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# LINK BETWEEN CONVEX ORDERING AND LTD ORDERING

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ABSTRACT. Two new characterizations of left tail decreasing (LTD) ordering for bivariate distributions with the identical marginals are given. Some properties of this ordering are discussed by using new characterization. Some comparisons of convex ordering and LTD ordering are made, along with applications to order statistics and Farlie- Gumbel- Morgerstern family of distributions.

## 1. INTRODUCTION

Convexity is a useful property to compare two real-valued functions, since functions with such a property protrude at end points of the interval on which they are defined. The notion of one function being convex relative to another is first introduced by Hardy *et al.* (1934). On the other hand, Chan *et al.* (1985) have utilized Hardy *et al.*'s ideas in a univariate statistical theory. Yilmaz and Tuncer (2004) have made to carry this ordering notion over to bivariate case.

For the purpose, we consider $\mathcal{F}_c(F_X, F_Y)$ that represents a bivariate family of continuous and increasing distribution functions with the identical marginals $F_X$ and $F_Y$. Also, D is assumed to denote a subset of $\Re$ such as $\{y : F_Y(y) > 0\}$ and also $\mathcal{K} = \{\mathcal{D} \cap \Re_2\}$ is given. For the pair $(X, Y)$ jointly distributed as $F(x, y) \in \mathcal{F}_c$, the conditional distribution of $X$ given $\mathcal{B}_y = \{Y \leq y\}$ is defined on $\mathcal{K}$ as the point function $\frac{F(x,y)}{F_Y(y)} = P(X \leq x \mid \mathcal{B}_y) = F_{\mathcal{B}_y}(x)$, or shortly $F_y(x)$.

Let $F_0(x, y) = F_X(x)F_Y(y)$ denote the independence case for the family $\mathcal{F}_c$. Since $F_0$ is unique in $\mathcal{F}_c$, it can be viewed as reference point and the distributions that belong the family can be ordered according to their positions relative to this reference point. This undoubtedly will also allow determination of dependence within the family.

The purpose of this note is to give new characterization of left tail decreasing (LTD) ordering introduced by Averous and Bernadet (2000) and to compare with convex ordering.

The characterizations of LTD ordering are presented in Section 2, some properties are also described. LTD ordering and convex ordering are compared in Section 3. Some dependence concepts are detected by using notion of convex ordering which is stronger than LTD ordering. Illustrative examples appear in section 4 to show the relation between convex ordering and LTD ordering.

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## 2. CHARACTERIZATION OF LTD ORDERING

Well known bivariate positive dependence notion is the left tail decreasing (LTD) concept introduced by Esary and Proschan (1972). Given a bivariate random vector $(X, Y)$, they defined $X$ to be left tail decreasing in $Y$ if $P(X \leq x \mid Y \leq y)$ is decreasing in $y$. Averous and Bernadet (2000) extended this concept by analogy with the work of Capéraà and Genest (1990); let $F_{y'y}$ stand for $F_{y'} \circ F_y^{-1}$ with $y < y'$ where $F_y^{-1}$ is the inverse function. If $F_y$ is continuous and increasing on its support for all $y$, then

$$\begin{aligned} F \text{ is } LTD &\iff \forall y < y' \; \forall x \;\; F_{y'}(x) \leq F_y(x) \\ &\iff \forall y < y' \; \forall 0 \leq u \leq 1 \;\; F_{y'y}(u) \leq u. \end{aligned}$$

Accordingly, they defined LTD ordering for two arbitrary bivariate distribution functions,

$$F <_{LTD} G \iff \forall y < y' \; \forall 0 \leq u \leq 1 \;\; G_{y'y}(u) \leq F_{y'y}(u).$$

The stronger notion of positive dependence ordering is stochastic increasing (SI), or positive regression dependence (PRD) ordering. An extension of this ordering is presented by Yanagimoto and Okamoto (1969); $F <_{SI} G$, if holds $F^{-1}(u \mid x') \geq F^{-1}(v \mid x)$ implies $G^{-1}(u \mid x') \geq G^{-1}(u \mid x)$ for any $x' > x$ and $0 < u, v < 1$. On the other hand, Hollander *et al.* (1987) proposed that $F \prec^{SI\ HPS} G$ if and only if $P_F(X \leq x \mid Y = y) - P_G(X \leq x \mid Y = y)$ is non decreasing in $y$ for all $x$. These lead us to propose new characterizations of LTD ordering by analogy. Following lemma which is the first result of this note gives an alternative form of the LTD ordering.

