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INTERVAL ESTIMATORS FOR THE PARAMETERS OF THE NORMAL DISTRIBUTION

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ABSTRACT. In this article we review unbiased, shortest, equally tailed and uniformly most accurate (UMA) interval estimators for the parameters of the normal distributon. MATLAB codes are presented for statistical computations.

1. INTRODUCTION

Let, $X_1, X_2, X_3, ..., X_n$ be i.i.d. random sample with p.d.f. $f(.;\theta)$, $\theta \in \Theta \subset \mathbb{R}$. For an observed value $\mathrm{x} = (x_1, x_2, x_3, ..., x_n)$ of the random sample $\mathrm{X} = (X_1, X_2, X_3, ..., X_n)$, an interval estimate of θ is any pair of functions $L(\mathrm{x})$ and $U(\mathrm{x})$, that satisfy $L(\mathrm{x}) \leq U(\mathrm{x})$ for all x values in the sample space, resulting with the inference, that the interval $[L(\mathrm{x}), U(\mathrm{x})]$ contains θ with some probability. The random interval $C(\mathrm{X}) = [L(\mathrm{X}), U(\mathrm{X})]$ is called an interval estimator for θ. The probability $P_\theta(\theta \in C(\mathrm{X}))$ is called the coverage probability of $C(\mathrm{X})$, and its infimum, $\inf_{\theta\in\Theta} P_\theta(\theta \in C(\mathrm{X}))$, is called confidence coefficient. An interval estimator together with a confidence coefficient is called a confidence interval. A confidence interval with confidence coefficient equal to some value, say $1-\alpha$, ($\alpha \in (0,1)$, is simply called a $1-\alpha$ confidence interval.

In using the standart for obtaining a confidence interval for θ, one seeks a random variable, called pivotal quantity, $Q(\mathrm{X}, \theta)$, whose distribution is independent of θ. Then, in a number of standart situations, the probability statement,

$$P_\theta(a \leq Q(\mathrm{X}, \theta) \leq b) = 1-\alpha$$

is converted to,

$$P_\theta(L(\mathrm{X}) \leq \theta \leq U(\mathrm{X})) = 1-\alpha.$$

A $1-\alpha$ confidence interval $C(\mathrm{X}) = [L(\mathrm{X}), U(\mathrm{X})]$ is called equally tailed interval for θ, if

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$$P_\theta\left(\theta \in (-\infty, L(\mathrm{X}))\right) = P_\theta\left(\theta \in (U(\mathrm{X}), \infty)\right) = 1 - \frac{\alpha}{2}.$$

The lenght of a confidence interval $C(\mathrm{X}) = [L(\mathrm{X}), U(\mathrm{X})]$ is the random variable,

$$\ell((C(\mathrm{X})) = U(\mathrm{X}) - L(\mathrm{X}).$$

It is quite possible that there exist more than one confidence interval for θ with the same confidence coefficient $1-\alpha$. In such a case, it is obvious that we will be interested in finding the shortest confidence interval within a certain class of confidence intervals.

A $1-\alpha$ confidence interval $C(\mathrm{X}) = [L(\mathrm{X}), U(\mathrm{X})]$ is unbiased for θ, if the probability of false coverage $P_\theta\left(\theta' \in [L(\mathrm{X}), U(\mathrm{X})]\right) \leq 1-\alpha$ for all $\theta' \in \Theta$, $\theta' \neq \theta$ [1,3,5].

A confidence interval that minimizes the probability of false coverage over a class of $1-\alpha$ confidence intervals is called a uniformly most accurate (UMA) confidence interval. Such confidence intervals can be constructed by inverting the acceptance region of uniformly most powerfull (UMP) tests. The correspondence between UMP test and UMA intervals carries over to UMP unbiased tests and UMA unbiased intervals [1,5].

2. Confidence intervals for the mean of the normal distribution

Let $\mathrm{X} = (X_1, X_2, ..., X_n)$be a sample from normal $N(\mu, \sigma^2)$ distribution with parameters $\mu \in \mathbb{R}$ and $\sigma^2 \in (0, \infty)$. Sufficient statistics for μ and σ^2 are $\bar{X}_n = \sum_{i=1}^{n} X_i/n$ and $S_{n-1}^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/n-1$.Obvious choices for a pivotal quantity for μ are,

$$Q_1(\mathrm{X}, \mu) = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

and

$$Q_2(\mathrm{X}, \mu) = \frac{\bar{X}_n - \mu}{S_{n-1}/\sqrt{n}} \sim t_{n-1}$$

depending upon whether σ^2 is known or not known. Now, from

$$P_\mu\left(a \leq Q_1(\mathrm{X}, \theta) \leq b\right) = 1-\alpha$$

we can write,

$$P_\mu\left(\bar{X}_n - b\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X}_n - a\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha.$$

