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ASYMPTOTIC UNBIASEDNESS OF SOME ESTIMATORS FOR RENEWAL AND VARIANCE FUNCTIONS

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ABSTRACT

In this paper some parametric and nonparametric estimators for the renewal function and variance function of a renewal process are considered. The asymptotic unbiasedness of these estimators is investigated.

KEYWORDS Renewal function; variance function; asymptotic unbiasedness.

1. INTRODUCTION

Renewal process arises in a wide variety of applications of probabilistic models such as reliability theory, inventory theory, continuous sampling plans, insurance applications and sequential analysis. The renewal function and variance function are basic tools in many of the applications involving renewal processes.

For the renewal function and variance function of a renewal process, Frees [3,4], Lin [6] and Aydoğdu and Öztürk [1] have presented some parametric and nonparametric estimators. Their consistency properties have been investigated [1,3,4], but it is still not known whether these estimators are asymptotically unbiased. In this study, the asymptotic unbiasedness of the estimators is established.

2. SOME ESTIMATORS FOR RENEWAL AND VARIANCE FUNCTIONS

Let $(X_k)_{k=1,2,...}$ be an independent, identically distributed sequence of positive random variables with distribution function F. Assume that F has mean μ and finite variance σ^2. For $t \geq 0$, the renewal process $\{N(t), t \geq 0\}$ is defined by

$$N(t) = \sup\{k : S_k \leq t\},$$

where $S_k = X_1 + ... + X_k$. $N(t)$ is the number of renewals up to time t. The renewal function and the variance function of the renewal process $\{N(t), t \geq 0\}$ are $M(t) = E(N(t))$, $t \geq 0$ and $V(t) = Var(N(t))$, $t \geq 0$, respectively. It is well known that

$$M(t) = \sum_{k=1}^{\infty} F^{k*}(t) \ , \ t \geq 0 \qquad (1)$$

and

$$V(t) = M(t)(1-M(t)) + 2M * M(t) \ , \ t \geq 0 \tag{2}$$

where $*$ denotes Stieltjes convolution and F^{k*} is the k-fold Stieltjes convolution of F [5]. Since $M * M(t) = \sum_{k=1}^{\infty} kF^{(k+1)*}(t)$, we can also write (2) as

$$V(t) = 2\sum_{k=1}^{\infty} kF^{k*}(t) - \sum_{k=1}^{\infty} F^{k*}(t)(1 + \sum_{k=1}^{\infty} F^{k*}(t)), t \geq 0. \tag{3}$$

For each fixed $t \geq 0$, the random variable $N(t)$ has finite moments of all orders [8]. Hence, $M(t)$ and $V(t)$ are finite for all $t \geq 0$.

Let $\theta_1, \theta_2 ..., \theta_r$ be the parameters of F and $\hat{F}_n = F(\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_r)$ be the resulting estimator of F based on a random sample $X_1, X_2, ..., X_n$ where $\hat{\theta}_1, \hat{\theta}_2 ..., \hat{\theta}_r$ are some estimators of $\theta_1, \theta_2 ..., \theta_r$, respectively. When the distribution function F is known but the parameters are unknown, Frees [3] suggests (motivated from (1)) the parametric estimator

$$\hat{M}_{1n}(t) = \sum_{k=1}^{\infty} \hat{F}_n^{k*}(t) \ , \ t \geq 0.$$

Similarly, from (3), a parametric estimator for the variance function $V(t)$ is

$$\hat{V}_{1n}(t) = 2\sum_{k=1}^{\infty} k\hat{F}_n^{k*}(t) - \sum_{k=1}^{\infty} \hat{F}_n^{k*}(t)(1 + \sum_{k=1}^{\infty} \hat{F}_n^{k*}(t)) \ , \ t \geq 0.$$

Suppose that $\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_r$ are strongly consistent estimators of $\theta_1, \theta_2, ..., \theta_r$ and the distribution function F is absolutely continuous with probability density function f. If f is continuous in each parameter $\theta_i, i = 1, ..., r$ then $\hat{M}_{1n}(t)$ and $\hat{V}_{1n}(t)$ are strongly consistent estimators for each fixed t [3,1], that is,

$$\hat{M}_{1n}(t) \xrightarrow[n \to \infty]{} M(t) \text{, with probability 1} \tag{4}$$

and

$$\hat{V}_{1n}(t) \xrightarrow[n \to \infty]{} V(t) \text{, with probability 1.} \tag{5}$$

