Commun.Fac.Sci.Univ.Ank.Series A1
Volume 54, Number 2, Pages 39–48 (2005)
ISSN 1303–5991

OSCULATING SPHERES AND OSCULATING CIRCLES OF A CURVE IN SEMI-RIEMANNIAN SPACE

E. SOYTÜRK, K. ILARSLAN AND D. SAĞLAM

ABSTRACT. In the Euclidean 3-space, there is a unique sphere for a curve $\alpha : I \to \mathbb{R}^3$ such that the sphere touches α at the third order at $\alpha(0)$. The intersection of the sphere with osculating plane is a circle which touches α at the second order at $\alpha(0)$ [5]. In this paper, the osculating sphere and the osculating circle of the curve are studied for each of timelike, spacelike and null (lightlike) curves in Semi-Riemannian Spaces; $\mathbb{R}^3_1$, $\mathbb{R}^4_1$ and $\mathbb{R}^4_2$.

1. INTRODUCTION

The Semi-Riemannian n-space $\mathbb{R}^n_\nu$, is the Euclidean n-space $\mathbb{R}^n$ with the Semi-Riemannian inner product

$$\langle X, Y\rangle = -\sum_{i=1}^{\nu} x_i y_i + \sum_{j=\nu+1}^{n} x_j y_j \tag{1.1}$$

where $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$.

An arbitrary vector $X = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n_\nu$ can have one of the three causal characters; it is spacelike if $\langle X, X\rangle > 0$ or $X = 0$, timelike if $\langle X, X\rangle < 0$, $X \neq 0$ and null (lightlike) if $\langle X, X\rangle = 0$, $X \neq 0$. Similarly, an arbitrary curve $\alpha : I \to \mathbb{R}^n_\nu$, $s \to \alpha(s)$ in $\mathbb{R}^n_\nu$, where s is a pseudo-arclength parameter, can locally be spacelike, timelike or null, if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null for every $s \in I$.

A *pseudosphere* of radius $r > 0$ in $\mathbb{R}^n_\nu$ is the hyperquadric

$$S^{n-1}_\nu(r) = \left\{p \in \mathbb{R}^n_\nu : \langle p, p\rangle = r^2\right\}.$$

Similarly, a *pseudohyperbolic space* of radius $r > 0$ in $\mathbb{R}^n_\nu$ is the hyperquadric

$$H^{n-1}_{\nu-1}(r) = \left\{p \in \mathbb{R}^n_\nu : \langle p, p\rangle = -r^2\right\}.$$

Let $\{T, N, B_1, B_2\}$ the moving Frenet frame along the curve α. Here, T is the tangent vector field, N is the principal normal vector field, B_1 and B_2 are the first

Received by the editors April 19, 2005, Accepted: Dce. 13, 2005.
1991 *Mathematics Subject Classification.* Primary 53B30; Secondary 53C50.
Key words and phrases. Osculating Spheres, Osculating Circles and Minkowski Space.

and second binormal vector fields of the curve α. Depending on the causal character of the curve α, we have the following Frenet formulae in $\mathbb{R}^3_1, \mathbb{R}^4_1$ and $\mathbb{R}^4_2$.

1.1. **Frenet Frame in R^3_1.**

Case 1: If α is a spacelike curve with a timelike principal normal N,

$T' = k_1 N,\ N' = k_1 T + k_2 B,\ B' = k_2 N,$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = -1$, $\langle B,B\rangle = 1$, $\langle T,N\rangle = 0$, $\langle T,B\rangle = 0$ and $\langle N,B\rangle = 0$.

If α is a spacelike curve with a spacelike principal normal N,

$T' = k_1 N,\ N' = -k_1 T + k_2 B,\ B' = k_2 N,$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = 1$, $\langle B,B\rangle = -1$ and $\langle T,N\rangle = 0$, $\langle T,B\rangle = 0$, $\langle N,B\rangle = 0$.

Case 2: If α is a timelike curve,

$T' = k_1 N,\ N' = k_1 T + k_2 B,\ B' = -k_2 N,$

where $\langle T,T\rangle = -1$, $\langle N,N\rangle = 1$, $\langle B,B\rangle = 1$ and $\langle T,N\rangle = 0$, $\langle T,B\rangle = 0$, $\langle N,B\rangle = 0$.

Case 3: If α is a null curve,

$T' = k_1 B,\ N' = -k_2 B,\ B' = -k_2 T + k_1 N,$

where $\langle T,T\rangle = 0$, $\langle T,N\rangle = 1$, $\langle T,B\rangle = 0$, $\langle N,N\rangle = 0$, $\langle N,B\rangle = 0$, $\langle B,B\rangle = 1$.

1.2. **Frenet Frame in R^4_1.**

Case 1: If α is a spacelike curve with a timelike principal normal N,

$T' = k_1 N,\ N' = k_1 T + k_2 B_1,\ B_1' = k_2 N + k_3 B_2$ and $B_2' = -k_3 B_1,$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = -1$, $\langle B_1,B_1\rangle = 1, \langle B_2,B_2\rangle = 1$, $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0$, $\langle T,B_2\rangle = 0$, $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

If α is a spacelike curve with a timelike first binormal B_1,

$T' = k_1 N,\ N' = -k_1 T + k_2 B_1,\ B_1' = k_2 N + k_3 B_2$ and $B_2' = k_3 B_1,$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = 1$, $\langle B_1,B_1\rangle = -1, \langle B_2,B_2\rangle = 1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$, $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