Then, the class of $1-\alpha$ confidence intervals based on the pivotal quantity $Q_1$is,

$$\mathbb{C}_1 = \left\{ C(\mathrm{X}, a, b) : C(\mathrm{X}, a, b) = \left[\bar{X}_n - b\frac{\sigma}{\sqrt{n}}, \bar{X}_n - a\frac{\sigma}{\sqrt{n}} \right], \right.$$
$$\left. a, b \in \mathbb{R}, a < b, P_\mu(\mu \in C(\mathrm{X}, a, b)) = 1 - \alpha \right\}$$

The length for an element of this class is

$$\ell\left(C(\mathrm{X}, a, b)\right) = \frac{\sigma}{\sqrt{n}}(b - a).$$

In order to find the shortest interval in the class $\mathbb{C}_1$ one has to solve the following problem.

$$\text{goal} : \min_{a,b} \frac{\sigma}{\sqrt{n}}(b - a)$$
$$\text{constraint} : \int_a^b f_{Q_1}(q)dq = 1 - \alpha$$

As a result, a and b are determined as, $b = -a = z_{1-\alpha/2}$ giving the shortest $1 - \alpha$ confidence interval $\left[\bar{X}_n - z_{1-\alpha/2}\sigma/\sqrt{n}, \bar{X}_n + z_{1-\alpha/2}\sigma/\sqrt{n}\right]$ [1,4].

By a similar discussion for Q_2, the shortest $1-\alpha$ confidence interval in the class,

$$\mathbb{C}_2 = \left\{ C(\mathrm{X}, a, b) : C(\underline{X}, a, b) = \left[\bar{X}_n - b\frac{S_{n-1}}{\sqrt{n}}, \bar{X}_n - a\frac{S_{n-1}}{\sqrt{n}} \right], \right.$$
$$\left. a, b \in \mathbb{R}, a < b, P_\mu\left(\mu \in C(\mathrm{X}, a, b)\right) = 1 - \alpha \right\}$$

can be obtained as $\left[\bar{X}_n - t_{n-1;1-\alpha/2}S_{n-1}/\sqrt{n}, \bar{X}_n + t_{n-1;1-\alpha/2}S_{n-1}/\sqrt{n}\right]$.

As it can be seen, these shortest $1-\alpha$ confidence intervals are also equally tailed confidence intervals for μ.

Now, do they are unbiased? Let, $\mu \neq \mu'$, $\mu' = \mu + \lambda\frac{\sigma}{\sqrt{n}}$ for some $\lambda \in \mathbb{R}$. Then we have,

$$P_\mu\left(\bar{X}_n - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu' \leq \bar{X}_n + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$
$$= P_\mu\left(\bar{X}_n - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu + \lambda\frac{\sigma}{\sqrt{n}} \leq \bar{X}_n + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$
$$= P\left(-z_{1-\alpha/2} + \lambda \leq Z \leq z_{1-\alpha/2} + \lambda\right)$$
$$\leq 1 - \alpha.$$

To see that the inequallity holds, let us consider the function,

$$h(\lambda) = P\left(-z_{1-\alpha/2} + \lambda \leq Z \leq z_{1-\alpha/2} + \lambda \right) \quad , \quad \lambda \in \mathbb{R}.$$

Note that,

$$\frac{dh(\lambda)}{d\lambda} = \frac{d}{d\lambda} \int_{-z_{1-\alpha/2}+\lambda}^{z_{1-\alpha/2}+\lambda} f_Z(z)dz = \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{\left(z_{1-\alpha/2}+\lambda\right)^2}{2}} - e^{-\frac{\left(-z_{1-\alpha/2}+\lambda\right)^2}{2}} \right) = 0$$

implies $\lambda = 0$. Thus $\lambda = 0$ is a critical value. Indeed $\lambda = 0$ maximizes the function. This implies that the interval is unbiased [3].

Hence, based upon the pivotal quantity Q_1, the shortest, equally tailed and unbiased $1-\alpha$ confidence intervals are identical. The same is true for Q_2.

