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# SOME SOLUTIONS AND DECOMPOSITIONS FOR A CLASS OF EQUATIONS

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ABSTRACT. This paper presents some solutions and decompositions for a class of singular partial differential equations which consist of iterated differential operator

## 1. INTRODUCTION

In this work, we are interested in a class of partial differential equations of the form

$$L_\alpha^m u = 0 \tag{1.1}$$

where $m$ is an arbitrary positive integer and

$$L_\alpha = \frac{\partial^2}{\partial y^2} + \frac{\alpha}{y}\frac{\partial}{\partial y} + P. \tag{1.2}$$

The m-times iterated operators $L_\alpha^m$ defined by the relations

$$L_\alpha^{k+1} u = L_\alpha(L_\alpha^k u), \quad k = 1, 2, \dots, m-1.$$

In (1.2), $\alpha$ is a real parameter and $P$ is any linear differential operator of arbitrary order $r$ and depends only on the variables $x_1, x_2, \dots, x_n$ such that

$$\frac{\partial}{\partial y}P(u) = P\left(\frac{\partial u}{\partial y}\right). \tag{1.3}$$

The domain of the operator $L_\alpha$ is the set of all real valued functions $u(x,y)$ of class $C^r(D_1)\bigcap C^2(D_2)$, where $x = (x_1, \dots, x_n)$, $D_1$ and $D_2$ are the regularity domains of u in $R^n$ and $R$, respectively. We denote any solution of equation $L_\alpha u = 0$ by $u^\alpha(x,y)$ or simply by $u^\alpha$. Clearly, the equation (1.1) contains some classical equations among which the best known are Laplace equation, wave equation, the

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generalized axially symmetric potential theory (GASPT) equation, Euler-Poisson-Darboux (EPD) equation and their iterated forms. As was pointed out by Weinstein [3] solutions of equation $L_\alpha u = 0$ satisfy the two fundamental recursion relations

$$u^\alpha = y^{1-\alpha} u^{2-\alpha}$$

and

$$\frac{1}{y}\frac{\partial u^\alpha}{\partial y} = u^{\alpha+2}.$$

The first relation establishes a one to one correspondence between solutions $u^\alpha$ and $u^{2-\alpha}$ and the second relation generates from a solution $u^\alpha$ to a solution $u^{\alpha+2}$. For some special cases of the operator $L_\alpha$, numerous solutions and decomposition formulas for equation (1.1) are well known. For example, in [5] for the operator

$$L_0 = \frac{\partial^2}{\partial y^2} + \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{1.4}$$

Almansi showed that the general solution of equation (1.1) in which $L_\alpha$ is denoted by (1.4) can be decomposed into

$$u = u_1^0 + y u_2^0 + \cdots + y^{m-1} u_m^0 \tag{1.5}$$

where all $u^0$ satisfy $L_0 u = 0$ and each term $y^{k-1}u_k^0$ $(k = 1, ..., m)$ satisfies an equation $L_\alpha^k u = 0$ of order $2k$. Later in [3], Weinstein proved that for the operator

$$L_\alpha = \frac{\partial^2}{\partial y^2} + \frac{\alpha}{y}\frac{\partial}{\partial y} \mp \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{1.6}$$

if the functions $u^\alpha$, $u^{\alpha-2}$, $u^{\alpha-4}$ etc. are arbitrary solutions of the equations $L_\alpha u = 0$, $L_{\alpha-2} u = 0$, $L_{\alpha-4} u = 0$ etc. respectively, then

$$u = \sum_{i=0}^{m-1} u^{\alpha-2i} = u^\alpha + u^{\alpha-2} + u^{\alpha-4} + \cdots + u^{\alpha-2(m-1)} \tag{1.7}$$

is a general solution of the iterated equation (1.1) in which $L_\alpha$ is denoted by (1.6). Further, in [6] Payne showed that the equation (1.1) in which $L_\alpha$ is denoted by (1.6), also admits the solution of the form

$$u = \sum_{i=0}^{m-1} y^{2i} u^{\alpha+2i} = u^\alpha + y^2 u^{\alpha+2} + y^4 u^{\alpha+4} + \cdots + y^{2(m-1)} u^{\alpha+2(m-1)}. \tag{1.8}$$

Moreover, it was pointed out that the representations (1.7) and (1.8) remain valid for solutions of equation (1.1) provided that $P$ is any linear differential operator satisfying (1.3). It will be one of the purposes of this paper to show that analogous to some known results, it is possible to find alternative decomposition formulas for solutions of equation (1.1). Finally, we note that Altın in [1] presented a generalization of the Almansi's expansion, a homogeneous function expansion and the Lord Kelvin principle for the solutions of a class of iterated elliptic or ultrahyperbolic equations which are included by the equation (1.1) and in [2] gave some particular solutions for equation (1.1), where $L_\alpha$ as given by (1.6), in terms of Bessel functions. More recently, in [4] we give same recursion relations which also hold for solutions of equation (1.1).