**Lemma 2.1.** *Let $F$ and $G \in \mathcal{F}_c(F_X, F_Y)$ then $F <_{LTD} G$ if and only if*

$$G_y(x') \geq F_y(x) \Rightarrow F_{y'}(x) \geq G_{y'}(x')$$

*for all $y, y'$ $x, x'$ with $y < y'$.*

*Proof.* (Sufficiency). Suppose that there exist $y < y'$ and $0 \leq u \leq 1$ such that $F_{y'y}(u) < G_{y'y}(u)$ and if $x' = G_y^{-1}(u)$ and $x = F_y^{-1}(u)$, then $G_y(x') = F_y(x)$ and $F_{y'}(x) < G_{y'}(x')$.

(Necessity). Suppose that $F_y(x) = u$ $\forall 0 \leq u \leq 1$ *i.e*, $x = F_y^{-1}(u)$, thereby $G_y(x') \geq u$ *i.e*, $x' \geq G_y^{-1}(u)$ such that $G_y(x') \geq G_{y'y}(u)$ where $F_{y'}(x) \geq G_{y'}(x') \geq G_{y'y}(u)$. Hence $F_{y'y}(u) \geq G_{y'y}(u)$. □

**Lemma 2.2.** *Under the same assumption above lemma, $F <_{LTD} G \iff F_y(x) - G_y(x)$ is non decreasing in $y$ for all $x$.*

*Proof.* From the above lemma, if $F <_{LTD} G$ then $F_y(x) - G_y(x') \leq 0 \Rightarrow F_{y'}(x) - G_{y'}(x') \geq 0$ holds for all $y < y'$ and $x, x'$. Hence for $x = x'$, $F_y(x) - G_y(x) \leq F_{y'}(x) - G_{y'}(x)$. Necessity part is similar. □

**Proposition 1.** *For any $F \in \mathcal{F}_c$, $F$ is $LTD$ if and only if, $F_0 <_{LTD} F$.*

*Proof.* $F$ is $LTD$ then $F_{y'}(x) \leq F_y(x)$ for all $y, y'$ $x$ with $y < y'$. Hence $F_{y'y}(u) \leq u = F_{0_{y'y}}(u) = F_Y(F_Y^{-1}(u))$, $F_0 <_{LTD} F$. Similarly, $F_0 <_{LTD} F$ then $F_y(x') \geq F_Y(y) \Rightarrow F_{y'}(x') \leq F_Y(y)$, it follows that $F_{y'}(x') \leq F_y(x')$ for all $y < y'$ and $x'$. □

**Proposition 2.** *Let $F$ and $G \in \mathcal{F}_c$, then $F <_{LTD} G$ implies $G$ is more positively quadrant dependent than $F$, i.e $F <_{PQD} G$.*

*Proof.* For any $F \in \mathcal{F}_c$, $F$ is a non decreasing function of each argument, namely $F(x',y')-F(x',y)-F(x,y')+F(x,y) \geq 0$ holds for all $x < x'$ and $y < y'$. $G_y(x') \geq F_y(x)$ and $F_{y'}(x) \geq G_{y'}(x')$ exist for all $y < y'$ and $x$, $x'$. Assume that $x < x'$ then $F(x',y') - F(x',y) - G(x',y') + G(x',y) \geq 0$, under the same margins condition, limiting $y' \to \infty$, $\lim_{y' \to \infty} F(x',y') - G(x',y') = 0$ then $G(x',y) - F(x',y) \geq 0$ hence $G(x',y) \geq F(x',y)$ for all $(x',y)$, *i.e* $F <_{PQD} G$. □

## 3. CONVEX ORDERING VERSUS LTD ORDERING

Before stating the results in this section we give the definition of the convex ordering introduced by Yilmaz and Tuncer (2004) for bivariate distribution functions. Let $F$,$G \in \mathcal{F}_c$, then for any $y \in \mathcal{D}$, we accordingly define $\phi_y(t) \equiv F_y \circ G_y^{-1}(t) : [0,1] \to [0,1]$. If the function $\phi_y(t)$ convex on $(0,1)$ for some $y \in \mathcal{D}$, then $F_y$ is said to be more convex relative to $G_y$ and is shortly expressed as $F_y \succsim G_y$. Similarly, if the function $\phi_y(t)$ convex on $(0,1)$ for all $y \in \mathcal{D}$, it is denoted as $F_\mathcal{D} \succsim G_\mathcal{D}$. Following lemma is the first result for detecting $LTD$ dependence.

