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ON GROUPS WHOSE EVERY PROPER SUBGROUP IS A B_n-GROUP

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ABSTRACT

Let G be a group in which every proper subgroup is a B_n-group. Also, $[G, x]$ has finite exponent, for all $x \in G$. B_n-group denotes the class of all group in which no subnormal subgroups has defect exceeding n, where n is a natural number. It is proved that the groups are soluble and Fitting groups.

KEYWORDS : Finite exponent, locally nilpotent groups, B_n-groups.

1. INTRODUCTION

Let n be a natural number. The class of all groups in which every subnormal subgroup has defect at most n is denoted by B_n. The groups in B_1 is studied in [11], [5], [12], and B_2, B_3, B_4 studied in [8], [3]. Moreover the general case B_n studied in [7],[4], [6] and [10].

Let G be a group and let X be a property of groups. If every proper subgroup of G satisfies X but itself does not satisfy it, then G is called a minimal non-X-group. Minimal non-B_1-groups are considered in [13] and given a classification of such groups. See also [2] for some other results related to the groups in which every subgroup is a B_1-group.

If we consider a locally nilpotent with every proper subgroup B_n-group G, where n is fixed, then we can see that G is nilpotent. Therefore, we consider locally nilpotent groups with every proper subgroup H of G a B_n-group for some natural number depending on H.

The following definitions are needed in the sequel. A group G is radicable if every elements of G has an *m*th root for every positive integer m. Let L denote the class of periodic radicable abelian groups.

A group G is a $\wp'\mathsf{L}$-group if and only if there is a transfinite ascending series $\{G_\alpha\}_{\alpha\leq\beta}$ in G with $G = G_\alpha$ and each $G_{\alpha+1}/G_\alpha$ is an L-group.

2. MAIN RESULTS

Theorem 2.1. Let G be a locally nilpotent hyperabelian p-group for some prime p such that every proper subgroup H of G a B_n-group for some natural number n depending on H. If $[G, x]$ has finite exponent for all $x \in G$ then,

(i) G is soluble,

(ii) G is a Fitting group and every subgroup of G is subnormal.

Proof. (i) Assume that G is not soluble. Let H be a proper subgroup of G then H is $\wp'\mathsf{L}$-by-nilpotent for all proper subgroups H of G by Corollary 6.4 [7]. Thus H has a normal $\wp'$ subgroup N such that H/N is nilpotent. Assume that N is not nilpotent. Since N is hyperabelian p-group N has a normal abelian series, $1 = N_0 \lhd N_1 \lhd N_2 \lhd \ldots N_\beta = N$.

Let λ be the least ordinal such that N_λ is not nilpotent. If λ is a limit ordinal then $N_\lambda = \bigcup_{\mu<\lambda} N_\mu$ N_μ is nilpotent for all $\mu < \lambda$. For every $x \in N_\lambda$ there exists $\mu < \lambda$ such that $x \in N_\mu$. Thus N_λ is a Fitting group and hence N_λ is nilpotent by Lemma 6.1 [7], but this is a contradiction. Thus $\lambda - 1$ exists and $N_{\lambda-1}$ is nilpotent. Then N_λ is a soluble p-group. Since $<x>^{N_\lambda}$ is soluble and it has finite exponent by hypothesis. Thus $<x>^{N_\lambda}$ is Baer group by Theorem 7.17 [14]. Now $<x>^{N_\lambda}$ is nilpotent by Lemma 6.1 [7] and hence H is soluble. In addition $<x>^H$ has finite exponent. Therefore $<x>^H$ is a Baer group by Theorem 7.17

[14]. This implies that H is nilpotent and that G is a Fitting group by Theorem 3.3 (ii) [16] and by Theorem 1.1 [1] G is soluble, a contradiction.

(ii) Assume that G is not a Fitting group. Then G cannot be nilpotent and hence every proper subgroup of G is nilpotent and thus G is soluble by (i). Since $<x>^G$ has finite exponent for all $x \in G$, $<x>^G$ is a Baer group by Theorem 7.17 [14] and hence $<x>^G$ is nilpotent by Lemma 6.1 [7]. Therefore G is a Fitting group. Assume that G has a maximal subgroup M. Then M is a normal subgroup of G, since G is locally nilpotent. Now there exists a finitely generated subgroup F of G such that $G = FM$. By Lemma 1 [9] G is nilpotent. If G has no maximal subgroup then every subgroup of G is subnormal by Theorem 3.1. (ii) [16].

Corollary 2.2. Let G be a periodic locally nilpotent hyperabelian group and let every proper subgroups H of G be a B_n-group for a natural number n depending on H. If $[G, x]$ has finite exponent for all $x \in G$ then,

(i) G is soluble,

(ii) G is a Fitting group and every subgroup of G is subnormal.

Proof. (i) Clearly G is the direct product of primary components by 12.1.1 [15]. If G is a p-group, then G is soluble by Theorem 2.1. If G is not a p-group then every primary components of G is soluble. Every primary components of G is nilpotent by the proof of Theorem 2.1. Let H be a proper subgroup of G. $<x>^H$ has finite exponent, for all $x \in G$ by hypothesis. This implies that $<x>^H$ has finitely primary components. Since every primary components of G is nilpotent, primary components of $<x>^H$ is nilpotent. This implies that $<x>^H$ is nilpotent by 5.2.8[15]. Thus H is a Baer group. H is nilpotent by Lemma 6.1.[7]. Thus, every proper subgroup of G is nilpotent. Therefore, G is a Fitting p-group for some prime p by Theorem 3.3 (i),(ii) [16]. G is soluble by Theorem 2.1.

(ii) G is soluble by (i). Also, G is the direct product of primary components by 12.1.1 [15]. If G is a p-group, then G is a Fitting group by Theorem 2.1. If G is not a p-group then every primary components of G is a Fitting group by Theorem 2.1. Thus every primary components of G is nilpotent by

Lemma 6.1.[7]. $<x>^G$ has finite exponent, for all $x \in G$ by hypothesis. This implies that $<x>^G$ is has finitely primary components. This primary components is nilpotent. Thus G is a Fitting group. If G has a maximal, then M is a normal subgroup of G, since G is locally nilpotent. Now there exists a finitely generated subgroup F of G such that $G = FM$. By Lemma 1 [9] G is nilpotent. If G has no maximal subgroup then every subgroup of G is subnormal by Theorem 3.1. (ii) [16].

Corollary 2.3. Let G be a periodic locally nilpotent hyperabelian group and let every proper subgroups H of G be a B_n-group for a natural number n depending on H. If G' has finite exponent then,

(i) G is soluble,

(ii) G is a Fitting group and every subgroup of G is subnormal.

Corollary 2.4. Let G be a locally nilpotent group and let every proper subgroup H of G be a B_n-group for a natural number n depending on H. If every proper subgroup H of G is soluble and has finite exponent then,

(i) G is soluble,

(ii) G is a Fitting group and every subgroup of G is subnormal.

ÖZET

n bir doğal sayı olmak üzere, altnormal altgruplarının defekti en fazla n olan grupların bir sınıfını B_n -grubu olarak ifade edelim. G, her özaltgrubu B_n -grup olan bir grup olsun. Ayrıca, her $x \in G$ için, $[G,x]$ in sonlu exponente sahip olduğunu Kabul edelim. Bu çalışmada, böyle grupların çözülebilir ve Fitting gruplar olduğu ispatlanmıştır.

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