

A NOTE ON SEIBERG-WITTEN MONOPOLE EQUATIONS ON \mathbf{R}^8

N. ÖZDEMİR AND N.DEĞİRMENÇİ

*Department of Mathematics, Faculty of Science, Anadolu University,
Eskişehir, TURKEY*

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ABSTRACT

Salamon's generalizations of the Seiberg-Witten equations are meaningful on any even-dimensional manifolds. In this work we show that there are no nontrivial solutions of these equations for any spin^c -structures on \mathbf{R}^8 .

1. INTRODUCTION

The Seiberg-Witten monopole equations are stated for 4-dimensional manifolds and these equations have great importance for the topology of smooth four-manifolds (see [7], [5]). There are also some analogues to these equations in 8-dimension (see [2], [7], [3]). In [1] it is shown that the one given by Salamon [7] have no nontrivial solutions for the standard spin^c -structures on \mathbf{R}^8 . In this work we show that Salamon's generalization of the Seiberg-Witten equations have no nontrivial solutions for any spin^c -structures on \mathbf{R}^8 .

2. PRELIMINARIES

In this section we give some basic definitions and facts about Seiberg-Witten monopole equations. For more details one can look in [7].

Definition 2.1. A spin^c -structure on a $2n$ -dimensional oriented real Hilbert space V is a pair (W, Γ) where W is a 2^n -dimensional complex Hermitian vector space and $\Gamma : V \rightarrow \text{End}(W)$ is a linear map which satisfies

$$\Gamma(v)^* + \Gamma(v) = 0, \quad \Gamma(v)^* \Gamma(v) = \|v\|^2$$

for every $v \in V$.

It is pointed out in [7] that such a map can be extended to an algebra isomorphism $Cl(V) \rightarrow \text{End}(W)$ which satisfies $\Gamma(\tilde{x}) = \Gamma(x)^*$, where

$Cl(V) \cong Cl(V) \otimes \mathbf{C}$ is complex Clifford algebra over V , \tilde{x} is conjugate of x in $Cl(V)$ and $\Gamma(x)^*$ denotes hermitian-conjugate of $\Gamma(x)$.

Let (W_1, Γ_1) and (W_2, Γ_2) be two spin^c -structures on V . If there exists a unitary isomorphism $U : W_1 \rightarrow W_2$ such that

$$U\Gamma_1(v)U^* = \Gamma_2(v)$$

for all $v \in V$, then the spin^c -structures (W_1, Γ_1) and (W_2, Γ_2) are said to be equivalent. It is known that such a unitary isomorphism always exists as a result of the following proposition (see [7]).

Proposition 2.2. Let (W_1, Γ_1) and (W_2, Γ_2) be two spin^c -structures on V . Then there exists a unitary isomorphism $U : W_1 \rightarrow W_2$ such that

$$U\Gamma_1(v)U^* = \Gamma_2(v)$$

for all $v \in V$.

Let (W, Γ) be a spin^c -structure on V . There is a natural splitting of W . Fix an orientation of V and denote by

$$\varepsilon = e_{2n} \cdots e_2 e_1 \in Cl(V)$$

the unique element of $Cl(V)$ which has degree $2n$ and is generated by a positively oriented orthonormal basis e_1, \dots, e_{2n} . Then $\varepsilon^2 = (-1)^n$ and hence

$$W = W^+ \oplus W^-$$

where the W^\pm are the eigen spaces of $\Gamma(\varepsilon)$

$$W^\pm = \{w \in W : \Gamma(\varepsilon)w = \pm i^n w\}.$$

Note that $\Gamma(v)W^+ \subset W^-$ and $\Gamma(v)W^- \subset W^+$ for every $v \in V$. So the restriction of $\Gamma(v)$ to W^+ for $v \in V$ determines a linear map $\gamma : V \rightarrow \text{Hom}(W^-, W^+)$ which satisfies

$$\gamma(v)^* \gamma(v) = |v|^2 1$$

for every $v \in V$.

