1].

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Definition 1.1. Let N be a near-ring, d a derivation of N. An additive mapping $f : N \to N$ is said to be right generalized derivation associated with d if

$$f(xy) = f(x)y + xd(y) \text{ for all } x, y \in R. \tag{1.1}$$

and f is said to be left generalized derivation associated with d if

$$f(xy) = d(x)y + xf(y) \text{ for all } x, y \in R. \tag{1.2}$$

f is said to be a generalized derivation associated with d if it is both a left and right generalized derivation associated with d.

Lemma 1.2. [1, Lemma 1] *Let N be a near-ring and d a derivation of N, then $a(yd(x) + d(y)x) = ayd(x) + ad(y)x$ for all $a, x, y \in N$.*

Lemma 1.3. *Let N be a prime near-ring, d a nonzero derivation of N and $a \in N$. If $ad(N) = 0$ $(d(N)a = 0)$, then $a = 0$.*

Proof. Suppose that $ad(N) = 0$. For arbitrary $x, y \in N$, we have
$0 = ad(xy) = ad(x)y + axd(y)$.
By the hypothesis,

$$axd(y) = 0, \quad \text{for all } x, y \in N.$$

Since N is prime near-ring and $d \neq 0$, we get $a = 0$.
Similarly argument works if $d(N)a = 0$. □

Lemma 1.4. *Let N be a 2− torsion free prime near-ring and d a derivation of N. If $d^2 = 0$, then $d = 0$.*

Proof. For arbitrary $x, y \in N$, we have

$$0 = d^2(xy) = d(d(xy)) = d(xd(y) + d(x)y)$$
$$= xd^2(y) + 2d(x)d(y) + d^2(x)y.$$

By the hypothesis,

$$2d(x)d(y) = 0, \text{ for all } x, y \in N.$$

Since N is 2− torsion free near- ring, we get

$$d(x)d(N) = 0, \text{ for all } x \in N.$$

Using Lemma 2, we get $d = 0$. □

Lemma 1.5. *Let N be a prime near-ring and d a nonzero derivation of N. If $d(N) \subset Z$, then $(N, +)$ is Abelian. Moreover, if N is 2−torsion free, then N is commutative ring.*

Proof. Suppose that $a \in N$ such that $d(a) \neq 0$. So, $d(a) \in Z \backslash \{0\}$ and $d(a) + d(a) \in Z \backslash \{0\}$. For all $x, y \in N$, we have

$$(d(a) + d(a))(x + y) = (x + y)(d(a) + d(a))$$

that is,

$$d(a)x + d(a)x + d(a)y + d(a)y = xd(a) + yd(a) + xd(a) + yd(a).$$

Since $d(a) \in Z$, we get

$$xd(a) + yd(a) = yd(a) + xd(a)$$

and so,

$$(x, y)d(a) = 0, \quad \text{for all } x, y \in N.$$

Since $d(a) \in Z\backslash\{0\}$ and N is a prime near-ring, we get $(x,y)=0$, for all $x,y \in N$. Thus $(N,+)$ is Abelian.

Now, using the hypothesis, for any $a,b,c \in N$,

$$cd(ab) = d(ab)c.$$

By Lemma 1, one can obtains

$$cad(b) + cd(a)b = ad(b)c + d(a)bc.$$

Using $d(N) \subset Z$ and $(N,+)$ is Abelian, we obtain that

$$cad(b) + cbd(a) = acd(b) + bcd(a)$$

which yields

$$([c,a])d(b) = [b,c]d(a), \quad \text{for all } a,b,c \in N.$$

Suppose now that N is not commutative. Choosing $b,c \in N$ such that $[b,c] \neq 0$ and replacing a by $d(a) \in Z$, we get

$$[b,c]d^2(a) = 0, \quad \text{for all } a,b,c \in N.$$

Since $d^2(a) \in Z$, we conclude that $d^2(a) = 0$, for all $a \in N$, and so $d = 0$ by Lemma 3. But it contradicts $d \neq 0$. This complates the proof. □

Lemma 1.6. *(i) Let f be right generalized derivation of N associated with d. Then $f(xy) = xd(y) + f(x)y$ for all $x,y \in N$.*

(ii) Let f be left generalized derivation of N associated with d. Then $f(xy) = xf(y) + d(x)y$ for all $x,y \in N$.