If α is a spacelike curve with a timelike second binormal B_2,

$T' = k_1 N,\ N' = -k_1 T + k_2 B_1,\ B_1' = -k_2 N + k_3 B_2$ and $B_2' = k_3 B_1,$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = 1$, $\langle B_1,B_1\rangle = -1, \langle B_2,B_2\rangle = -1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0$, $\langle T,B_2\rangle = 0$, $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

Case 2: If α is a timelike curve,

$T' = k_1 N,\ N' = k_1 T + k_2 B_1,\ B_1' = -k_2 N + k_3 B_2,\ B_2' = -k_3 B_1,$

where $\langle T,T\rangle = -1$, $\langle N,N\rangle = 1$, $\langle B_1,B_1\rangle = 1$, $\langle B_2,B_2\rangle = 1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$ $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

Case 3: If α is a null curve,

$T' = k_1 B_1,\ N' = -k_2 B_1 - k_3 B_2,\ B_1' = -k_2 T + k_1 N$ and $B_2' = -k_3 T,$

where $\langle T,T\rangle = 0$, $\langle T,N\rangle = -1$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$, $\langle N,N\rangle = 0$, $\langle N,B_1\rangle = 0$, $\langle N,B_2\rangle = 0$, $\langle B_1,B_1\rangle = 1$ and $\langle B_2,B_2\rangle = 1$.

1.3. **Frenet Frame in R^4_2.**

Case 1: If α is a spacelike curve with timelike principal normal N and first timelike binormal B_1,

$T' = k_1 N,\ N' = k_1 T + k_2 B_1,\ B_1' = -k_2 N + k_3 B_2$ and $B_2' = k_3 B_1,$

where $\langle T,T\rangle = 1$, $\langle N,N\rangle = -1$, $\langle B_1,B_1\rangle = -1, \langle B_2,B_2\rangle = 1$, $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0$, $\langle T,B_2\rangle = 0$, $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

If α is a spacelike curve with a timelike principal N and second binormal B_2,
$T' = k_1 N,\ N' = k_1 T + k_2 B_1,\ B_1' = k_2 N + k_3 B_2$ and $B_2' = k_3 B_1$,
where $\langle T,T\rangle = 1$, $\langle N,N\rangle = -1$, $\langle B_1,B_1\rangle = 1, \langle B_2,B_2\rangle = -1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$, $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

If α is a spacelike curve with a timelike first binormal B_1 and second normal B_2,
$T' = k_1 N,\ N' = -k_1 T + k_2 B_1,\ B_1' = k_2 N + k_3 B_2$ and $B_2' = -k_3 B_1$,
where $\langle T,T\rangle = 1$, $\langle N,N\rangle = 1$, $\langle B_1,B_1\rangle = 1, \langle B_2,B_2\rangle = -1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$, $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

Case 2: If α is a timelike curve with timelike principal normal N,
$T' = k_1 N,\ N' = -k_1 T + k_2 B_1,\ B_1' = k_2 N + k_3 B_2,\ B_2' = -k_3 B_1$,
where $\langle T,T\rangle = -1$, $\langle N,N\rangle = -1$, $\langle B_1,B_1\rangle = 1$, $\langle B_2,B_2\rangle = 1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$ $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

If α is a timelike curve with timelike first binormal B_1,
$T' = k_1 N,\ N' = k_1 T + k_2 B_1,\ B_1' = k_2 N + k_3 B_2,\ B_2' = k_3 B_1$,
where $\langle T,T\rangle = -1$, $\langle N,N\rangle = 1$, $\langle B_1,B_1\rangle = -1$, $\langle B_2,B_2\rangle = 1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$ $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

If α is a timelike curve with timelike second binormal B_2,
$T' = k_1 N,\ N' = k_1 T + k_2 B_1,\ B_1' = -k_2 N + k_3 B_2,\ B_2' = k_3 B_1$,
where $\langle T,T\rangle = -1$, $\langle N,N\rangle = 1$, $\langle B_1,B_1\rangle = 1$, $\langle B_2,B_2\rangle = -1$ and $\langle T,N\rangle = 0$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$ $\langle N,B_1\rangle = 0, \langle N,B_2\rangle = 0$ and $\langle B_1,B_2\rangle = 0$.

Case 3: If α is a null curve with timelike first binormal B_1,
$T' = -k_1 B_1,\ N' = k_2 B_1 - k_3 B_2,\ B_1' = -k_2 T + k_1 N$ and $B_2' = -k_3 T$,
where $\langle T,T\rangle = 0$, $\langle T,N\rangle = -1$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$, $\langle N,N\rangle = 0$, $\langle N,B_1\rangle = 0$, $\langle N,B_2\rangle = 0$, $\langle B_1,B_1\rangle = -1$ and $\langle B_2,B_2\rangle = 1$.

If α is a null curve with timelike second binormal B_2,
$T' = k_1 B_1,\ N' = -k_2 B_1 + k_3 B_2,\ B_1' = -k_2 T + k_1 N$ and $B_2' = -k_3 T$,
where $\langle T,T\rangle = 0$, $\langle T,N\rangle = -1$, $\langle T,B_1\rangle = 0, \langle T,B_2\rangle = 0$, $\langle N,N\rangle = 0$, $\langle N,B_1\rangle = 0$, $\langle N,B_2\rangle = 0$, $\langle B_1,B_1\rangle = 1$ and $\langle B_2,B_2\rangle = -1$. [1], [2], [3] and [4].