Let (W, Γ) be a spin^c structure on V . Such a structure gives an action of the space of 2-forms $\Lambda^2 V$ on W . This action is defined by the following:

Firstly, identify $\Lambda^2 V$ with the space of second order elements of Clifford algebra $C_2(V)$ via the map

$$\Lambda^2 V \rightarrow C_2(V), \eta = \sum_{i < j} \eta_{ij} e_i \wedge e_j \mapsto \sum_{i < j} \eta_{ij} e_i e_j .$$

Compose this map with Γ to obtain a map $\rho : \Lambda^2 V \rightarrow \text{End}(W)$ given by

$$\rho \left(\sum_{i < j} \eta_{ij} e_i \wedge e_j \right) = \sum_{i < j} \eta_{ij} \Gamma(e_i) \Gamma(e_j)$$

for any orthonormal basis e_1, \dots, e_{2n} of V . This map is independent of the choice of the orthonormal basis e_1, \dots, e_{2n} . The spaces W^\pm are invariant under $\rho(\eta)$ for every 2-form $\eta \in \Lambda^2 V$. So we can define

$$\rho^\pm(\eta) = \rho(\eta)|_{W^\pm}$$

for $\eta \in \Lambda^2 V$. In 4-dimensions $\rho^+(\eta) = \rho^+(\eta^+)$ for every 2-form $\eta \in \Lambda^2 V$, where η^+ is the self-dual part of η . The map ρ extends to a map

$$\rho : \Lambda^2 V \otimes \mathbb{C} \rightarrow \text{End}(W)$$

on the space of complex valued 2-forms. If η is a real valued 2-form, then $\rho(\eta)$ is skew-Hermitian and if η is imaginary valued then $\rho(\eta)$ is Hermitian.

Globalizing above Γ to $2n$ -dimensional oriented manifold X defines a spin^c structure $\Gamma : TX \rightarrow \text{End}(W)$, W being a 2^n -dimensional complex Hermitian vector bundle on X . Such a structure exists iff $w_2(X)$ has an integral lift (see [4]). Γ extends to an isomorphism between the complex Clifford algebra bundle $Cl(TX)$ and $\text{End}(W)$. There is a natural splitting $W = W^+ \oplus W^-$ into the $\pm i^n$ eigenspaces of $\Gamma(e_{2n} e_{2n-1} \dots e_1)$ where e_1, e_2, \dots, e_{2n} is any positively oriented local orthonormal frame of TX .

A Hermitian connection ∇ on W is called a spin^c connection (compatible with the Levi-Civita connection) if

$$\nabla_v (\Gamma(w)\Psi) = \Gamma(w)\nabla_v \Psi + \Gamma(\nabla_v w)\Psi$$

where Ψ is a spinor (section of W), v and w are vector fields on X and ∇_v is the Levi-Civita connection on X . ∇ preserves the subbundles W^\pm .

There is a principal $\text{Spin}^c(2n)$ -bundle P on X such that the bundle W of spinors, the tangent bundle TX , and the line bundle L_Γ can be recovered as the associated bundles

$$W = P \times_{\text{Spin}^c(2n)} \mathbf{C}^{2n}, \quad TX = P \times_{Ad} \mathbf{R}^{2n}$$

where Ad is the adjoint action of

$$\text{Spin}^c(2n) = \{e^{i\theta} x : \theta \in \mathbf{R}, x \in \text{Spin}(2n)\} \subset \text{Cl}_{2n}$$

on \mathbf{R}^{2n} . Then one can obtain a complex line bundle $L_\Gamma = P \times_\delta \mathbf{C}$ where

$$\delta : \text{Spin}^c(2n) \rightarrow S^1 \text{ by } \delta(e^{i\theta} x) = e^{2i\theta}.$$

There is a one-to-one correspondence between spin^c connections on W and $\text{spin}^c(2n) = \text{Lie}(\text{Spin}^c(2n)) = \text{spin}(2n) \oplus i\mathbf{R}$ -valued connection 1-forms $\widehat{A} \in \mathbf{A}(P) \subset \Omega^1(P, \text{spin}^c(2n))$ on P . Hence every spin^c connection \widehat{A} decomposes as

$$\widehat{A} = \widehat{A}_0 + \frac{1}{2^n} \text{trace}(\widehat{A})$$

where \widehat{A}_0 is the traceless part of \widehat{A} . Let $A = \frac{1}{2^n} \text{trace}(\widehat{A})$. This is an imaginary valued 1-form in $\Omega^1(P, i\mathbf{R})$ which satisfies

$$A_{pq}(vg) = A_p(v), \quad A_p(p.\xi) = \frac{1}{2^n} \text{trace}(\xi) \quad (1)$$

for $v \in T_p P$, $g \in \text{Spin}^c(2n)$, and $\xi \in \text{spin}^c(2n)$. Let

$$\mathbf{A}(\Gamma) = \{A \in \Omega^1(P, i\mathbf{R}) : A \text{ satisfies (1)}\}$$