3. Confidence intervals for the variance of the normal distribution

A pivotal quantity for σ^2 is,

$$Q(\mathrm{X}, \sigma^2) = \frac{(n-1)S_{n-1}^2}{\sigma^2} \sim \chi_{n-1}^2.$$

Converting the statement,

$$P_{\sigma^2}\left(a \leq \frac{(n-1)S_{n-1}^2}{\sigma^2} \leq b\right) = 1 - \alpha$$

we can write,

$$P_{\sigma^2}\left(\frac{(n-1)S_{n-1}^2}{b} \leq \sigma^2 \leq \frac{(n-1)S_{n-1}^2}{a}\right) = 1 - \alpha.$$

Then, the class of $1-\alpha$ confidence intervals for σ^2 based on the pivotal quantity Q is,

$$\mathbb{C} = \Big\{ C(\mathrm{X}, a, b) : C(\mathrm{X}, a, b) = \left[\frac{(n-1)S_{n-1}^2}{b}, \frac{(n-1)S_{n-1}^2}{a}\right], \\ a, b > 0, a < b, P_{\sigma^2}\left(\sigma^2 \in C(\mathrm{X}, a, b)\right) = 1 - \alpha \text{-} \Big\}$$

with,

$$\ell\left(C(\mathrm{X}, a, b)\right) = (n-1)S_{n-1}^2\left(\frac{1}{a} - \frac{1}{b}\right).$$

In order to find the shortest interval in the class $\mathbb{C}$, one has to solve the following problem.

$$\text{goal}: \min_{a,b} \ (n-1)S_{n-1}^2\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\text{constraint}: \int_a^b f_Q(q)dq = 1-\alpha$$

where, f_Q is the p.d.f. of χ^2_{n-1} distribution. As a result [1,2,4], the shortest $1-\alpha$ confidence intervals for σ^2 based on the pivotal quantity Q is determined by the values a and b satisfying,

$$a^2 f_Q(a) = b^2 f_Q(b) \quad \text{and} \quad \int_a^b f_Q(q)dq = 1-\alpha.$$

Tables are constructed by Tate and Klett (1959) for the numerical solutions of these equations,. It is a simple task to solve these equations numerically with nowadays computer facilities. A short MATLAB program is as follows.

```
alfa=.05;n=26;
aa=chi2inv(alfa,n-1);
for a=.1:.01:aa
b=chi2inv((1-alfa+chi2cdf(a,n-1)),n-1);
if a 2*chi2pdf(a,n-1)-b 2*chi2pdf(b,n-1)>=0;break;end
end
[a b]
ans = 14.2640 45.7100
```

Tate and Klett's table gives, a=14.2636 and b=45.7051.

The equally tailed $1-\alpha$ confidence interval for σ^2 based on the pivotal quantity Q is,

$$\left[\frac{(n-1)S^2_{n-1}}{\chi^2_{n-1;1-\alpha/2}}, \frac{(n-1)S^2_{n-1}}{\chi^2_{n-1;\alpha/2}}\right].$$

Now, let us look for an unbiased $1-\alpha$ confidence interval for $\theta=\sigma^2$ in the class $\mathbb{C}$. Let, $\theta \neq \theta'$, $\theta' = \lambda\theta$ for some $\lambda \in (0,\infty)$. Then for an $C(\mathrm{X},a,b) \in \mathbb{C}$ to be unbised, we must have,

$$P_\theta\left(\frac{(n-1)S^2_{n-1}}{b} \leq \theta' \leq \frac{(n-1)S^2_{n-1}}{a}\right) = \begin{cases} 1-\alpha & , \quad \theta=\theta' \\ \leq 1-\alpha & , \quad \theta\neq\theta' \end{cases}$$

or

$$h(\lambda,a,b) = P_\theta\left(\frac{(n-1)S^2_{n-1}}{\lambda b} \leq \theta \leq \frac{(n-1)S^2_{n-1}}{\lambda a}\right)$$

$$= P_\theta\left(\lambda a \leq \frac{(n-1)S^2_{n-1}}{\theta} \leq \lambda b\right) = \begin{cases} 1-\alpha & , \quad \lambda=1 \\ \leq 1-\alpha & , \quad \lambda\neq 1 \end{cases}$$

So, we need,

$$\frac{\partial h(\lambda,a,b)}{\partial\lambda}\Big|_{\lambda=1} = \frac{\partial}{\partial\lambda}\int_{\lambda a}^{\lambda b} f_Q(q)dq\,\Big|_{\lambda=1}$$
$$= bf_Q(\lambda b) - af_Q(\lambda a)\,\Big|_{\lambda=1}$$
$$= bf_Q(b) - af_Q(a) = 0$$

and

$$\int_a^b f_Q(q)dq = 1-\alpha.$$

Let, a^* and b^* satisfy these conditions, i.e. $b^* f_Q(b^*) = a^* f_Q(a^*)$ and

$$\int_{a^*}^{b^*} f_Q(q)dq = 1-\alpha.$$

Then, for $\lambda \in (0,\infty)$

$$h(\lambda,a^*,b^*) \le h(1,a^*,b^*) = 1-\alpha$$

which implies that the interval,

$$\left[\frac{(n-1)S_{n-1}^2}{b^*}, \frac{(n-1)S_{n-1}^2}{a^*}\right]$$

is an unbiased $1-\alpha$ confidence interval for $\theta = \sigma^2$. The graph of $h(\lambda,a^*,b^*)$ as a function of $\lambda \in (0,5)$ is in the Figure 1.