**Lemma 3.1.** *For any $F \in \mathcal{F}_c$, if $F_{0_\mathcal{D}} \succsim F_\mathcal{D}$ then $Y$ is $LTD$ in $X$.*

*Proof.* Recall that a function convex (concave) if and only if, its inverse is concave (convex). Thus $\psi_y(t) = \phi_y^{-1}(t) = F_y F_X^{-1}(t)$ is concave on $(0,1)$. Hence $\frac{\psi_y(t)}{t}$ is non increasing in $t$, *i.e*, $\frac{F_y F_X^{-1}(t)}{t} \geq \frac{F_y F_X^{-1}(t')}{t'}$, for all $t,t' \in (0,1)$ with $t < t'$. By setting $F_X^{-1}(t) = x$ and $F_X^{-1}(t') = x'$, $x < x'$ then the latter inequality can be rewritten as

$$\frac{F_y(x)}{F_X(x)} \geq \frac{F_y(x')}{F_X(x')}, \tag{3.1}$$

using definition of $F_y(x) = \frac{F(x,y)}{F_y(y)}$, then (3.1) implies $F_x(y) \geq F_{x'}(y)$ for all $x, x'$ $y$ with $x < x'$. □

In the next lemma, we obtain the strongest notion of positive dependence which is called totally positive of order two ($TP_2$) introduced by Karlin (1968). It is emphasized that $PQD$, $LTD$ ($RTI$) and $SI$ ($PRD$), well known positive dependence concepts, are exhibited since $F$ is $TP_2$ (Barlow and Proschan (1975), pp. 143).

**Lemma 3.2.** *For any $F \in \mathcal{F}_c$, if $F_{y'} \succsim F_y$ for all $y < y'$, $y,y' \in \mathcal{D}$ then $F$ is $TP_2$ and $F$ is $LTD$.*

*Proof.* $F_{y'y}(u)$ is convex on $(0,1)$ with $F_{y'y}(0) = 0$ and $F_{y'y}(1) = 1$. Hence $F_{y'y}(u) \leq u$ for all $u \in [0,1]$, *i.e*, $F$ is $LTD$. Furthermore, $\frac{F_{y'y}(u)}{u}$ is non decreasing in $u$. Since $F_y^{-1}(u)$ is increasing and continuous $F_y^{-1}(u) = x$ and $F_y^{-1}(u') = x'$, $x < x'$ can be taken, then $\frac{F_{y'}(x)}{F_y(x)} \leq \frac{F_{y'}(x')}{F_y(x')}$ can be written. It follows that $F(x',y)F(x,y') \leq F(x',y')F(x,y)$ is satisfied for all $x < x'$ and $y < y'$, $(x,y) \in \Re_2$, *i.e* $F$ is $TP_2$. □

Following lemma gives a relation between two ordering concepts:

**Lemma 3.3.** *For any $F,G \in \mathcal{F}_c$, $F_\mathcal{D} \succsim G_\mathcal{D}$ implies $F <_{PQD} G$ and $F <_{LTD} G$.*

*Proof.* $\phi_y(t) = F_y G_y^{-1}(t)$ is convex for all $y \in \mathcal{D}$, then $\phi_y(t) \leq t$, since $\phi_y(0) = 0$ and $\phi_y(1) = 1$. Using continuity of $G_y^{-1}(t)$, take $G_y^{-1}(t) = x$ hence $F_y(x) \leq G_y(x)$. This implies $F(x,y) \leq G(x,y)$ for all $(x,y) \in \Re_2$.

Proof of second part is similar to proof of lemma 3; if $F_D \succsim G_D$ then $\frac{F_y(x)}{G_y(x)}$ is non decreasing in $x$. Hence $\frac{F_x(y)}{G_x(y)} \le \frac{F_{x'}(y)}{G_{x'}(y)}$ can be obtained. Recalling the first lemma, $F <_{LTD} G \Leftrightarrow G_x(y') \ge F_x(y) \Rightarrow F_{x'}(y) \ge G_{x'}(y')\ \forall x < x',\ y, y'$ then for $y = y'$, $\frac{F_x(y)}{G_x(y)} \le 1 \Longrightarrow \frac{F_{x'}(y)}{G_{x'}(y)} \ge 1$ can be written, which implies $\frac{F_x(y)}{G_x(y)} \le \frac{F_{x'}(y)}{G_{x'}(y)}$. Consequently, $F <_{LTD} G$. □

*Remark* 3.4. $X$ $LTD$ in $Y$ and $Y$ $LTD$ in $X$ differ from each other. If one holds, the other does not need to hold.