There is a one-to-one correspondence between these 1-forms and spin^c connections on W . Let ∇_A be the spin^c connection corresponding to A . $\mathbf{A}(\Gamma)$ is an affine space with parallel vector space $\Omega^1(X, i\mathbf{R})$. Let $F_A \in \Omega^2(P, i\mathbf{R})$ be the curvature of the 1-form A and D_A denote the Dirac operator corresponding to $A \in \mathbf{A}(\Gamma)$,

$$D_A : C^\infty(X, W^+) \rightarrow C^\infty(X, W^-)$$

defined by

$$D_A(\Psi) = \sum_{i=1}^{2n} \Gamma(e_i) \nabla_{A, e_i}(\Psi)$$

where $\Psi \in C^\infty(X, W^+)$ and e_1, e_2, \dots, e_{2n} is any local orthonormal frame.

The Seiberg-Witten equations can now be expressed as follows:

Let $\Gamma : TX \rightarrow \text{End}(W)$ be a fixed spin^c structure on X and consider the pair $(A, \Psi) \in \mathbf{A}(\Gamma) \times C^\infty(X, W^+)$. The Seiberg-Witten equations read

$$D_A \Psi = 0, \quad \rho^+(F_A) = (\Psi \Psi^*)_0$$

where $(\Psi \Psi^*)_0 \in C^\infty(X, \text{End}(W^+))$ is defined by $(\Psi \Psi^*)(\tau) = \langle \Psi, \tau \rangle \Psi$ for $\tau \in C^\infty(X, W^+)$ and $(\Psi \Psi^*)_0$ is the traceless part of $(\Psi \Psi^*)$.

3. MONOPOLE EQUATIONS ON \mathbf{R}^8 WITH DIFFERENT Spin^c -STRUCTURES AND THEIR RELATIONS

One can find the explicit expressions of the Seiberg-Witten monopole equations on \mathbf{R}^4 in [6] and [7].

In our case $X = \mathbf{R}^8$, $W_1 = W_2 = \mathbf{C}^{16}$ and $L_\Gamma = \mathbf{R}^8 \times \mathbf{C}$, (W_1, Γ_1) and (W_2, Γ_2) spin^c -structures on \mathbf{R}^8 and we consider the unitary map U from W_1 to W_2 that satisfies

$$U \circ \Gamma_1(v) \circ U^* = \Gamma_2(v) \tag{2}$$

for all $v \in \mathbf{R}^8$.

In [1] they consider standard spin^c -structure which is obtained from the well-known isomorphism of the complex Clifford algebra Cl_{2n} with $\text{End}(\Lambda^* \mathbf{C}^n)$ and they express following theorem:

Theorem 3.1. There are no nontrivial solutions of the Seiberg-Witten equations on \mathbf{R}^8 with constant standard spin^c -structure, i. e. $\rho^+(F_A) = (\Psi \Psi^*)_0$ (alone) implies $F_A = 0$ and $\Psi = 0$.

Our goal is to state a similar theorem for any spin^c-structure on \mathbf{R}^8 . To do this we need some lemmas.

Lemma 3.2. If a unitary isomorphism U from W_1 to W_2 satisfies (2), then U maps W_1^\pm onto W_2^\pm .

Proof. Let $\Psi \in C^\infty(\mathbf{R}^8, W_1^+)$. Then $\Gamma_1(\varepsilon)\Psi = \Psi$ where

$$\begin{aligned} \varepsilon &= e_{2n} \cdots e_1. \\ \Psi &= \Gamma_1(e_{2n} \cdots e_1)\Psi \\ &= \Gamma_1(e_{2n}) \cdots \Gamma_1(e_1)\Psi \\ &= U^* \Gamma_2(e_{2n}) U \cdots U^* \Gamma_2(e_1) U \Psi \\ &= U^* \Gamma_2(e_{2n} \cdots e_1) U \Psi \end{aligned}$$

From the last equality $\Gamma_2(e_{2n} \cdots e_1) U \Psi = U \Psi$ that is, $U \Psi \in C^\infty(\mathbf{R}^8, W_2^+)$. Thus U maps W_1^+ onto W_2^+ . It can be shown in a similar way that U maps W_1^- onto W_2^- .

Lemma 3.3. The maps $\rho_1 : \Lambda^2(T^*\mathbf{R}^8) \otimes \mathbf{C} \rightarrow \text{End}(W_1)$ and $\rho_2 : \Lambda^2(T^*\mathbf{R}^8) \otimes \mathbf{C} \rightarrow \text{End}(W_2)$ satisfy $\rho_1(\eta) = U \rho_2(\eta) U^*$ for any 2-form $\eta = \sum_{i < j} \eta_{ij} e_i \wedge e_j$ in $\Lambda^2(T^*\mathbf{R}^8) \otimes \mathbf{C}$.