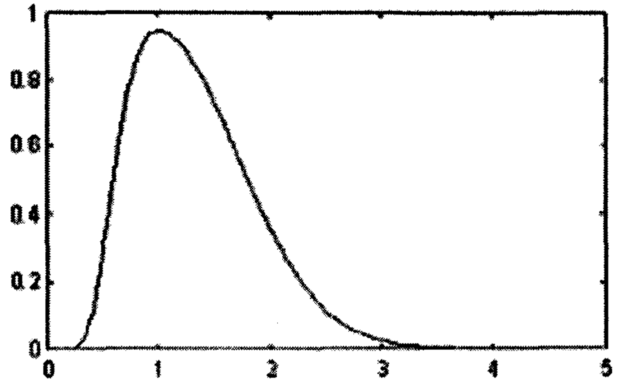

Figure1: The graph of $h(\lambda,a^*,b^*)$ as a function of $\lambda \in (0,5)$

4. UMA UNBIASED CONFIDENCE INTERVALS FOR μ AND σ^2

Consider, $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. The acceptance region of the UMP unbiased test of size α is,

$$A_1(\mu_0) = \left\{ \mathrm{X} : \mu_0 - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \bar{X}_n \leq \mu_0 + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \right\}$$

for σ^2 known, and

$$A_2(\mu_0) = \left\{ \mathrm{X} \ : \mu_0 - t_{n-1;1-\alpha/2}\frac{S_{n-1}}{\sqrt{n}} \leq \bar{X}_n \leq \mu_0 + t_{n-1;1-\alpha/2}\frac{S_{n-1}}{\sqrt{n}} \right\}$$

when σ^2 is unknown. The corresponding UMA unbiased intervals are,

$$\left[\bar{X}_n - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X}_n + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \right]$$

and

$$\left[\bar{X}_n - t_{n-1;1-\alpha/2}\frac{S_{n-1}}{\sqrt{n}}, \bar{X}_n + t_{n-1;1-\alpha/2}\frac{S_{n-1}}{\sqrt{n}} \right]$$

respectively.

Consider, $H_0 : \sigma^2 = \sigma_0^2$ versus $H_0 : \sigma^2 \neq \sigma_0^2$. The acceptance region of the UMP unbiased test of size α is,

$$A_1(\sigma_0^2) = \left\{ \mathrm{X} : \frac{\sigma_0^2 \chi^2_{n-1;\alpha/2}}{n-1} \leq S^2_{n-1} \leq \frac{\sigma_0^2 \chi^2_{n-1;1-\alpha/2}}{n-1} \right\}$$

The correponding UMA unbiased interval is,

$$\left[\frac{(n-1)S^2_{n-1}}{\chi^2_{n-1;1-\alpha/2}}, \frac{(n-1)S^2_{n-1}}{\chi^2_{n-1;\alpha/2}} \right].$$

Now, let us construct a confidence interval based on the likelihood ratio test in order to compare it with previously obtained intervals. The likelihood ratio test of $H_0 : \sigma^2 = \sigma_0^2$ versus $H_0 : \sigma^2 \neq \sigma_0^2$ is,

$$\Psi(\mathrm{X}) = \begin{cases} 1 \quad , & \left(\frac{(n-1)S^2_{n-1}}{n\sigma_0^2}\right)^{\frac{n}{2}} \exp\left(-\frac{(n-1)S^2_{n-1}}{\sigma_0^2}\right) < c \\ 0 \quad , & \left(\frac{(n-1)S^2_{n-1}}{n\sigma_0^2}\right)^{\frac{n}{2}} \exp\left(-\frac{(n-1)S^2_{n-1}}{\sigma_0^2}\right) \geq c \end{cases}$$

where, $c = c_\alpha$ is chosen to give a size α test [1]. If we denote $T = \frac{(n-1)S^2_{n-1}}{\sigma_0^2}$, then $T \sim \chi^2_{n-1}$ under H_0. The acceptance region of the test is,

$$A(\sigma_0^2) = \left\{ \mathrm{X} : (\frac{T}{n})^{\frac{n}{2}} \exp(-\frac{T}{2}) \geq c_\alpha \right\}.$$