## 4. Applications

There are two applications, the first illustrated by Averous and Bernadet (2000) is given for univariate case, the second is given for bivariate case:

This can be viewed as an illustration from the theory of order statistics, consider two random samples $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ from univariate distributions $F$ and $G$, respectively. If $X_{(i)}$ and $Y_{(i)}$ denote $i$ th smallest order statistic from their respective sample, let $K(x,y)$ and $H(x,y)$ stand for the distributions of $(X_{(1)},\ X_{(n)})$ and $(Y_{(1)},\ Y_{(n)})$, respectively. It is then straightforward to prove that

$$F \succsim G \Rightarrow G \prec_{rh} F \Leftrightarrow H \prec_{LTD} K$$

where "$\prec_{rh}$" denotes reversed hazard order (see Shaked and Shanthikumar, pp. 24).

$$K_y(x) = P_F\left(X_{(1)} \le x \mid X_{(n)} \le y\right) = 1 - \left[1 - \frac{F(x)}{F(y)}\right]^n, \qquad x \le y$$

and

$$H_y(x) = P_G\left(Y_{(1)} \le x \mid Y_{(n)} \le y\right) = 1 - \left[1 - \frac{G(x)}{G(y)}\right]^n, \qquad x \le y$$

are defined. Hence

$$K_{y'y}(u) = 1 - \left[1 - \frac{F(y)}{F(y')}\left(1 - (1-u)^{\frac{1}{n}}\right)\right]^n,$$

and

$$H_{y'y}(u) = 1 - \left[1 - \frac{G(y)}{G(y')}\left(1 - (1-u)^{\frac{1}{n}}\right)\right]^n, \qquad y < y'.$$

$F \succsim G \Rightarrow G \prec_{rh} F$. $\phi(t) = FG^{-1}(t)$ is convex on $(0,1)$ then, $\frac{F(y)}{G(y)}$ is non decreasing in $y$, $\frac{F(y)}{G(y)} \le \frac{F(y')}{G(y')}$, $y < y'$, it can be rewritten as

$$\frac{G(y')}{F(y')} \le \frac{G(y)}{F(y)},$$

which gives us $G \prec_{rh} F$.

$G \prec_{rh} F \Leftrightarrow H \prec_{LTD} K$. By using the latter inequality, we can also rewrite that form $\frac{F(y)}{F(y')} \le \frac{G(y)}{G(y')}$. Hence $K_{y'y}(u) \le H_{y'y}(u)$. Similarly, $H \prec_{LTD} K \Rightarrow G \prec_{rh} F$.

Let $F$ and $G$ be continuous and increasing on their supports and belong to FGM family of distributions, where

$$\begin{aligned} F(x,y) &= F_X(x)F_Y(y)\left[1 + \alpha\left(1 - F_X(x)\right)\left(1 - F_Y(y)\right)\right] \\ G(x,y) &= F_X(x)F_Y(y)\left[1 + \beta\left(1 - F_X(x)\right)\left(1 - F_Y(y)\right)\right], \end{aligned}$$

with $\alpha,\ \beta \in [-1,1]$. $F_\alpha$ and $F_\beta$ stand for conditional distribution functions with $\{Y \le y\}$, *i.e.*,

$$F_\alpha(x) = F_X(x)\left[1+\alpha\left(1-F_X(x)\right)\left(1-F_Y(y)\right)\right]$$
$$F_\beta(x) = F_X(x)\left[1+\beta\left(1-F_X(x)\right)\left(1-F_Y(y)\right)\right].$$

then for all $\beta > \alpha$, $F_\alpha \succsim F_\beta$ implies $F <_{LTD} G$.

For $\beta \neq 0$, $\phi_y(t)$ can be obtained as

$$\phi_y(t) = F_\alpha F_\beta^{-1}(t) = \begin{cases} \frac{k_\alpha}{k_\beta}t + \frac{(k_\beta-k_\alpha)}{2k_\beta^2}\left[1+k_\beta-\sqrt{\Delta}\right] & ,\ \beta \neq 0 \\ t\left[1+k_\alpha(1-t)\right] & ,\ \beta = 0 \end{cases},$$