2. Osculating Sphere of a Timelike Curve

We shall assume that the timelike curve $\alpha : I \to \mathbb{R}^3_1$ is parametrized such that $\|\alpha'(t)\| = 1$. Then we have $\alpha'(t) = T$. Let (y_1, y_2, y_3) be the Euclidean coordinate system in $\mathbb{R}^3_1$. We take a sphere $\langle y-d, y-d\rangle = r^2$, with origin and radius d and r respectively, where $y = (y_1, y_2, y_3)$. Let $f(t) = \langle \alpha(t) - d, \alpha(t) - d\rangle - r^2$. If we have the following equations

$$f(0) = 0,\ f'(0) = 0,\ f''(0) = 0,\ f'''(0) = 0 \tag{2.1}$$

then we say that the sphere touches α at the third order to the curve at $\alpha(0)$.

Theorem 2.1. *Let $k_1(0)$ and $k_2(0)$, the curvatures of a timelike curve $\alpha : I \to \mathbb{R}^3_1$ at $\alpha(0)$, be different from zero. Then there exists a sphere which touches at the third order to the curve at $\alpha(0)$, and the equation of the sphere according to the frame $\{T_0, N_0, B_0\}$ is*

$$-x_1^2 + (x_2 + \rho_o)^2 + (x_3 + \rho_o'\sigma_o)^2 = \rho_o^2 + (\rho_o'\sigma_o)^2, \tag{2.2}$$

where $\rho_o = \dfrac{1}{k_1(0)}$ and $\sigma_o = \dfrac{1}{k_2(0)}$.

Proof. If $f(0) = 0$ then $\langle \alpha(0) - d, \alpha(0) - d \rangle = r^2$. Since we have $f' = 2\langle \alpha', \alpha - d \rangle$, $f'(0) = 0$ implies $\langle T_o, \alpha(0) - d \rangle = 0$. Similarly we have $f'' = 2[\langle \alpha'', \alpha - d \rangle + \langle \alpha', \alpha' \rangle]$, $f''(0) = 0$ implies $\langle k_1(0)N_o, \alpha(0) - d \rangle + \langle T_o, T_o \rangle = 0$. Substituting, $\langle T_o, T_o \rangle = -1$ in this equation we obtain $\langle N_o, \alpha(0) - d \rangle = \rho_o$ is obtained.

Considering $f''' = 2[\langle \alpha''', \alpha - d \rangle + 3\langle \alpha'', \alpha' \rangle]$ and $f'''(0) = 0$ we get
$\langle k_1^2(0)T_o + k_1'(0)N_o + k_1(0)k_2(0)B_o, \alpha(0) - d \rangle + 3\langle k_1(0)N_o, T_o \rangle = 0.$
Consequently,

$$k_1^2(0)\langle T_o, \alpha(0) - d \rangle + k_1'(0)\langle N_o, \alpha(0) - d \rangle$$
$$+k_1(0)k_2(0)\langle B_o, \alpha(0) - d \rangle + 3k_1(0)\langle N_o, T_o \rangle = 0.$$

Here, Substituting $\langle T_o, \alpha(0) - d \rangle = 0$, $\langle N_o, \alpha(0) - d \rangle = \rho_o$, $\langle N_o, T_o \rangle = 0$, we obtain

$$\langle B_o, \alpha(0) - d \rangle = \frac{-\rho_o k_1'(0)}{k_1(0)k_2(0)} = \frac{-k_1'(0)}{k_1^2(0)k_2(0)} = \rho_o'\sigma_o.$$

Now we investigate the numbers u_1, u_2, u_3 such that $\alpha(0) - d = u_1T_o + u_2N_o + u_3B_o$. Since $\langle T_o, \alpha(0) - d \rangle = -u_1$ and $\langle T_o, \alpha(0) - d \rangle = 0$, then we find $u_1 = 0$. Since $\langle N_o, \alpha(0) - d \rangle = u_2$ and $\langle N_o, \alpha(0) - d \rangle = \rho_o$ then we find $u_2 = \rho_o$. Since $\langle B_o, \alpha(0) - d \rangle = u_3$ and $\langle B_o, \alpha(0) - d \rangle = \rho_o'\sigma_o$, then we find $u_3 = \rho_o'\sigma_o$. Also, the origin of the sphere that contacts at the third order to the curve at the point $\alpha(0)$ is

$$d = \alpha(0) - \rho_oN_o - \rho_o'\sigma_oB_o. \tag{2.3}$$

Given a variable P on the osculating sphere, suppose $P = \alpha(0) + x_1T_o + x_2N_o + x_3B_o$. Hence,

$$P - d = x_1T_o + (x_2 + \rho_o)N_o + (x_3 + \rho_o'\sigma_o)B_o$$

also

$$\langle P - d,\ P - d \rangle = -x_1^2 + (x_2 + \rho_o)^2 + (x_3 + \rho_o'\sigma_o)^2$$

using (2.3), we obtain

$$r^2 = \langle \alpha(0) - d, \alpha(0) - d \rangle = \rho_o^2 + (\rho_o'\sigma_o)^2.$$

□

Now, we show that the circle which is the intersection of the osculating sphere at $\alpha(0)$ with the plane $Sp\{T_o, N_o\}$, contacts at the second order to the curve at $\alpha(0)$. This circle is called osculating circle of the curve at $\alpha(0)$.

Theorem 2.2. *For each timelike curve $\alpha : I \to \mathbb{R}_1^3$, there exist a curve $\gamma : \mathbb{R} \to \mathbb{R}_1^3$,*

$$\gamma(\theta) = \alpha(0) + (\rho_o \sinh\theta)T_o + \rho_o(-1 + \cosh\theta)N_o \tag{2.4}$$

which contacts α at the second order at $\alpha(0)$.