The corresponding $1-\alpha$ confidence interval is,

$$\left\{ \sigma^2 : \left(\frac{(n-1)S_{n-1}^2}{n\sigma^2} \right)^{\frac{n}{2}} \exp(-\frac{(n-1)S_{n-1}^2}{\sigma^2}) \geq c_\alpha \right\}.$$

Since the function $g(t) = (\frac{t}{n})^{\frac{n}{2}} \exp(-\frac{t}{2}), t > 0$ is unimodal, it follows that the confidence set is in the form,

$$\left\{ \sigma^2 : a \leq \frac{(n-1)S_{n-1}^2}{\sigma^2} \leq b \right\}$$

where, a and b satisfy $a^{\frac{n}{2}} \exp(-\frac{a}{2}) = b^{\frac{n}{2}} \exp(-\frac{b}{2})$. Thus, the values of a and b, that give the $1-\alpha$ confidence interval must satisfy, $a^{\frac{n}{2}} \exp(-\frac{a}{2}) = b^{\frac{n}{2}} \exp(-\frac{b}{2})$ and $\int_a^b f_T(t)dt = 1-\alpha$.

5. Computational considerations

There are no computational troubles in obtaining confidence intervals for μ. For σ^2, some specific calculations have to be done in order to get a confidence interval with the prescribed property. All confidence intervals considered are of the form,

$$\left\{ \sigma^2 : \frac{(n-1)S_{n-1}^2}{b} \leq \sigma^2 \leq \frac{(n-1)S_{n-1}^2}{a} \right\}.$$

Different a and b will be assigned for each kind of interval. Let, a_s and b_s, denote the a and b values for the shortest, a_u and b_u for the unbiased, a_{et} and b_{et} for the equally tailed, a_{LR} and b_{LR} for the likelihood ratio test based $1-\alpha$ confidence intervals. These values can be calculated running the following MATLAB codes.

```
alfa=.05;n=26;
aa=chi2inv(alfa,n-1);
for a=.1:.01:aa
b=chi2inv((1-alfa+chi2cdf(a,n-1)),n-1);
if a 2*chi2pdf(a,n-1)-b 2*chi2pdf(b,n-1)>=0;break;end
end
as=a
bs=b
for a=.1:.01:aa
b=chi2inv((1-alfa+chi2cdf(a,n-1)),n-1);
if a*chi2pdf(a,n-1)-b*chi2pdf(b,n-1)>=0;break;end
end
au=a
```

```
bu=b
aet=chi2inv(alfa/2,n-1)
bet=chi2inv(1-alfa/2,n-1)
for a=.1:.01:aa
b=chi2inv((1-alfa+chi2cdf(a,n-1)),n-1);
if abs(a (n/2)*exp(-a/2)- b (n/2)*exp(-b/2))<=0.01;break;end
end
aLR=a
bLR=b
as = 14.2700 bs = 45.7722
au = 13.5200 bu = 41.6758
aet= 13.1197 bet= 40.6465
aLR= 14.6100 bLR= 63.9172
```

For an example, a random sample of size $n = 26$ from $N(\mu = 5, \sigma^2 = 4)$ is generated by the MATLAB's code *randn(26,1)*2+5*.The generated numbers, estimates $\bar{x}$, s_{n-1}^2 and %95 confidence intervals are as follows.

X :

4.8413	3.1929	4.5926	1.8392	5.5776
8.0703	5.0718	0.8914	4.8427	4.1414
3.7870	3.7449	5.2651	3.6367	5.1116
2.3053	6.0708	8.1859	2.9509	4.2643
5.9388	6.1058	7.0368	2.5313	4.0701
				5.7419

$\bar{x} = 4.6080 \quad s_{n-1}^2 = 3.1564$

Parameter	Type of Interval	Estimated Interval	Lenght
μ	shortest, unbiased, equally tailed, likelihood ratio based		
	σ^2 known	[3.8392,5.3768]	1.5376
	σ^2 unknown	[3.8904,5.3256]	1.4352
σ^2	shortest	[1.7240,5.5297]	3.8058
σ^2	unbiased	[1.8934,5.8365]	3.9431
σ^2	equally tailed	[1.9414,6.0146]	4.0732
σ^2	likelihood ratio based	[1.2346,5.4011]	4.1665

ÖZET. Bu makalede normal dağılım parametreleri için yansız, en kısa eşit kuyruklu ve düzgün en isabetli (UMA) aralık tahmin edicileri gözden geçirilmiştir. İstatistiksel hesaplamalarda kullanılmak üzere MATLAB kodları sunulmuştur.

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