Commun. Fac. Sci. Univ. Ank. Series A1
V. 53.(No.2) pp.7 -12 (2004)

ON POISSON MACHINE INTERFERENCE MODEL: M/M/2/k/N with BALKING, RENEGING and HETEROGENEOUS REPAIRMEN

A. I. SHAWKY*

Faculty of Eng. at Shoubra, P. O. Box 1206, El Maadi 11728, Cairo, EGYPT

(Received Dec. 26, 2004; Revised Nov. 8, 2004; Accepted Nov. 19, 2004)

ABSTRACT

The aim of this paper is to derive the analytical solution of the queue; M/M/2/k/N for machine interference system with balking, reneging and two heterogeneous repairmen. A modified queue discipline to the classical one FIFO is used with a more general condition. The steady-state probabilities and some measures of effectiveness are derived in explicit forms. Also some special cases are deduced.

KEYWORDS

Heterogeneous repairmen, machine interference, balking, reneging.

1. INTRODUCTION

Kleinrock [3] studied the system: M/M/c/k/N, Gross and Harris [2] discussed the queue: M/M/c/k/k with spares only, Medhi [5], Bunday [1] and White et al. [8] are treated the system: M/M/c/k/k. These studies were obtained without balking and reneging concepts while the service rates are homogeneous. The present paper treats the analytical solution of the queue: M/M/2/k/N for machine interference model with balking, reneging and two heterogeneous repairmen.

2. DESCRIPTION OF THE SYSTEM

A unit arriving for service with rate λ and the service time distribution is an exponential with service rates μ_1 and μ_2. We assume that we have a finite source (population) of N customers, two heterogeneous servers (repairmen) are available and the system has a finite storage room such that the total number of customers (machines) in the system is no more than k. The queue discipline considered here is a modification of both Singh [7] and Krishnamoorthi [4], and it is:

* Now at Girls College of Education, P.O. Box 19043, Jeddah 21435, Saudi Arabia.

i) If both repairmen are free, the head unit (machine) of the queue goes to the first repairman with probability π_1 or to the second repairman with probability $\pi_2, \pi_1 + \pi_2 = 1$.
ii) If only one repairman is free, the head unit goes directly to it.
iii) If the two repairmen are busy, the units wait in their order until any repairman becomes vacant.

Consider the balk concept with probability:
$\beta = \text{prob.}\ \{\text{a unit joins the queue}\}$,
where $0 \le \beta < 1$ if $n = 2,3,...,k$ and $\beta = 1$ if $n = 0,1$.

We assume that the units may renege according to an exponential distribution, $f(t) = \alpha\, e^{-\alpha t}$, $t > 0$, with parameter α. The probability of reneging in a short period of time Δt is given by $r_n = (n-2)\alpha\,\Delta t$, for $2 < n \le k$ and $r_n = 0$, for $n = 0,1,2$.

3. THE STEADY-STATE EQUATIONS AND THEIR SOLUTION

We define the equilibrium probabilities:
$P_{0,0}$ = prob. {there is no unit in the system}
$P_{1,0}$ = prob. {there is one unit in repairman I},
$P_{0,1}$ = prob. {there is one unit in repairman II},
P_n = prob. {there are n units in the system}, $n = 2,3,...,k$.
Also, $P_0 = P_{0,0}$, $P_1 = P_{1,0} + P_{0,1}$ and $P_2 = P_{1,1}$.

Consequently, the steady-state probability difference equations are:

$$N\lambda P_0 = \mu_1 P_{1,0} + \mu_2 P_{0,1}, \qquad n = 0, \tag{1}$$

$$\left.\begin{aligned}[(N-1)\lambda + \mu_1]P_{1,0} &= \mu_2 P_{1,1} + N\lambda\pi_1 P_0,\\ [(N-1)\lambda + \mu_2]P_{0,1} &= \mu_1 P_{1,1} + N\lambda\pi_2 P_0\end{aligned}\right\} \qquad n = 1, \tag{2}$$

$$[(N-2)\beta\lambda + \mu]P_2 = (\mu + \alpha)P_3 + (N-1)\lambda P_1, \qquad n = 2, \tag{3}$$

$$[(N-n)\beta\lambda + \mu + (n-2)\alpha]P_n = [\mu + (n-1)\alpha]P_{n+1} + (N-n+1)\beta\lambda P_{n-1}, \qquad n = 3,4,...,k-1, \tag{4}$$

$$[\mu + (k-2)\alpha]P_k = (N-k+1)\beta\lambda P_{k-1}, \qquad n = k, \tag{5}$$

where $\mu = \mu_1 + \mu_2$.
From equations (1) and (2) we have:

$$P_{1,0} = \frac{N\lambda\{(N-1)\lambda + \pi_1\mu\}}{\mu_1\{2(N-1)\lambda + \mu\}} P_0, \tag{6}$$

$$P_{0,1} = \frac{N\lambda\{(N-1)\lambda + \pi_2\mu\}}{\mu_2\{2(N-1)\lambda + \mu\}} P_0. \tag{7}$$