Proof. The equation of the intersection of the plane $Sp\{T_o, N_o\}$ with the sphere which is given by (2.2) according to the frame $\{T_o, N_o, B_o\}$ is,

$$-x_1^2 + (x_2 + \rho_o)^2 = \rho_o^2.$$

From this equation we can write $x_1 = \rho_o \sinh\theta$, $x_2 = \rho_o(-1 + \cosh\theta)$. Thus the intersection circle can be given by (2.4). Clearly $\gamma(0) = \alpha(0)$.
Since $\gamma'(\theta) = (\rho_o \cosh\theta)T_o + (\rho_o \sinh\theta)N_o$, we have $\gamma'(0) = \rho_oT_o = \rho_o\alpha'(0)$. Also, $\gamma''(\theta) = (\rho_o \sinh\theta)T_o + (\rho_o \cosh\theta)N_o$ implies $\gamma''(0) = \rho_oN_o = \rho_o^2\alpha''(0)$.

The equalities $\gamma(0) = \alpha(0)$, $\gamma'(0) = \rho_o\alpha'(0)$ and $\gamma''(0) = \rho_o^2\alpha''(0)$ show that the curve γ touches α at the second order at $\alpha(0)$. □

Corollary 1. *Osculating circle of a timelike curve* $\alpha : I \to \mathbb{R}_1^3$ *at* $\alpha(0)$ *is also a timelike curve.*

Proof. It is easy to see that for every $\theta \in R$, $\langle\gamma'(\theta), \gamma'(\theta)\rangle = -\rho_o^2(\cosh\theta)^2 + \rho_o^2(\sinh\theta)^2 = -\rho_o^2 < 0$. □

We can state the following theorems for the osculating sphere of a timelike curve in $\mathbb{R}_1^4$ and $\mathbb{R}_2^4$ as follows:

Theorem 2.3. *Let* $k_1(0)$, $k_2(0)$ *and* $k_3(0)$, *the curvatures of a timelike curve* $\alpha : I \to \mathbb{R}_1^4$ *at* $\alpha(0)$, *be different from zero. Then there exist a sphere which touches at the fourth order to at* $\alpha(0)$ *and equation of the sphere according to the frame* $\{T_o, N_o, B_{1_0}, B_{2_0}\}$ *is*

$$-x_1^2 + (x_2 + \lambda_1)^2 + (x_3 + \lambda_2)^2 + (x_4 + \lambda_3)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{2.5}$$

where $\lambda_1 = \rho_o$, $\lambda_2 = \rho_o'\sigma_o$, $\lambda_3 = \left((\rho_o'\sigma_o)' + \frac{\rho_o}{\sigma_o}\right)\omega_0$ and $\rho_o = \dfrac{1}{k_1(0)}$, $\sigma_o = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

Theorem 2.4. *Let* $k_1(0)$, $k_2(0)$ *and* $k_3(0)$, *the curvatures of a timelike curve* $\alpha : I \to \mathbb{R}_2^4$ *at* $\alpha(0)$ *with timelike principal normal* N, *be different from zero. Then there exist a sphere which touches at the fourth order to* α *at* $\alpha(0)$ *and the equation of the sphere according to the frame* $\{T_o, N_o, B_{1_0}, B_{2_0}\}$ *is*

$$-x_1^2 - (x_2 - \lambda_1)^2 + (x_3 + \lambda_2)^2 + (x_4 + \lambda_3)^2 = -\lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{2.6}$$

where $\lambda_1 = \rho_o$, $\lambda_2 = \rho_o'\sigma_o$, $\lambda_3 = \left((\rho_o'\sigma_o)' - \frac{\rho_o}{\sigma_o}\right)\omega_0$ and $\rho_o = \dfrac{1}{k_1(0)}$, $\sigma_o = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

Theorem 2.5. *Let* $k_1(0)$, $k_2(0)$ *and* $k_3(0)$,*the curvatures of a timelike curve* $\alpha : I \to \mathbb{R}_2^4$ *at* $\alpha(0)$ *with timelike first binormal* B_1, *be different from zero. Then there exist a sphere which touches at the fourth order to* α *at* $\alpha(0)$ *and the equation of the sphere according to the frame* $\{T_o, N_o, B_{1_0}, B_{2_0}\}$ *is*

$$-x_1^2 + (x_2 + \lambda_1)^2 - (x_3 + \lambda_2)^2 + (x_4 + \lambda_3)^2 = \lambda_1^2 - \lambda_2^2 + \lambda_3^2, \tag{2.7}$$

where $\lambda_1 = \rho_o$, $\lambda_2 = \rho_o'\sigma_o$, $\lambda_3 = \left((\rho_o'\sigma_o)' - \frac{\rho_o}{\sigma_o}\right)\omega_0$ and $\rho_o = \dfrac{1}{k_1(0)}$, $\sigma_o = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

Theorem 2.6. *Let* $k_1(0)$, $k_2(0)$ *and* $k_3(0)$, *the curvatures of a timelike curve* $\alpha : I \to \mathbb{R}_2^4$ *at* $\alpha(0)$ *with timelike second binormal* B_2, *be different from zero. Then there exist a sphere which touches fourth order to* α *at* $\alpha(0)$ *and the equation of the sphere according to the frame* $\{T_o, N_o, B_{1_0}, B_{2_0}\}$ *is*

$$-x_1^2 + (x_2 + \lambda_1)^2 + (x_3 - \lambda_2)^2 - (x_4 - \lambda_3)^2 = \lambda_1^2 + \lambda_2^2 - \lambda_3^2, \tag{2.8}$$

where $\lambda_1 = \rho_o$, $\lambda_2 = \rho_o' \sigma_o$, $\lambda_3 = \left((\rho_o' \sigma_o)' + \frac{\rho_o}{\sigma_o} \right) \omega_0$ and $\rho_o = \dfrac{1}{k_1(0)}$, $\sigma_o = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