Proof. (i) For any $x,y \in N$, we get

$$\begin{aligned} f((x+x)y) &= f(x+x)y + (x+x)d(y) \\ &= f(x)y + f(x)y + xd(y) + xd(y) \end{aligned}$$

and

$$f(xy+xy) = f(x)y + xd(y) + f(x)y + xd(y).$$

Comparing these equations, one can obtain

$$f(x)y + xd(y) = xd(y) + f(x)y, \quad \text{for all } x,y \in N.$$

That is $f(xy) = xd(y) + f(x)y$.

(ii) Similarly. □

Lemma 1.7. *Let f be generalized derivation of N associated with d. Then $a(xd(y)+f(x)y) = axd(y) + af(x)y$ for all $a,x,y \in N$.*

Proof. The proof can be given using a similar approach as in the proof of [2, Lemma 1]. For any $a,x,y \in N$, we get

$$f(a(xy)) = af(xy) + d(a)xy = a(xd(y) + f(x)y) + d(a)xy.$$

On the other hand,

$$\begin{aligned} f((ax)y) &= axd(y) + f(ax)y \\ &= axd(y) + (af(x) + d(a)x)y = axd(y) + af(x)y + d(a)xy. \end{aligned}$$

For two expressions of $f(axy)$, we obtain that

$$a(xd(y) + f(x)y) = axd(y) + af(x)y, \quad \text{for all } a,x,y \in N.$$

□

Lemma 1.8. *Let N be a prime near-ring, f a nonzero generalized derivation of N associated with nonzero derivation d and $a \in N$.*

(i) If $af(N) = 0$, *then* $a = 0$.
(ii) If $f(N)a = 0$, *then* $a = 0$.

Proof. (i)For all $x, y \in N$, we get

$$0 = af(xy) = axd(y) + af(x)y$$

and so,

$$aNd(N) = 0.$$

Since N is prime near-ring and $d \neq 0$, we obtain $a = 0$.
ii) A similar argument works if $f(N)a = 0$. □

Theorem 1.9. *Let f be a generalized derivation of N associated with nonzero derivation d. If N is a 2−torsion free prime near- ring and $f^2 = 0$, then $f = 0$.*

Proof. For arbitrary $x, y \in N$, we have

$$\begin{aligned} 0 = f^2(xy) &= f(f(xy)) = f(f(x)y + xd(y)) \\ &= f^2(x)y + 2f(x)d(y) + xd^2(y). \end{aligned}$$

By the hypothesis,

$$2f(x)d(y) + xd^2(y) = 0 \text{ for all } x, y \in N. \tag{1.3}$$

Writing $f(x)$ by x in (1.3), we get

$$f(x)d^2(y) = 0 \text{ for all } x, y \in N.$$

By Lemma 7 (ii), we obtain that $d^2(N) = 0$ or $f = 0$. If $d^2(N) = 0$ then $d = 0$ from Lemma 3, a contradiction. So, we find $f = 0$. □

Theorem 1.10. *Let N be a prime near-ring with a nonzero generalized derivation f associated with nonzero derivation d. If $f(N) \subset Z$, then $(N, +)$ is Abelian. Moreover, if N is 2−torsion free, then N is a commutative ring.*

Proof. The same argument used in the proof of Lemma 4 shows that both $f(a) \in Z \backslash \{0\}$ and $f(a) + f(a) \in Z \backslash \{0\}$, then we have.

$$f(a)(x, y) = 0 \text{ for all } x, y \in N.$$

Since $f(a) \in Z \backslash \{0\}$ and N is a prime near-ring, it follows that $(x, y) = 0$, for all $x, y \in N$. Thus $(N, +)$ is abelian.

Using the hypothesis, for any $x, y, z \in N$,

$$zf(xy) = f(xy)z.$$

By Lemma 6, we have

$$\begin{aligned} z(xd(y) + f(x)y) &= (f(x)y + xd(y))z \\ zxd(y) + zf(x)y &= f(x)yz + xd(y)z. \end{aligned}$$

Using $f(N) \subset Z$ and $(N, +)$ is Abelian, we obtain that

$$zxd(y) - xd(y)z = f(x)yz - zf(x)y$$

and so,

$$zxd(y) - xd(y)z = f(x)[y, z], \quad \text{for all } x, y, z \in N. \tag{1.4}$$

Substituting $f(y)$ for y in (1.4) and using $f(N) \subset Z$, we get

$$[z, x]d(f(y)) = 0, \quad \text{for all } x, y, z \in N.$$

Since $f(y) \in Z$ and so $d(f\ (y)) \in Z$, we have

$$d(f(y)) = 0, \text{ for all } y \in N \quad \text{or} \quad N \text{ is commutative ring.}$$