Therefore,

$$P_1 = P_{1,0} + P_{0,1} = N\Delta P_0, \tag{8}$$

where

$$\Delta = \frac{\lambda[(N-1)\lambda + \pi_2\mu_1 + \pi_1\mu_2]}{\mu_1\mu_2[2(N-1)\rho + 1]}, \qquad \rho = \frac{\lambda}{\mu} = \frac{\lambda}{\mu_1 + \mu_2}. \tag{9}$$

Summing up equations (1), (2) and using (8) we obtain:

$$P_2 = N(N-1)\rho\Delta P_0. \tag{10}$$

From equations (3) and (10) we get:

$$P_3 = \frac{N(N-1)(N-2)\beta\lambda^2}{\mu(\mu+\alpha)} \Delta P_0. \tag{11}$$

Using (4) for $n = 3, 4, \ldots, k-1$ and (11) we deduce that:

$$P_n = \frac{N_{(n)}\beta^{n-2}\gamma^{n-1}}{(\delta)_{n-1}} \Delta P_0, \qquad 2 \le n \le k, \tag{12}$$

where

$$\gamma = \frac{\lambda}{\alpha},\ \delta = \frac{\mu}{\alpha},\ N_{(n)} = N(N-1)\ldots(N-n+1), n \ge 1, N_{(0)} = 1,$$

and $(\delta)_n = \delta(\delta+1)\ldots(\delta+n-1), n \ge 1, (\delta)_0 = 1.$

From the boundary condition: $\sum_{n=0}^{k} P_n = 1,$ we get

$$P_0^{-1} = 1 + N\Delta + \Delta \sum_{n=2}^{k} \frac{N_{(n)}\beta^{n-2}\gamma^{n-1}}{(\delta)_{n-1}}. \tag{13}$$

Thus, the expected number of units in the system and in the queue are, respectively,

$$L = \sum_{n-1}^{k} nP_n = \left\{ N + \gamma \sum_{n=2}^{k} n(\beta\gamma)^{n-2} \frac{N_{(n)}}{(\delta)_{n-1}} \right\} \Delta P_0,$$

$$L_q = L + 2P_0 + P_1 - 2,$$

and the expected waiting time in the queue and in the system are, respectively,

$$W = \frac{L}{\lambda'}, \qquad W_q = \frac{L_q}{\lambda'},$$

where

$$\lambda' = \frac{\mu}{2}(L - L_q), \qquad \mu = \mu_1 + \mu_2 .$$

The machine availability (rate of production per machine) is: $M.A. = 1 - L/k$.

The operative efficiency (utilization) is: $O.E. = 1 - P_0 - \frac{1}{2}P_1$.

4. SPECIAL CASES

Some queueing systems can be obtained as special cases of this model.

i) If $\mu_1 = \mu_2$ and $\pi_1 = \pi_2 = \frac{1}{2}$, we get the homogeneous repairmen model: M/M/2/k/N with balking and reneging, which studied by Shawky [6] at c=2.

ii) Moreover, if we put $\alpha = 0$ and $\beta = 1$, we can obtain the model: M/M/2/k/N without balking and reneging, which studied by Keinrock [3]. Also, if $N = k$, the system becomes: M/M/2/k/k without any concept, which was discussed by White et al. [1], Medhi [5], Gross and Harris [2] and Bunday [1].

5. NUMERICAL EXAMPLE

In the above system, letting *k=4* and *N=7*, i.e., the queue: M/M/2/4/7, the results are:

$$P_0 = 1 / \left\{ 1 + 7\Delta + \rho\Delta \left[42 + \frac{210\beta\lambda}{\mu + \alpha} + \frac{840\beta^2\lambda^2}{(\mu+\alpha)(\mu+2\alpha)} \right] \right\}$$

$$L = \left\{ 7 + \rho \left[84 + \frac{630\beta\lambda}{\mu+\alpha} + \frac{3360\beta^2\lambda^2}{(\mu+\alpha)(\mu+2\alpha)} \right] \right\} \Delta P_0 ,$$

$$L_q = L - 2 + (2 + 7\Delta)P_0 , \qquad M..A. = 1 - \frac{L}{4}, \qquad O.E. = 1 - (1 + \frac{7}{2}\Delta)P_0 ,$$

where:

$$\Delta = \frac{\lambda(6\lambda + \mu_1\pi_2 + \mu_2\pi_1)}{\mu_1\mu_2(12\rho + 1)}, \quad \rho = \frac{\lambda}{\mu} \quad \text{and} \quad \mu = \mu_1 + \mu_2 .$$