3. Osculating Sphere of a Null Curve

Theorem 3.1. *Let $k_1(0) \neq 0$. The sphere that contacts at the third order to null (lightlike) curve $\alpha : I \to \mathbb{R}_1^3$ at $\alpha(0)$ is the pseudosphere at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_0, N_0, B_0\}$ is*

$$-2x_1x_2 + x_3^2 = 0. \tag{3.1}$$

Proof. If $f(0) = 0$, then $\langle \alpha(0) - d, \alpha(0) - d \rangle = r^2$. Since $f' = 2 \langle \alpha', \alpha - d \rangle$ $f'(0) = 0$ implies $\langle T_o, \alpha(0) - d \rangle = 0$. Since $f'' = 2k_1 \langle B, \alpha - d \rangle$, $f''(0) = 0$ implies

$2k_1(0) \langle B_o, \alpha(0) - d \rangle = 0$. Since $k_1(0) \neq 0$ then we have $\langle B_o, \alpha(0) - d \rangle = 0$. Since

$$f''' = 2[k_1' \langle B, \alpha - d \rangle - k_1 k_2 \langle T, \alpha - d \rangle + k_1^2 \langle N, \alpha - d \rangle + k_1 \langle B, T \rangle]$$

then $f'''(0) = 0$ implies $k_1^2(0) \langle N_o, \alpha(0) - d \rangle = 0$. And we obtain $\langle N_o, \alpha(0) - d \rangle = 0$

.

Now we investigate the numbers u_1, u_2, u_3 such that $\alpha(0) - d = u_1 T_o + u_2 N_o + u_3 B_o$. Considering $\langle T_o, \alpha(0) - d \rangle = 0$, $\langle B_o, \alpha(0) - d \rangle = 0$, $\langle N_o, \alpha(0) - d \rangle = 0$ and Frenet frame, we have $\langle T_o, \alpha(0) - d \rangle = u_1 \langle T_o, T_o \rangle + u_2 \langle T_o, N_o \rangle + u_3 \langle T_o, B_o \rangle$ then $u_2 = 0$. $\langle N_o, \alpha(0) - d \rangle = u_1 \langle N_o, T_o \rangle + u_2 \langle N_o, N_o \rangle + u_3 \langle N_o, B_o \rangle$ then $u_1 = 0$. Similarly $\langle B_o, \alpha(0) - d \rangle = u_1 \langle B_o, T_o \rangle + u_2 \langle B_o, N_o \rangle + u_3 \langle B_o, B_o \rangle$
then $u_3 = 0$. Thus $d = \alpha(0)$. Since $\langle \alpha(0) - d, \alpha(0) - d \rangle = r^2$ then we must have $r = 0$. Also the equation of the pseudosphere which contacts at the third order to α at $\alpha(0)$ is $\langle y - \alpha(0), y - \alpha(0) \rangle = 0$. □

We can state the following theorems for the osculating sphere of a null curve in $\mathbb{R}_1^4$ and $\mathbb{R}_2^4$ as follows:

Theorem 3.2. *Let $k_1(0) \neq 0$. The sphere that contacts at the third order to null (lightlike) curve $\alpha : I \to \mathbb{R}_1^4$ at $\alpha(0)$ is the pseudosphere at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_o, N_o, B_{1_0}, B_{2_0}\}$ is*

$$-2x_1x_2 + x_3^2 + (x_4 - \frac{1}{k_3})^2 = \left(\frac{1}{k_3} \right)^2. \tag{3.2}$$

Theorem 3.3. *Let $k_1(0) \neq 0$. The sphere that contacts at the fourth order to null (lightlike) curve $\alpha : I \to \mathbb{R}_2^4$ at $\alpha(0)$ with timelike first binormal B_1 is the pseudosphere at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_o, N_o, B_{1_0}, B_{2_0}\}$ is*

$$-2x_1x_2 - x_3^2 + (x_4 - \frac{1}{k_3})^2 = \left(\frac{1}{k_3} \right)^2. \tag{3.3}$$

Theorem 3.4. *Let $k_1(0) \neq 0$. The sphere that contacts at the fourth order to null (lightlike) curve $\alpha : I \to \mathbb{R}_2^4$ at $\alpha(0)$ with timelike second binormal B_2 is the pseudosphere at $\alpha(0)$ and the equation of the sphere according to the frame*

$\{T_o, N_o, B_{1_0}, B_{2_0}\}$ *is*

$$-2x_1x_2 + x_3^2 - (x_4 - \frac{1}{k_3})^2 = \left(\frac{1}{k_3}\right)^2. \tag{3.4}$$

4. Osculating Sphere of a Spacelike Curve

Theorem 4.1. *Let $k_1(0)$ and $k_2(0)$, the curvatures of a spacelike curve $\alpha : I \to \mathbb{R}_1^3$ with timelike principal normal N at $\alpha(0)$, be different from zero. Then there exists a sphere which touches at the third order to the curve at $\alpha(0)$, and the equation of the sphere according to the frame $\{T_0, N_0, B_{1_0}\}$ is*

$$x_1^2 - (x_2 + \rho_o)^2 + (x_3 - \rho_o'\sigma_o)^2 = -\rho_o^2 + (\rho_o'\sigma_o)^2, \tag{4.1}$$

where $\rho_o = \dfrac{1}{k_1(0)}$ and $\sigma_o = \dfrac{1}{k_2(0)}$.