## 2. Some Solutions

The main object of this section is to introduce some particular solutions of equation (1.1).

Now, we first establish the following lemma.

**Lemma 2.1.** *Let $k$ be arbitrary positive integer. Then*

$$L_\alpha \left(P^k u\right) = P^k \left(L_\alpha u\right) \tag{2.1}$$

*where $P^k$ denotes the successive iterations of the operator $P$, $k$ times onto itself.*

**Proof.** We will proof this lemma by induction on $k$. Since $P$ is free of the variable $y$, it is easily seen that expression (2.1) is correct for $k = 1$, that is,

$$L_\alpha \left(Pu\right) = P\left(L_\alpha u\right). \tag{2.2}$$

Now we suppose that it holds for $k-1$, i.e. let

$$L_\alpha \left(P^{k-1} u\right) = P^{k-1} \left(L_\alpha u\right). \tag{2.3}$$

Hence by making use of (2.2) and (2.3), we then find

$$\begin{aligned} L_\alpha \left(P^k u\right) &= L_\alpha \left[P^{k-1}\left(Pu\right)\right] \\ &= P^{k-1}\left[L_\alpha\left(Pu\right)\right] \\ &= P^{k-1}\left[P\left(L_\alpha u\right)\right] \\ &= P^k\left(L_\alpha u\right) \end{aligned}$$

which is the required result.

**Lemma 2.2.** *Let $m$ and $k$ be arbitrary positive integers. Then*

$$L_\alpha^m \left(P^k u\right) = P^k \left(L_\alpha^m u\right). \tag{2.4}$$

**Proof.** From Lemma1, we already know that $L_\alpha \left(P^k u\right) = P^k \left(L_\alpha u\right)$. Application of the operator $L_\alpha$ on both sides of this gives

$$\begin{aligned} L_\alpha^2 \left(P^k u\right) &= L_\alpha \left[P^k\left(L_\alpha u\right)\right] \\ &= P^k\left[L_\alpha\left(L_\alpha u\right)\right] \\ &= P^k\left(L_\alpha^2 u\right). \end{aligned}$$

Hence, in a similar manner by applying the operator $L_\alpha$ consecutively $m-2$ times on both sides of the last equality we immediately obtain the formula (2.4).

Now, using Lemma2 we can prove our first theorem.

**Theorem 2.3.** *If $u$ is a solution of the equation* (1.1), *then for $k = 1, 2, \ldots$ each of the functions $P^k u$ is also a solution of the equation* (1.1).

**Proof.** By the hypothesis, since $u$ is a solution of equation (1.1), from (2.4) we have

$$L_\alpha^m \left(P^k u\right) = 0$$

which means that the functions $Pu$, $P^2u$, $P^3u, \ldots$ satisfy the equation (1.1).

**Example 1.** In equation (1.1), taking $m = 1, \alpha = -2$ and $P = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$, we consider the GASPT equation

$$L_{-2}u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial y^2} - \frac{2}{y}\frac{\partial u}{\partial y} = 0. \tag{2.5}$$

A simple computation shows that

$$u^{-2}(x, y) = u = (x_1^2 + x_2^2 + y^2)^{\frac{1}{2}}$$

satisfies (2.5). Hence by Theorem1,

$$P^k u = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)^k (x_1^2 + x_2^2 + y^2)^{\frac{1}{2}}, \quad k = 1, 2, \ldots$$

are also solutions of the same equation. For instance,

$$Pu = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)(x_1^2+x_2^2+y^2)^{\frac{1}{2}} = 2\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{1}{2}} - \left(x_1^2 + x_2^2 + y^2\right)^{-\frac{3}{2}}\left(x_1^2 + x_2^2\right)$$

and

$$\begin{aligned} P^2u &= \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)^2 (x_1^2 + x_2^2 + y^2)^{\frac{1}{2}} \\ &= -8\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{3}{2}} + 24\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{5}{2}}\left(x_1^2 + x_2^2\right) \\ &\quad - 15\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{7}{2}}\left(x_1^2 + x_2^2\right)^2 \end{aligned}$$

are two solutions of equation (2.5).

**Theorem 2.4.** *Let $T$ be an operator denoted by $T = y\frac{\partial}{\partial y}$, and let $m \geq 2$ be arbitrary positive integer. Suppose also that $u$ is a solution of equation $L_\alpha^{m-1}u = 0$. Then $Tu$ is a solution of equation* (1.1).