When the parametric form of F is not known, a nonparametric estimator motivated from (1) is defined as

$$\hat{M}_{2n}(t) = \sum_{k=1}^{m} F_n^{k*}(t) \ , \ t \geq 0$$

where

$$F_n^{k*}(t) = \frac{1}{\binom{n}{k}} \sum I(X_{i_1} + ... + X_{i_k} \leq t),$$

the sum extends over all subsamples without replacement of size k from $\{X_1, ..., X_n\}$ and $m = m(n)$ is a positive integer depending on n such that $m \leq n$ and $m \uparrow \infty$ as

$n \uparrow \infty$ [3,4]. The design parameter m was introduced by Frees [3] to reduce the amount of computation, since, in the case of $m=n$, one has to evaluate $2^n - 1$ indicators. The design parameter m depends on n in order of the asymptotic results to become valid. From (3), a nonparametric estimator for the variance function $V(t)$ is

$$\hat{V}_{2n}(t) = 2\sum_{k=1}^{m_1} kF_n^{k*}(t) - \sum_{k=1}^{m} F_n^{k*}(t)(1+\sum_{k=1}^{m} F_n^{k*}(t)), t \geq 0$$

where $m_1 = m_1(n)$ is a positive integer depending on n such that $m_1 \leq n$ and $m_1 \uparrow \infty$ as $n \uparrow \infty$ [1].

The estimator $\hat{M}_{2n}(t)$ is weakly consistent for the case $m < n$ and it is strongly consistent for the case $m=n$ [4]. $\hat{V}_{2n}(t)$ is weakly consistent estimator of $V(t)$ in the case of $m<n$ and $m_1 < n$ and in the case of $m = m_1 = n$ this estimator is strongly consistent [1].

Another nonparametric estimator for the renewal function $M(t)$ motivated from (1) is defined in Frees [4] by the formula

$$\hat{M}_{3n}(t) = \sum_{k=1}^{\infty} \tilde{F}_n^{k*}(t), t \geq 0$$

where $\tilde{F}_n^{k*}$ is the k-fold Stieltjes convolution of the empirical distribution function F_n. $\hat{M}_{3n}(t)$ is called the empirical renewal function by Schneider et al. [7]. Correspondingly, from (3),

$$\hat{V}_{3n}(t) = 2\sum_{k=1}^{\infty} k\tilde{F}_n^{k*}(t) - \sum_{k=1}^{\infty} \tilde{F}_n^{k*}(t)(1+\sum_{k=1}^{\infty} \tilde{F}_n^{k*}(t)), t \geq 0$$

is a nonparametric estimator for the variance function $V(t)$. $\hat{V}_{3n}(t)$ is called the empirical variance function. The estimators $\hat{M}_{3n}(t)$ and $\hat{V}_{3n}(t)$ are strongly consistent [1].

3. ASYMPTOTIC UNBIASEDNESS

In the previous section we have considered some consistency properties of the estimators. Their relative performance empirically for small sample sizes has been investigated by Aydoğdu and Öztürk [2]. In this section, the asymptotic unbiasedness of the estimators is established.

Let us first investigate the unbiasedness and asymptotic unbiasedness of the estimators $\hat{M}_{1n}(t)$ and $\hat{V}_{1n}(t)$ for each fixed t ($t \geq 0$). Let F be the exponential distribution function with parameter $\theta > 0$, that is, $F(x) = 1 - e^{-x/\theta}$, $x \geq 0$. Consider a

random sample $X_1,...,X_n$ of size n from this distribution. $\overline{X}_n$ is strongly consistent estimator for θ. Since $M(t) = V(t) = t/\theta$,

$$\hat{M}_{1n}(t) = \frac{t}{\overline{X}_n}, t \geq 0$$

and

$$\hat{V}_{1n}(t) = \frac{t}{\overline{X}_n}, t \geq 0.$$

It is clear that $\hat{M}_{1n}(t)$ and $\hat{V}_{1n}(t)$ are strongly consistent estimators of $M(t)$ and $V(t)$ for each fixed t. We have

$$E\left(\frac{t}{\overline{X}_n}\right) = \int_0^\infty \frac{nt}{\Gamma(n)\theta^n} x^{n-2} e^{-x/\theta} dx$$

$$= \frac{nt}{(n-1)\theta}, n > 1.$$

Then, both $\hat{M}_{1n}(t)$ and $\hat{V}_{1n}(t)$ are asymptotically unbiased estimators even though they are not unbiased. Therefore, $\hat{M}_{1n}(t)$ and $\hat{V}_{1n}(t)$ are not in general unbiased. Their asymptotic unbiasedness is established under some conditions as given by the following theorem.