Proof. Let $f(t) = \langle \alpha(t) - d, \alpha(t) - d\rangle - r^2$. If $f(0) = 0$ then $\langle \alpha(0) - d, \alpha(0) - d\rangle = r^2$. Since $f' = 2\langle \alpha', \alpha - d\rangle$ then $f'(0) = 0$ implies to $\langle T_o, \alpha(0) - d\rangle = 0$. Since $f'' = 2\left[\langle \alpha'', \alpha - d\rangle + \langle \alpha', \alpha'\rangle\right], f''(0) = 0$ implies to $[\langle k_1(0)N_o, \alpha(0) - d\rangle + \langle T_o, T_o\rangle] = 0$ and we get $\langle N_o, \alpha(0) - d\rangle = -\rho_o$. Since $f''' = 2\left[\langle \alpha''', \alpha - d\rangle + 3\langle \alpha'', \alpha'\rangle\right]$ and the equality $f'''(0) = 0$ implies to

$\left\langle k_1^2(0)T_o + k_1'(0)N_o + k_1(0)k_2(0)B_o, \alpha(0) - d\right\rangle + 3\langle k_1(0)N_o, T_o\rangle = 0$

Let us consider $\langle T_o, \alpha(0) - d\rangle = 0$, $\langle N_o, \alpha(t) - d\rangle = -\rho_o$, $\langle N_o, T_o\rangle = 0$ then we obtain

$$\langle B_o, \alpha(0) - d\rangle = \frac{\rho_o k_1'(0)}{k_1(0)k_2(0)} = \frac{k_1'(0)}{k_1^2(0)k_2(0)} = -\rho_o'\sigma_o.$$

Now we investigating the numbers u_1, u_2, u_3 such that $\alpha(0) - d = u_1T_o + u_2N_o + u_3B_o$, we obtain

$$\alpha(0) - d = \rho_o N_o - \rho_o'\sigma_o B_o.$$

Thus, the origin of the sphere which contacts at the third order to the curve α at point $\alpha(0)$ is

$$d = \alpha(0) - \rho_o N_o + \rho_o'\sigma_o B_0.$$

When a variable P is given on this sphere, we suppose $P = \alpha(0) + x_1T_o + x_2N_o + x_3B_o$. Hence, we get

$P - d = x_1T_o + (x_2 + \rho_o)N_o + (x_3 - \rho_o'\sigma_o)B_o,$

then

$$\langle P - d,\, P - d\rangle = x_1^2 - (x_2 + \rho_o)^2 + (x_3 - \rho_o'\sigma_o)^2.$$

Also $r^2 = \langle \alpha(0) - d, \alpha(0) - d\rangle = -\rho_o^2 + (\rho_o'\sigma_o)^2$ then the equation (4.1) is obtained. □

Corollary 2. *If $-\rho_o^2 + (\rho_o'\sigma_o)^2 > 0$ at $\alpha(0)$ for the spacelike curve $\alpha : I \to \mathbb{R}_1^3$ whose principal normal vector is timelike, then osculating sphere is a one-sheet hyperboloid. If $-\rho_o^2 + (\rho_o'\sigma_o)^2 < 0$, then osculating sphere is a two-sheet hyperboloid.*

Now, we show that the circle which is the intersection of the osculating sphere at $\alpha(0)$ for a spacelike curve $\alpha : I \to \mathbb{R}^3_1$ whose principal normal vector is timelike the plane $Sp\{T_o, N_o\}$, contacts at the second order to the curve at $\alpha(0)$. The circle is called osculating circle of the spacelike curve at $\alpha(0)$.

Theorem 4.2. *A spacelike curve $\alpha : I \to \mathbb{R}^3_1$ whose principal normal vector is timelike has a circle $\gamma : \mathbb{R} \to \mathbb{R}^3_1$ which contacts at the second order to the curve at $\alpha(0)$ and*

$$\gamma(\theta) = \alpha(0) + (\rho_o \sinh\theta)T_o + \rho_o(-1 + \cosh\theta)N_o. \tag{4.2}$$

Proof. The equation of the intersection of the plane $Sp\{T_o, N_o\}$ with the sphere which is given in (4.1) according to the frame $\{T_o, N_o, B_o\}$ is

$$-x_1^2 + (x_2 + \rho_o)^2 = \rho_o^2.$$

Then we have $x_1 = \rho_o \sinh\theta$, $x_2 = \rho_o(-1+\cosh\theta)$. Thus, the intersection circle can be given as in the equality in (4.2). Clearly $\gamma(0) = \alpha(0)$. Since

$$\gamma'(\theta) = (\rho_o \cosh\theta)T_o + (\rho_o \sinh\theta)N_o$$

then we get,

$$\gamma'(0) = \rho_o T_0 = \rho_o \alpha'(0).$$

Since $\gamma''(\theta) = (\rho_o \sinh\theta)T_o + (\rho_o \cosh\theta)N_o$, then we obtain,

$$\gamma''(0) = \rho_o N_o = \rho_o^2 \alpha''(0).$$

The equalities $\gamma(0) = \alpha(0)$, $\gamma'(0) = \rho_o\alpha'(0)$ and $\gamma''(0) = \rho_o^2\alpha''(0)$ show that the curve γ contacts at the second order to the curve α at $\alpha(0)$. □

Corollary 3. *Osculating circle of a spacelike curve $\alpha : I \to \mathbb{R}^3_1$ whose principal normal vector is timelike at $\alpha(0)$ is also a spacelike curve.*

Proof. It is easy to see that for every $\theta \in \mathbb{R}$, $\langle\gamma'(\theta), \gamma'(\theta)\rangle = \rho_o^2(\cosh\theta)^2 - \rho_o^2(\sinh\theta)^2 = \rho_o^2 > 0$. □

Theorem 4.3. *Let $\alpha : I \to \mathbb{R}^3_1$ be a spacelike curve with timelike binormal vector field and the curvatures of the curve at point $\alpha(0)$; $k_1(0)$ and $k_2(0)$ different from zero. Thus there exist a sphere which contacts at the third order to the curve α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_o, N_o, B_o\}$ is*

$$x_1^2 + (x_2 - \rho_o)^2 - (x_3 - \rho_o'\sigma_o)^2 = \rho_o^2 - (\rho_o'\sigma_o)^2, \tag{4.3}$$

where $\rho_o = \dfrac{1}{k_1(0)}$ and $\sigma_o = \dfrac{1}{k_2(0)}$.