**Proof.** By direct calculation, it can be shown that

$$L_\alpha (Tu) = (2 + T) L_\alpha u - 2Pu. \tag{2.6}$$

Application of the operator $L_\alpha$ on both sides of expression (2.6) yields

$$\begin{aligned} L_\alpha^2 (Tu) &= L_\alpha \left[(2 + T) L_\alpha u - 2Pu\right] \\ &= 2L_\alpha^2 u + L_\alpha \left[T \left(L_\alpha u\right)\right] - 2L_\alpha (Pu) \end{aligned}$$

and by means of (2.6),

$$L_\alpha^2 (Tu) = 2L_\alpha^2 u + (2+T) L_\alpha^2 u - 2P (L_\alpha u) - 2L_\alpha (Pu).$$

Further, from Lemma1 since $L_\alpha P = PL_\alpha$, we can write

$$\begin{aligned} L_\alpha^2 (Tu) &= (4+T) L_\alpha^2 u - 4P (L_\alpha u) \\ &= [(4+T) L_\alpha - 4P] L_\alpha u. \end{aligned}$$

Hence, applying the operator $L_\alpha$ repeatedly on both sides of this, by induction we readily obtain the formula

$$L_\alpha^m (Tu) = [(2m+T) L_\alpha - 2mP] L_\alpha^{m-1} u. \tag{2.7}$$

By the hypothesis, since $u$ is a solution of the equation $L_\alpha^{m-1} u = 0$, (2.7) gives $L_\alpha^m (Tu) = 0$ which means that the function $Tu$ satisfies equation (1.1).

**Example 2.** We again consider the equation (2.5) and its solutions in example 1. Thus, from Theorem2, two-times iterated GASPT equation

$$\begin{aligned} L_{-2}^2 u = {} & \frac{\partial^4 u}{\partial x_1^4} + \frac{\partial^4 u}{\partial x_2^4} + \frac{\partial^4 u}{\partial y^4} + 2\frac{\partial^4 u}{\partial x_1^2 x_2^2} + 2\frac{\partial^4 u}{\partial x_1^2 \partial y^2} + 2\frac{\partial^4 u}{\partial x_2^2 \partial y^2} - \frac{4}{y}\frac{\partial^3 u}{\partial x_1^2 \partial y} - \frac{4}{y}\frac{\partial^3 u}{\partial x_2^2 \partial y} \\ & - \frac{4}{y}\frac{\partial^3 u}{\partial y^3} + \frac{8}{y^2}\frac{\partial^2 u}{\partial y^2} - \frac{8}{y^3}\frac{\partial u}{\partial y} = 0 \end{aligned}$$

has the solutions

$$Tu = y\frac{\partial}{\partial y}\left[\left(x_1^2 + x_2^2 + y^2\right)^{\frac{1}{2}}\right] = y^2\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{1}{2}}$$

$$\begin{aligned} T(Pu) &= y\frac{\partial}{\partial y}\left[2\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{1}{2}} - \left(x_1^2 + x_2^2 + y^2\right)^{-\frac{3}{2}}\left(x_1^2 + x_2^2\right)\right] \\ &= y^2\left[-2\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{3}{2}} + 3\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{5}{2}}\left(x_1^2 + x_2^2\right)\right] \end{aligned}$$

and

$$\begin{aligned} T(P^2 u) = {} & y\frac{\partial}{\partial y}\Big[-8\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{3}{2}} + 24\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{5}{2}}\left(x_1^2 + x_2^2\right) \\ & - 15\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{7}{2}}\left(x_1^2 + x_2^2\right)^2\Big] \\ = {} & y^2\Big[24\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{5}{2}} - 120\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{7}{2}}\left(x_1^2 + x_2^2\right) \\ & + 105\left(x_1^2 + x_2^2 + y^2\right)^{-\frac{9}{2}}\left(x_1^2 + x_2^2\right)^2\Big]. \end{aligned}$$

## 3. Applications of Theorem 2

In this section, by making use of Theorem 2, we shall give alternative decompositions for solutions of the iterated equation (1.1).

**Theorem 3.1.** *Let $m \geq 2$ be arbitrary integer. If $u^{\alpha}$ any solution of $L_{\alpha}u = 0$, then*

$$u_1 = y^2 \sum_{i=0}^{m-2} u^{\alpha+2(1-i)} \tag{3.1}$$

*is a solution of equation* (1.1).