Theorem 1. *Let* $f = f(\theta_1,\theta_2,...,\theta_r)$ *be probability density function of* $F = F(\theta_1,\theta_2,...,\theta_r)$ *for F absolutely continuous. Suppose that f is continuous in each parameter* θ_i, *i=1,...,r and* $\hat{\theta}_1,\hat{\theta}_2,...,\hat{\theta}_r$ *are strongly consistent estimators of* $\theta_1,\theta_2,...,\theta_r$, *respectively. If* $F(t_0) < 1$ *for any* $t_0 \geq 0$ *then for* $t \leq t_0$,

$$\lim_{n\to\infty} E(\hat{M}_{1n}(t)) = M(t)$$

and

$$\lim_{n\to\infty} E(\hat{V}_{1n}(t)) = V(t),$$

that is, $\hat{M}_{1n}(t)$ *and* $\hat{V}_{1n}(t)$ *are asymptotically unbiased estimators of M(t) and V(t) for* $t \leq t_0$.

Proof. Since $F(t_0) < 1$, $\hat{F}_n(t_0) < 1$, then there exists some $c > 0$ such that $\hat{F}_n(t_0) \leq \frac{c}{1+c}$. Therefore, $\hat{F}_n^{k*}(t_0) \leq (\frac{c}{1+c})^k$ for all $k \geq 1$ and so

$$\hat{M}_{1n}(t_0) \leq c. \tag{6}$$

From the expression (2) of $V(t)$, we have

$$\hat{V}_{1n}(t) = \hat{M}_{1n}(t)(1 - \hat{M}_{1n}(t)) + 2\hat{M}_{1n} * \hat{M}_{1n}(t), t \geq 0.$$

It is clear that $0 \le \hat{M}_{1n} * \hat{M}_{1n}(t) \le (\hat{M}_{1n}(t))^2$. Hence, it follows from (6) that

$$\hat{V}_{1n}(t) \le c(1+c). \tag{7}$$

Considering (4), (6), (5) and (7), from the bounded convergence theorem we can obtain that, for $t \le t_0$

$$\lim_{n\to\infty} E(\hat{M}_{1n}(t)) = M(t)$$

and

$$\lim_{n\to\infty} E(\hat{V}_{1n}(t)) = V(t).$$

Thus, the proof is completed.

Consider the nonparametric estimator $\hat{M}_{2n}(t)$ for $M(t)$. It is clear that $F_n^{k*}(t)$ is an unbiased estimator of $F^{k*}(t)$ for each fixed t. Then,

$$E(\hat{M}_{2n}(t)) = \sum_{k=1}^{m} F^{k*}(t)$$

and

$$\begin{aligned}\lim_{n\to\infty} E(\hat{M}_{2n}(t)) &= \sum_{k=1}^{\infty} F^{k*}(t) \\ &= M(t).\end{aligned}$$

Therefore, the estimator $\hat{M}_{2n}(t)$ is asymptotically unbiased even though it is not unbiased.

Let us now investigate the unbiasedness and asymptotic unbiasedness of the estimator $\hat{V}_{2n}(t)$ for each fixed t. From the unbiasedness of $F_n^{k*}(t)$, it is easily seen that

$$E(\hat{V}_{2n}(t)) = 2\sum_{k=1}^{m_1} kF^{k*}(t) - \sum_{k=1}^{m} F^{k*}(t) - (\sum_{k=1}^{m} F^{k*}(t))^2 - \mathrm{Var}(\sum_{k=1}^{m} F_n^{k*}(t)).$$

Hence, $\hat{V}_{2n}(t)$ is not in general unbiased. The asymptotic unbiasedness of this estimator is established by the following theorem.