Proof. The proof is similar to the proof of the Theorem 4.1. □

Theorem 4.4. *For each spacelike curve $\alpha : I \to \mathbb{R}^3_1$, whose binormal vector field is timelike, there exist a circle $\gamma : R \to \mathbb{R}^3_1$ which contacts at the second order to the curve at $\alpha(0)$ and*

$$\gamma(\theta) = \alpha(0) + \rho_o \sinh(\theta + \pi))T_o + (\rho_o + \rho_o \cosh(\theta + \pi)N_o. \tag{4.4}$$

Proof. The proof is similar to the proof of the Theorem 4.3. □

Corollary 4. *Osculating circle of a spacelike curve $\alpha : I \to \mathbb{R}^3_1$ whose binormal vector field is timelike at point $\alpha(0)$ is also a spacelike curve.*

Proof. It can be made in a similar way to the proof of Theorem4.2. □

We can state the following theorems for the osculating sphere of a spacelike curve in $\mathbb{R}^4_1$ and $\mathbb{R}^4_2$ as follows:

Theorem 4.5. *Let $\alpha : I \to \mathbb{R}^4_1$ be a spacelike curve with timelike principal vector field N and the curvatures of the curve at $\alpha(0)$; $k_1(0)$, $k_2(0)$ and $k_3(0)$ be different from zero. Thus there exist a sphere which contacts at the fourth order to the curve α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_0, N_0, B_{1_0}, B_{2_0}\}$ is*

$$x_1^2 - (x_2+\lambda_1)^2 + (x_3-\lambda_2)^2 + (x_4+\lambda_3)^2 = -\lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad (4.5)$$

where $\lambda_1 = \rho_0$, $\lambda_2 = \rho_0'\sigma_0$, $\lambda_3 = \left(\left(-(\rho_0'\sigma_0)' + \frac{\rho_0}{\sigma_0}\right)\omega_0\right)$ *and* $\rho_0 = \dfrac{1}{k_1(0)}$, $\sigma_0 = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

Theorem 4.6. *Let $\alpha : I \to \mathbb{R}^4_1$ be a spacelike curve with timelike first binormal B_1 and the curvatures of the curve at $\alpha(0)$; $k_1(0)$, $k_2(0)$ and $k_3(0)$ be different from zero. Thus there exist a sphere which contacts at the fourth order to the curve α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_0, N_0, B_{1_0}, B_{2_0}\}$ is*

$$x_1^2 + (x_2-\lambda_1)^2 - (x_3+\lambda_2)^2 + (x_4+\lambda_3)^2 = \lambda_1^2 - \lambda_2^2 + \lambda_3^2, \qquad (4.6)$$

where $\lambda_1 = \rho_0$, $\lambda_2 = \rho_0'\sigma_0$, $\lambda_3 = \left(\left(-(\rho_0'\sigma_0)' + \frac{\rho_0}{\sigma_0}\right)\omega_0\right)$ and $\rho_0 = \dfrac{1}{k_1(0)}$, $\sigma_0 = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

Theorem 4.7. *Let $\alpha : I \to \mathbb{R}^4_2$ be a spacelike curve with timelike principal normal N and timelike first binormal B_1 and the curvatures of the curve at point $\alpha(0)$; $k_1(0)$, $k_2(0)$ and $k_3(0)$ be different from zero. Thus there exist a sphere which contacts fourth order to the curve α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_0, N_0, B_{1_0}, B_{2_0}\}$ is*

$$x_1^2 - (x_2+\lambda_1)^2 - (x_3+\lambda_2)^2 + (x_4-\lambda_3)^2 = -\lambda_1^2 - \lambda_2^2 + \lambda_3^2, \qquad (4.7)$$

where $\lambda_1 = \rho_0$, $\lambda_2 = \rho_0'\sigma_0$, $\lambda_3 = -\left(\left((\rho_0'\sigma_0)' + \frac{\rho_0}{\sigma_0}\right)\omega_0\right)$ and $\rho_0 = \dfrac{1}{k_1(0)}$, $\sigma_0 = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

Theorem 4.8. *Let $\alpha : I \to \mathbb{R}^4_2$ be a spacelike curve with timelike principal normal N and timelike second binormal B_2 and the curvatures of the curve at point $\alpha(0)$; $k_1(0)$, $k_2(0)$ and $k_3(0)$ be different from zero. Thus there exist a sphere which contacts fourth order to the curve α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_0, N_0, B_{1_0}, B_{2_0}\}$ is*

$$x_1^2 - (x_2 + \lambda_1)^2 + (x_3 - \lambda_2)^2 - (x_4 + \lambda_3)^2 = -\lambda_1^2 + \lambda_2^2 - \lambda_3^2, \tag{4.8}$$

where $\lambda_1 = \rho_0$, $\lambda_2 = \rho_0'\sigma_0$, $\lambda_3 = -\left(\left((\rho_0'\sigma_0)' + \frac{\rho_0}{\sigma_0}\right)\omega_0\right)$and $\rho_0 = \dfrac{1}{k_1(0)}$, $\sigma_0 = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