**Proof.** For the equation (1.1) the following decomposition formula is well-known

$$u = \sum_{i=0}^{m-1} u^{\alpha-2i} = u^{\alpha} + u^{\alpha-2} + u^{\alpha-4} + \cdots + u^{\alpha-2(m-1)} \tag{3.2}$$

(see [3]). In (3.2) replacing $m$ by $m-1$ we immediately find that

$$u = \sum_{i=0}^{m-2} u^{\alpha-2i} = u^{\alpha} + u^{\alpha-2} + u^{\alpha-4} + \cdots + u^{\alpha-2(m-2)}$$

is a solution of equation $L_{\alpha}^{m-1}u = 0$. Thus by Theorem2,

$$u_1 = Tu = y\frac{\partial}{\partial y}\left(\sum_{i=0}^{m-2} u^{\alpha-2i}\right) = y\sum_{i=0}^{m-2}\frac{\partial u^{\alpha-2i}}{\partial y} \tag{3.3}$$

satisfies equation (1.1). On the other hand, If we take $\alpha - 2i$ in place of $\alpha$ in the recursion relation

$$\frac{1}{y}\frac{\partial u^{\alpha}}{\partial y} = u^{\alpha+2}$$

which expresses $u^{\alpha+2}$ in terms of $u^{\alpha}$, then it becomes

$$\frac{1}{y}\frac{\partial u^{\alpha-2i}}{\partial y} = u^{\alpha+2(1-i)}$$

or

$$\frac{\partial u^{\alpha-2i}}{\partial y} = yu^{\alpha+2(1-i)}.$$

Substituting this on the right side of (3.3) we receive the solution (3.1) which is the desired result. We note that by virtue of the recursion relation $u^{\alpha} = y^{1-\alpha}u^{2-\alpha}$ that relates a $u^{\alpha}$ to every $u^{2-\alpha}$ and vice versa, the solution (3.1) can be expressed in the alternative form

$$u_1 = y^{1-\alpha}\sum_{i=0}^{m-2} y^{2i}u^{-\alpha+2i}.$$

**Theorem 3.2.** *Let $m \geq 2$ be arbitrary integer. If $u^\alpha$ any solution of $L_\alpha u = 0$, then*

$$u_2 = \sum_{i=1}^{m-2} 2iy^{2i}u^{\alpha+2i} + \sum_{i=0}^{m-2} y^{2i+2}u^{\alpha+2(1+i)} \tag{3.4}$$

*is a solution of equation* (1.1).

**Proof.** The proof is similar to that of the processing theorem. As we mentioned before, another decomposition formula for solutions of equation (1.1) is given by

$$u = \sum_{i=0}^{m-1} y^{2i}u^{\alpha+2i} = u^\alpha + y^2u^{\alpha+2} + y^4u^{\alpha+4} + \cdots + y^{2(m-1)}u^{\alpha+2(m-1)} \tag{3.5}$$

(see [6]). If we take $m-1$ in place of $m$ in (3.5), we then find that

$$u = \sum_{i=0}^{m-2} y^{2i}u^{\alpha+2i} = u^\alpha + y^2u^{\alpha+2} + y^4u^{\alpha+4} + \cdots + y^{2(m-2)}u^{\alpha+2(m-2)}$$

is a solution of equation $L_\alpha^{m-1}u = 0$. Hence, from Theorem 2 it follows that

$$\begin{aligned} u_2 = Tu &= y\frac{\partial}{\partial y}\left(\sum_{i=0}^{m-2} y^{2i}u^{\alpha+2i}\right) \\ &= y\sum_{i=0}^{m-2}\frac{\partial}{\partial y}\left(y^{2i}u^{\alpha+2i}\right) \\ &= \sum_{i=0}^{m-2}\left(2iy^{2i}u^{\alpha+2i} + y^{2i+1}\frac{\partial u^{\alpha+2i}}{\partial y}\right) \end{aligned} \tag{3.6}$$

satisfies equation (1.1). By using the recursion formula

$$\frac{1}{y}\frac{\partial u^\alpha}{\partial y} = u^{\alpha+2}$$

we can write (3.6) as follows:

$$u_2 = \sum_{i=1}^{m-2} 2iy^{2i}u^{\alpha+2i} + \sum_{i=0}^{m-2} y^{2i+2}u^{\alpha+2(1+i)}.$$

Thus, the proof is completed. It should be note that by aid of the relation $u^\alpha = y^{1-\alpha}u^{2-\alpha}$ the solution (3.4) also can be represented in the form

$$u_2 = y^{1-\alpha}\left(\sum_{i=1}^{m-2} 2iu^{-\alpha+2(1-i)} + \sum_{i=0}^{m-2} u^{-\alpha-2i}\right).$$

**ÖZET:**Bu çalışmada ardışık türev operatörlerini içeren kısmi türevli denlemlerin bir sınıfı için bazı çözümler ve dekomposizyon formülleri verilmiştir.

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