Now, we introduce the three tables for some measures of effectiveness at $\mu_1 = 8$, $\mu_2 = 10$ and $\lambda = 2$ for the different values of π_1, β and α when two of them are fixed.

Table I: $\alpha = 0.5$ and $\beta = 0.25$

π_1	0	0.25	0.5	0.75	1.0
P_0	0.2470207	0.269136	0.2644224	0.2598711	0.2554738
L	1.085753	1.093059	1.100108	1.106915	1.113491
L_q	0.0448256	0.0451272	0.04541815	0.04569936	0.0459705
$M.A.$	0.7285617	0.7267354	0.724973	0.7232713	0.7216273
$O.E.$	0.5204638	0.5239657	0.527345	0.5306078	0.5337603

Table II: $\pi_1 = 0.25$ and $\alpha = 0.5$

β	0	0.25	0.5	0.75	1.0
P_0	0.2807018	0.269136	0.2565518	0.2433526	0.2298935
L	1.007018	1.093059	1.193445	1.30413	1.421387
L_q	0.0	0.0451272	0.1009967	0.1649901	0.2346353
$M.A.$	0.7482456	0.7267354	0.7016388	0.6739675	0.6446533
$O.E.$	0.5035088	0.5239657	0.546224	0.5695701	0.5933758

Table III: $\pi_1 = 0.25$ and $\beta = 0.25$

α	0	0.25	0.5	0.75	1.0
P_0	0.2687682	0.2689553	0.269136	0.2693107	0.2694797
L	1.095998	1.0945	1.093059	1.091669	1.090329
L_q	0.04676479	0.045929612	0.04512722	0.04435581	0.04361296
$M.A.$	0.7260007	0.726375	0.7267354	0.7270827	0.7274178
$O.E.$	0.5246164	0.5242853	0.5239657	0.5236567	0.5233579

Tables I and II show that π_1 and β are increasing as L, L_q and $O.E.$ are increasing and as P_0 and $M.A.$ are decreasing. While from Table III, α increases as P_0 and $M.A.$ are increasing and as L, L_q and $O.E.$ are decreasing.

6. CONCLUSION

In this paper, the machine interference model: M/M/2/k/N is studied with balking, reneging and heterogeneous repairmen. The steady-state probabilities and some measures of effectiveness are derived in explicit forms. We discussed the numerical example and deduced the expected number of units in the system, in the queue, the machine availability and operative efficiency.

ÖZET

Bu bildirinin amacı, vazgeçme opsiyonlu, engelsiz ve iki değişik tamircisi bulunan M/M/2/k/N türü bir makina girişim (parazit) sistemine ilişkin kuyruk sorununa analitik bir çözüm bulmaktır. Çok daha genel koşullar altında, klasik FIFO kuyruk disiplininin değişik bir şekli uygulanmıştır. Denge durumuna ilişkin olasılıklarla bazı etkinlik ölçütleri açık şekilleri ile verilmiştir. Ayrıca, özel durumlara ilişkin bazı analitik sonuçlar elde edilmiştir.

REFERENCES

[1] B. D. Bunday, "An introduction to queueing theory". John Wiley, New York, 1996.
[2] D. Gross and C. M. Harris, "Fundamentals of queueing theory". John Wiley, New York, 1974.
[3] L. Kleinrock, "Queueing systems". Vol. 1, John Wiley, New York, 1975.
[4] B. Krishnamoorthi, "On Poisson queue with two heterogeneous servers". Oper. Res. 11, pp.321-330, 1963.
[5] J. Medhi, "Stochastic models in queueing theory". Academic Press, New York, 1991.
[6] A. I. Shawky, "The machine interference model: M/M/C/K/N with balking, reneging and spares". Opsearch, 37, pp. 25-35, 2000.
[7] V. P. Singh, "Two-server Markovian queues with balking: heterogeneous vs. homogeneous servers", Oper. Res. 18, pp. 145-159, 1970.
[8] J. A. White, J. W. Schmidt and G. K. Bennett, "Analysis of queueing systems". Academic Press, New York, 1975.