Theorem 4.9. *Let $\alpha : I \to \mathbb{R}_2^4$ be a spacelike curve with timelike first binormal B_1 and timelike second binormal B_2 and the curvatures of the curve at point $\alpha(0)$; $k_1(0)$, $k_2(0)$ and $k_3(0)$ be different from zero. Thus there exist a sphere which contacts at the fourth order to the curve α at $\alpha(0)$ and the equation of the sphere according to the frame $\{T_0, N_0, B_{1_0}, B_{2_0}\}$ is*

$$x_1^2 + (x_2 - \lambda_1)^2 - (x_3 + \lambda_2)^2 - (x_4 - \lambda_3)^2 = \lambda_1^2 - \lambda_2^2 + \lambda_3^2, \tag{4.9}$$

where $\lambda_1 = \rho_0$, $\lambda_2 = \rho_0'\sigma_0$, $\lambda_3 = \left(\left(-(\rho_0'\sigma_0)' + \frac{\rho_0}{\sigma_0}\right)\omega_0\right)$and $\rho_0 = \dfrac{1}{k_1(0)}$, $\sigma_0 = \dfrac{1}{k_2(0)}$, $\omega_0 = \dfrac{1}{k_3(0)}$.

ÖZET. Üç boyutlu Öklid uzayında bir $\alpha : I \to R^3$ eğrisinin $\alpha(0)$ noktasında eğriye üçüncü basamaktan değen bir ve yalnız bir küre vardır. Oskülatör düzlemiyle bu kürenin arakesiti, eğriye $\alpha(0)$ noktasında ikinci basamaktan değen bir çemberdir [5]. Bu çalışmada R_1^3, R_1^4 ve R_2^4 yarı Riemann uzaylarında zamansı, uzaysı ve boşluksu (ışıksı) eğrilerin her biri için eğrinin oskülatör küresi ve eğrilik çemberi incelenmiştir.

References

[1] Duggal, K. L. and Bejancu, A., Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications. Kluwer Academic Publisher. 1996.

[2] Duggal, K. L. and Jin, D.H., Geometry of Null Curves. Math. J. Toyoma Univ., Vol.22; 95-120, 1999.

[3] Ekmekci, N. and İlarslan, K., Higher Curvatures of a Regular Curve in Lorentzian Space. Jour. of Inst. of Math & Comp. Sci. (Math. Ser.), Vol.11, No 2, 97-102, 1998.

[4] Ikawa, T., On Curves and Submanifolds in an indefinite-Riemannian Manifold. Tsukuba Math. J., Vol.9, 353-371, 1985.

[5] O'Neill, B., Semi-Riemannian Geometry, with Applications to Relavity. Academic Press. New York. 1988.

[6] Struik, D. J., Lectures on Clasical Differential Geometry, New york,1951.

Current address: E. SOYTURK, K. ILARSLAN Department of Mathematics, Faculty of Science-Literature, Afyon Kocatepe University, 03200,ANS Campus, Afyon-Turkey, D. SAĞLAM. Department of Mathematics, Faculty of Science, Ankara University, 06100,, Tandogan, Ankara -Turkey

INFORMATION FOR AUTHORS

Series A1of COMMUNICATIONS accepts original articles written in English in the fields of Mathematics and Statistics. Review articles written by eminent scientists can also be invited by the Editor.

Three copies of manuscripts must be submitted in AMS Article Tex format. They should contain a discriptive title, the author's name, an address, an abstract and the AMS Subject classification with a list of key words and phrases.

It is a fundamental condition that the articles submitted have not been previously published and will not be simultaneously submitted and published elsewhere.

After the manuscript has been accepted for publication, i.e., after referre-recommended revision completed, the author will not be permitted to make any new additions to the manuscript.

After the acceptance of the manuscripts for publication you will be asked to submit a revised electronic copy of the manuscript written in AMS tex format. Then it will solely be the author's responsibility for any typographical mistakes which occur in their article as it appears in the Journal.

1. The title of the paper should not be long but informative. A running title must also be given.
2. The abstract should not exceed 200 words and it should condense the essential features of the articles with the focus on the major advances in the field.
3. References must be listed in alphabetical order. Within the manuscript, refer to the references by their given number in brackets. They should be styled and punctuated according to the following examples:
   [1] Kelley, J.L. *General Topology*, 1970, New York, Van Nostrand.
   [2] Maddox, I,J., Some inclusion theorems, Proc.Glasgow Math. 6, (1964)161-168.
   Abstracts, unpublished data and personal communications should not be given among the
   references, but they may be mentioned in the text.
4. Footnotes, except a possible acknowledgement of assistance or financial support on the first page, should be avoided by being incorporated into the text.
5. All tables and figures must be numbered consecutively throughout the paper (Table 1, Figure 2) and also have a caption or legend.
6. Acknowledgements should be given as short as possible at the end of the text.

7. 25 reprints will be provided free of charge.
8. Irrespective of their acceptance, manuscripts will not be returned to the authors.
9. Each paper is due to be charged for the amount determined by the administiration each year.
10. Texts should be sent to

Prof.Dr.Öner Çakar
Editor
Communications
University of Ankara
Faculty of Science
06100 Tandoğan
Ankara / TURKEY
E-mail: commun@science.ankara.edu.tr
http://math.science.ankara.edu.tr/dergi/dergi.htm