where $k_i = i\,(1-F_Y(y))$, $i=\alpha,\beta$ and $\Delta = (1+k_\beta)^2 - 4k_\beta t$. Obviously, $\phi_y(t)$ is continuous and twice differentiable function of $t$, so that $\dfrac{\partial^2\phi_y(t)}{\partial t^2} \ge 0$. Hence $\phi_y(t)$ is convex, *i.e*, $F_\alpha \succsim F_\beta$. From Lemma 5, this implies $F <_{LTD} G$. If someone wants to desired result, $F_{x'x}(u)$ and $G_{x'x}(u)$ defined as

$$F_{x'x}(u) = \begin{cases} u\frac{1-F_X(x')}{1-F_X(x)} + \alpha\left(F_X(x')-F_X(x)\right)\left[\frac{1+m_\alpha-\sqrt{\Delta_\alpha}}{2m_\alpha^2}\right] & ,\alpha \neq 0 \\ u & ,\alpha = 0 \end{cases}$$

and

$$G_{x'x}(u) = \begin{cases} u\frac{1-F_X(x')}{1-F_X(x)} + \beta\left(F_X(x')-F_X(x)\right)\left[\frac{1+m_\beta-\sqrt{\Delta_\beta}}{2m_\beta^2}\right] & ,\beta \neq 0 \\ u & ,\beta = 0 \end{cases}$$

where $m_i = i\,(1-F_X(x))$ and $\Delta_i = (1+m_i)^2 - 4m_i u$, $i=\alpha,\beta$. Let $H_{x'x}$ stand for $F_{x'x} - G_{x'x}$, then $H_{x'x}(0) = 0$ and $H_{x'x}(1) = 0$. There are only two roots of $H_{x'x}$ for all $\alpha,\beta$ and $x,x'$. Stationary point of $H_{x'x}(u)$ is $u^* = \frac{1}{2} + \frac{(\alpha+\beta)(1-F_X(x))}{4}$, $u^* \in [0,1]$. This point is unique for fixed $\alpha,\beta$ and $x$ and $\left.\frac{d^2H_{x'x}(u)}{du^2}\right|_{u=u^*} \le 0$, maximizes $H_{x'x}(u)$. Therefore $H_{x'x}(u)$ is positive on $[0,1]$. Hence follows the result.

ÖZET:Bu çalışmada, ortak marjinallere sahip iki boyutlu dağılımlar için tanımlanmış olan sol kuyruktan azalan bağımlılık sıralaması için iki yeni karakterizasyon verilmiştir. Bu yeni karakterizasyonların özellikleri belirtilmiş, bazı konveks dönüşümler yardımı ile bu bağımlılık sıralaması ile konveks sıralama arasındaki ilişkiden uygulamalı örnekler verilerek bahsedilmiştir.

## REFERENCES

[1] Averous, J. and Dortet-Bernadet, J., LTD and RTI dependence orderings. The Canadian Journal of Statistics., Vol 28. No:1, (2000). 151-157.

[2] Barlow, R. and Proschan, F., Statistical Theory of Reliability and Life Testing, 1975. Holt, Rinehart and Winston, New York.

[3] Capéraà, P. and Genest, C., Concepts de dépendance et ordres stochastiques pour des lois bidimensionnelles. The Canadian Journal of Statistics., Vol 18, (1990). 315-326.

[4] Chan, W., Proschan, F. and Sethuraman, J., Convex ordering among funcions, with applications to reliability and mathematical statistics. Topics in Statistical Dependence (Somerset, PA, 1987), (1985). 121-134. IMS Lecture Notes Monogr. Ser.,16 Inst. Math. Statistics, Hayward, CA, 1990.

[5] Esary, J.D. and Proschan, F., Relationships among some concepts of bivariate dependence. Annals of Mathematical Statistics, 43; (1972). 651-665.
[6] Hardy, G.H., Littlewood, J.E. and Pólya, G., Inequalities, 1934. Cambridge at The University Press.
[7] Hollander M., Proschan F. and Scaning J., Information, censoring, and dependence. IMS Lecture Notes Monograph Ser., 16, (1990). 315-326.
[8] Karlin, S., Total Positivity, 1968. Standford University Press.
[9] Shaked M. and Shanthikumar, J., Stochastic Orders and Their Applications, 1994. Academic Press, New York.
[10] Yanagimoto, T. and Okamoto, M., Stochastically larger component of a random variable. Ann. Inst. Statist. Math., 21, (1969). 489-506.
[11] Yılmaz, M., Tuncer, Y., Bir ve İki Boyutlu Dağılımlar için Konveks Sıralama, (2004). "İstatistik Günleri 2004 Sempozyumu", Kuşadası-Aydın.

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