Proof.

$$\begin{aligned} \rho_2(\eta) &= \sum_{i < j} \eta_{ij} \Gamma_2(e_i) \Gamma_2(e_j) \\ &= \sum_{i < j} \eta_{ij} U \Gamma_2(e_i) U^* U \Gamma_2(e_j) U^* \quad (\text{Since } U U^* = I) \\ &= \sum_{i < j} U \eta_{ij} \Gamma_2(e_i) \Gamma_2(e_j) U^* \\ &= U \left(\sum_{i < j} \eta_{ij} \Gamma_2(e_i) \Gamma_2(e_j) \right) U^* \\ &= U(\rho_1(\eta)) U^*. \end{aligned}$$

Note that $\rho_2^+(\eta) = U(\rho_1^+(\eta)) U^*$.

Lemma 3.4. If $\Psi \in C^\infty(\mathbf{R}^8, \mathcal{W}_1^+)$, then the equality $((U\Psi)(U\Psi)^*)_0 = U(\Psi\Psi^*)_0 U^*$ holds for any unitary isomorphism $U : \mathbf{C}^{16} \rightarrow \mathbf{C}^{16}$.

Proof.

$$\begin{aligned}
 (U(\Psi\Psi^*)_0 U^*)\tau &= (U(\Psi\Psi^*)_0)(U^*\tau) \\
 &= U\langle\Psi, U^*\tau\rangle\Psi - \text{trace}(\Psi\Psi^*)U^*\tau \\
 &= \langle\Psi, U^*\tau\rangle U\Psi - \text{trace}(\Psi\Psi^*)\tau \\
 &= \langle\Psi, U^*\tau\rangle U\Psi - \text{trace}((U\Psi)(U\Psi)^*) \\
 &= ((U\Psi)(U\Psi)^*)_0 \tau
 \end{aligned}$$

for all $\tau \in C^\infty(\mathbf{R}^8, \mathcal{W}_1^+)$. Note that,

$$\text{trace}(\Psi\Psi^*) = \|\Psi\|^2 = \|U\Psi\|^2 = \text{trace}((U\Psi)(U\Psi)^*), \text{ since } U \text{ is unitary.}$$

Lemma 3.5. Let $(\Gamma_1, \mathcal{W}_1)$ and $(\Gamma_2, \mathcal{W}_2)$ be two spin^c -structures on \mathbf{R}^8 and $U : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ be a unitary isomorphism such that $U \circ \Gamma_1(v) \circ U^* = \Gamma_2(v)$ for all $v \in \mathbf{R}^8$. If the pair (A, Ψ) is a solution of the monopole equations with respect to Γ_1 , then the pair $(A, U\Psi)$ is a solution of the monopole equations with respect to Γ_2 .

Proof. Let (A, Ψ) be a solution of the equations

$$\begin{aligned}
 D_A \Psi &= \sum_{i=1}^8 \Gamma_1(e_i) \nabla_i(\Psi) = 0. \\
 \rho_1^+(F_A) &= \sum_{i < j} F_{ij} \Gamma_1(e_i) \Gamma_1(e_j) = (\Psi\Psi^*)_0
 \end{aligned}$$

Then

$$\begin{aligned}
D_A(U\Psi) &= \sum_{i=1}^8 \Gamma_2(e_i) \nabla_i(U\Psi) \\
&= \sum_{i=1}^8 U \Gamma_1(e_i) U^* \nabla_i(U\Psi) \\
&= \sum_{i=1}^8 U \Gamma_1(e_i) U^* U \nabla_i(\Psi) \quad (\text{since } \nabla_i(U\Psi) = U \nabla_i(\Psi)) \\
&= U \sum_{i=1}^8 \Gamma_1(e_i) \nabla_i(\Psi) = U(D_A \Psi) = 0.
\end{aligned}$$

The equality $\nabla_i(U\Psi) = U \nabla_i(\Psi)$ holds for all $\Psi \in C^\infty(\mathbf{R}^8, \mathcal{W}_1^+)$,

$U\Psi = \left(\sum_{i=1}^{16} u_{1i} \psi_i, \dots, \sum_{i=1}^{16} u_{(16)i} \psi_i \right)$ where $U = (u_{ij})$ is the matrix notation of the unitary map U .