Let assume that $d(f\ (y)) = 0$, for all $y \in N$. Then

$$\begin{aligned} 0 &= d(f(xy)) = d(d(x)y + xf(y)) \\ &= d^2(x)y + d(x)d(y) + d(x)f(y) = 0, \quad \text{for all } x, y \in N. \end{aligned}$$

Replacing y by yz in this equation and using this, we obtain that
$0 = d^2(x)yz + d(x)d(yz) + d(x)f(yz)$
$= d^2(x)yz + d(x)d(y)z + d(x)yd(z) + d(x)f(y)z + d(x)yd(z)$
$= \{d^2(x)y + d(x)d(y) + d(x)f(y)\}z + 2d(x)yd(z) = 2d(x)yd(z).$
Since N is a 2−torsion free near-ring, we get

$$d(N)Nd(N) = 0.$$

Thus, we obtain that $d = 0$. It contradicts $d \neq 0$. So we must have N is commutative ring. □

Theorem 1.11. *Let N be a prime near-ring and f be a generalized derivation of N associated with d. If f acts as a homomorphism on N, then $d = 0$.*

Proof. Let f acts as a homomorphism on N. Then

$$f(xy) = f(x)f(y) = xd(y) + f(x)yx, \quad \text{for all } x, y \in N. \tag{1.5}$$

Taking yx by y in (1.5), we get
$xd(yx) + f(x)yx = f(x)f(yx) = f(x)(yd(x) + f(y)x) = f(x)yd(x) + f(x)f(y)x$
$= f(x)yd(x) + f(xy)x = f(x)yd(x) + xd(y)x + f(x)yx$
and so,

$$xd(yx) = f(x)yd(x) + xd(y)x, \quad \text{for all } x, y \in N.$$

Using Lemma 1, we obtain that

$$xyd(x) = f(x)yd(x), \quad \text{for all } x, y \in N. \tag{1.6}$$

Replacing $f(y)$ by y in (1.6), then
$xf(y)d(x) = f(x)f(y)d(x) = f(xy)d(x) = d(x)yd(x) + xf(y)d(x)$
and so,

$$d(x)Nd(x) = 0, \quad \text{for all } x \in N.$$

Since N is a prime near-ring, we have $d = 0$. □

Theorem 1.12. *Let N be a prime near-ring and f be a generalized derivation of N associated with d. If f acts as a anti-homomorphism on N, then $d = 0$.*

Proof. By the hypothesis,we get

$$f(y)f(x) = xd(y) + f(x)y, \quad \text{for all } x, y \in N. \tag{1.7}$$

Replacing x by xy in (1.7), then
$xyd(y) + f(xy)y = f(y)f(xy) = f(y)(xd(y) + f(x)y) = f(y)xd(y) + f(y)f(x)y$
$= f(y)xd(y) + f(xy)y$
and so

$$xyd(y) = f(y)xd(y), \quad \text{for all } x, y \in N. \tag{1.8}$$

If we take rx instead of x in (1.8), we have
$f(y)rxd(y) = rxyd(y) = rf(y)xd(y)$

and so

$$[r, f(y)]xd(y) = 0, \quad \text{for all } x, y, r \in N.$$

Since N is a prime near-ring, we arrive at $f(y) \in Z$ or $d(y) = 0$, for all $y \in N$. Let's define $A = \{x \in N \mid d(x) = 0\}$ and $B = \{x \in N \mid f(x) \in Z\}$. Clearly each of A and B is additive subgroup of N such that $N = A \cup B$. But, a group can not be the set-theoretic union of two proper subgroups. Hence $N = A$ or $N = B$. In the latter case, $f(N) \subset Z$, which forces f acts as homomorphism on N, and so $d = 0$ by Theorem 3. If $N = A$ then $d = 0$. The proof is complated. □

ÖZET: N merkezi Z olan bir sağ asal near-halka, $f : N \to N$ tanımlı d ile ilgili bir genelleştirilmiş türev olsun. Bu durumda: (i) Eğer $f^2(N) = 0$ ise $f = 0$ dır. (ii) Eğer $f(N) \subset Z$ ise N değişmeli bir halkadır. (iii) Eğer her $x, y \in N$ için $f(xy) = f(x)f(y)$ veya $f(xy) = f(y)f(x)$ ise $d = 0$ dır.

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