Theorem 2. *Suppose that either* $n = O(m^{2r-4})$ *for* $r > 2$ *or* $\log n = o(m)$. *Then, for each fixed t,*

$$\lim_{n\to\infty} E(\hat{V}_{2n}(t)) = V(t),$$

that is, $\hat{V}_{2n}(t)$ *is asymptotically unbiased estimator of V(t).*

Proof. Since $\mathrm{Var}(\sum_{k=1}^{m} F_n^{k*}(t)) = \sum_{k,r=1}^{m} \mathrm{Cov}(F_n^{k*}(t), F_n^{r*}(t))$,

$$E(\hat{V}_{2n}(t)) = 2\sum_{k=1}^{m_1} kF^{k*}(t) - \sum_{k=1}^{m} F^{k*}(t) - (\sum_{k=1}^{m} F^{k*}(t))^2 - \sum_{k,r=1}^{m} \mathrm{Cov}(F_n^{k*}(t), F_n^{r*}(t)).$$

Define

$$\gamma_{kr}(i) = \mathrm{Cov}(F^{(k-i)*}(t-(X_1+\ldots+X_i)), F^{(r-i)*}(t-(X_1+\ldots+X_i))).$$

It is known from Frees [4] that

$$\mathrm{Cov}(F_n^{k*}(t), F_n^{r*}(t)) = \frac{1}{\binom{n}{k}} \sum_{i=1}^{k} \binom{r}{i}\binom{n-r}{k-i}\gamma_{kr}(i)$$

and if $n = O(m^{2r-4})$ for $r > 2$ or $\log n = o(m)$ then

$$\lim_{n\to\infty} \sum_{k,r=1}^{m} kr\gamma_{kr}(1) < \infty \tag{8}$$

and

$$\lim_{n\to\infty} \sum_{k,r=1}^{m} \left\{ \frac{n}{\binom{n}{k}} \sum_{i=1}^{k} \binom{r}{i}\binom{n-r}{k-i}\gamma_{kr}(i) - kr\gamma_{kr}(1) \right\} = 0. \tag{9}$$

Let $a_n = \sum_{k,r=1}^{m} \frac{1}{\binom{n}{k}} \sum_{i=1}^{k} \binom{r}{i}\binom{n-r}{k-i}\gamma_{kr}(i)$ and $b_n = \sum_{k,r=1}^{m} kr\gamma_{kr}(1)$. From (8) and (9), it is clear that $\lim_{n\to\infty} b_n < \infty$ and $\lim_{n\to\infty} na_n - b_n = 0$. Then, $\lim_{n\to\infty} a_n = 0$, that is,

$$\lim_{n\to\infty} \sum_{k,r=1}^{m} \frac{1}{\binom{n}{k}} \sum_{i=1}^{k} \binom{r}{i}\binom{n-r}{k-i}\gamma_{kr}(i) = 0.$$

Further, $\lim_{n\to\infty} \sum_{k=1}^{m} F^{k*}(t) = \sum_{k=1}^{\infty} F^{k*}(t)$ and $\lim_{n\to\infty} \sum_{k=1}^{m_1} kF^{k*}(t) = \sum_{k=1}^{\infty} kF^{k*}(t)$. Hence, we obtain

$$\lim_{n\to\infty} E(\hat{V}_{2n}(t)) = V(t).$$

We now consider the other nonparametric estimators $\hat{M}_{3n}(t)$ and $\hat{V}_{3n}(t)$. $\widetilde{F}_n^{k*}(t)$ is not an unbiased estimator of $F^{k*}(t)$ for $k \geq 2$. So, it is clear that $\hat{M}_{3n}(t)$ and $\hat{V}_{3n}(t)$ are not unbiased. Their asymptotic unbiasedness is established by the following theorem.

Theorem 3. *If* $F_n(t_0) < 1$ *for any* $t_0 \geq 0$ *then for* $t \leq t_0$,

$$\lim_{n\to\infty} E(\hat{M}_{3n}(t)) = M(t)$$

and

$$\lim_{n\to\infty} E(\hat{V}_{3n}(t)) = V(t),$$

that is, $\hat{M}_{3n}(t)$ *and* $\hat{V}_{3n}(t)$ *are asymptotically unbiased estimators of M(t) and V(t) for* $t \leq t_0$.

Proof. The proof is completed by taking $\tilde{F}_n(t)$, $\hat{M}_{3n}(t)$, $\hat{V}_{3n}(t)$ instead of $\hat{F}_n(t)$, $\hat{M}_{1n}(t)$, $\hat{V}_{1n}(t)$ in the proof of Theorem 1.

ÖZET

Bu çalışmada bir yenileme sürecinin yenileme ve varyans fonksiyonları için bazı parametrik ve parametrik olmayan tahmin ediciler ele alınır. Bu tahmin edicilerin asimptotik yansızlığı incelenir.

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