$$\begin{aligned}
 \nabla_i(U\Psi) &= \nabla_i \begin{bmatrix} u_{11}\psi_1 + \cdots + u_{1(16)}\psi_{(16)} \\ u_{21}\psi_1 + \cdots + u_{2(16)}\psi_{(16)} \\ \vdots \\ u_{(16)1}\psi_1 + \cdots + u_{(16)(16)}\psi_{(16)} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{\partial}{\partial x_i}(u_{11}\psi_1 + \cdots + u_{1(16)}\psi_{(16)}) + A_i(u_{11}\psi_1 + \cdots + u_{1(16)}\psi_{(16)}) \\ \frac{\partial}{\partial x_i}(u_{21}\psi_1 + \cdots + u_{2(16)}\psi_{(16)}) + A_i(u_{21}\psi_1 + \cdots + u_{2(16)}\psi_{(16)}) \\ \vdots \\ \frac{\partial}{\partial x_i}(u_{(16)1}\psi_1 + \cdots + u_{(16)(16)}\psi_{(16)}) + A_i(u_{(16)1}\psi_1 + \cdots + u_{(16)(16)}\psi_{(16)}) \end{bmatrix} \\
 &= \begin{bmatrix} u_{11} \frac{\partial_1 \psi_1}{\partial x_i} + \cdots + u_{1(16)} \frac{\partial_1 \psi_{(16)}}{\partial x_i} + u_{11} A_i \psi_1 + \cdots + u_{1(16)} A_i \psi_{(16)} \\ u_{21} \frac{\partial_1 \psi_1}{\partial x_i} + \cdots + u_{2(16)} \frac{\partial_1 \psi_{(16)}}{\partial x_i} + u_{21} A_i \psi_1 + \cdots + u_{2(16)} A_i \psi_{(16)} \\ \vdots \\ u_{(16)1} \frac{\partial_1 \psi_1}{\partial x_i} + \cdots + u_{(16)(16)} \frac{\partial_1 \psi_{(16)}}{\partial x_i} + u_{(16)1} A_i \psi_1 + \cdots + u_{(16)(16)} A_i \psi_{(16)} \end{bmatrix} \\
 &= \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1(16)} \\ u_{21} & u_{22} & \cdots & u_{2(16)} \\ \vdots & \vdots & \ddots & \vdots \\ u_{(16)1} & u_{(16)2} & \cdots & u_{(16)(16)} \end{bmatrix} \begin{bmatrix} \frac{\partial_1 \psi_1}{\partial x_i} + A_i \psi_1 \\ \frac{\partial_1 \psi_2}{\partial x_i} + A_i \psi_2 \\ \vdots \\ \frac{\partial_1 \psi_{(16)1}}{\partial x_i} + A_i \psi_{(16)} \end{bmatrix} \\
 &= U \nabla_i(\Psi)
 \end{aligned}$$

For the second equation:

$$\begin{aligned}
 \rho_2^+(F_A) &= U(\rho_1^+(\eta))U^* \text{ (from Lemma)} \\
 &= U(\Psi\Psi^*)_0 U^* \text{ (since } \Psi \text{ is a solution)} \\
 &= ((U\Psi)(U\Psi)^*)_0 \text{ (from Lemma)}
 \end{aligned}$$

To summarise, we can express the following theorem:

Theorem 3.6. Let (Γ, \mathcal{W}) be any spin^c -structure on \mathbf{R}^8 . Then there are no nontrivial solutions of the Seiberg-Witten equations on \mathbf{R}^8 with arbitrary spin^c -structure, i. e. $\rho^+(F_A) = (\Psi\Psi^*)_0$ implies $F_A = 0$ and $\Psi = 0$.

Proof. Let (A, Ψ) be a solution to the Seiberg-Witten equations on \mathbf{R}^8 with respect to (Γ, \mathcal{W}) . Since standard spin^c -structure is equivalent to the any spin^c -structure (Γ, \mathcal{W}) , there exists a unitary isomorphism U which satisfies the equation (2). Then the pair $(A, U\Psi)$ is a solution for the Seiberg-Witten equations on \mathbf{R}^8 with respect to standard spin^c -structure and from Theorem 3.1., $A = 0$ and $U\Psi = 0$. Since U is a isomorphism we get $\Psi = 0$.

ÖZET

Salamon'un genelleştirdiği Seiberg-Witten denklemleri herhangi bir çift boyutta anlamlıdır. Bu çalışmada \mathbf{R}^8 üzerindeki herhangi bir spin^c yapısı için Salamon tarafından verilen Seiberg-Witten denklemlerinin nontrivial çözümünün olmadığı gösterilmiştir.

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