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ON THE ROTATION MATRICES IN THE SEMI-EUCLIDEAN SPACE

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ABSTRACT. Chong and Andrews studied on the rotations matrix in E^3[2]. In this study, Cayley formula in E_1^3 is obtained from a semi skew-symmetric matrix, generalizating Andrews' result with some changes. Moreover, some results about Cayley formula are given.

1. INTRODUCTION

In E_1^3, let $R(s,\theta)$ denote a right handed Lorentzian rotation about a unit spacelike vector s through an angle θ. Note that a left Lorentzian rotation can be expressed as a right handed Lorentzian rotation about $-s$ and that

$$R^{-1}(s,\theta) = R(-s,\theta).$$

If $r \longrightarrow r'$ is under a rotation, we may write

$$R(s,\theta)r = r' = Ar.$$

where A is a real 3×3 semi-orthogonal matrix which will be proved to be proper semi-orthogonal (i.e., $A^{-1} = \epsilon A^T \epsilon$, $\det A = 1$ and $\epsilon = dia\,(-1,1,1)$[4]). Besides, it will also be proved that every real 3×3 proper semi-orthogonal matrix represents a Lorentzian rotation.

The Lorentzian or Minkowski 3-space E_1^3 is the Euclidean 3-space provided with the Lorentzian inner product

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle := -x_1y_1 + x_2y_2 + x_3y_3$$

and Lorentzian vector product

$$\overrightarrow{x} \wedge \overrightarrow{y} := (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$$

where $\overrightarrow{x} = (x_1,x_2,x_3)$ and $\overrightarrow{y} = (y_1,y_2,y_3)$.

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An arbitrary vector $\overrightarrow{x} = (x_1, x_2, x_3)$ in E_1^3 can have one of three Lorentzian causal characters: it is spacelike if $\langle \overrightarrow{x}, \overrightarrow{x} \rangle > 0$ or $\overrightarrow{x} = 0$, timelike if $\langle \overrightarrow{x}, \overrightarrow{x} \rangle < 0$ and null (light-like) if $\langle \overrightarrow{x}, \overrightarrow{x} \rangle = 0$. Recall that the pseudo-norm of an arbitrary vector $\overrightarrow{x} \in E_1^3$ is given by $\|\overrightarrow{x}\| = \sqrt{|\langle \overrightarrow{x}, \overrightarrow{x} \rangle|}$.

Let us take the plane through O perpendicular to s as the Lorentz plane for timelike vector ξ. We shall use ξ to denote a vector in E_1^3 and allow operations on it by matrices such as A, as well. For each unit vector s, we define the associated semi skew-symmetric matrix

$$S = \begin{bmatrix} 0 & c & -b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \leftrightarrow s = (a, b, c)$$

($S^T = -\epsilon S \epsilon$, matrix $\epsilon = dia\,(-1, 1, 1)$[4]) so that s and S determine each other uniquely [1].

Also observe that $Sr = s \wedge r$, $S\overrightarrow{s} = 0$ and $s \wedge \xi = S\xi$. Here notation "$\wedge$" represents vectorial product in Lorentzian E_1^3 and $s \wedge \xi$ is perpendicular to both s and ξ[1].

Lemma 1.1. *Let λ be an eigenvalue of a matrix A such that*

$$Au = \lambda u, u \neq 0.$$

Then

$$A^2 u = \lambda A u = \lambda^2 u$$

and

$$A^3 u = \lambda^3 u, etc.$$

It follows that if f is a polynomial function, $f(A)u = f(\lambda)u$, i.e., $f(\lambda)$ is an eigenvalue of $f(A)$ [3].

Lemma 1.2. *Let $S_{3\times 3}$ be a real skew-symmetric matrix. Then,*

a) *If $S \leftrightarrow \overrightarrow{s}$ is timelike vector, the eigenvalues of matrix S all lie on the imaginary axis [1].*

b) *If $S \leftrightarrow \overrightarrow{s}$ is spacelike vector, the eigenvalues of matrix S all lie on the real axis [1].*

2. Quadratic Representations

Theorem 2.1. *In E_1^3, let $R(\overrightarrow{s}, \theta)$ denote a righ handed Lorentzian rotation about a unit spacelike vector s through an angle θ. Then,*

$$R(\overrightarrow{s}, \theta) = Ar,$$

where $A = S^0 + S(\sinh\theta) - S^2(1 - \cosh\theta) = f(s)$ say; also A is a proper semi-orthogonal matrix.

Proof. An arbitrary vector in E_1^3 is expressible as $\overrightarrow{r} = k\overrightarrow{s} + \overrightarrow{\xi}, k \in IR$. Let s be a spacelike axis (vector) and ξ be a timelike vector. Let us find the semi-orthogonal matrix to be obtained by rotating the vector ξ which is perpendicular to the s axis through an angle θ.

(i) Let r be in E_1^2. At the Lorentz plane, if ξ rotates about a unit spacelike vector s through an angle θ, we get

$$R(s,\theta)\xi = (\cosh\theta)\xi + \sinh\theta(s \wedge \xi)$$
$$R(s,\theta)\xi = (\cosh\theta)\xi + \sinh\theta(S\xi)$$
$$R(s,\theta)\xi = (\cosh\theta)\xi + \sinh\theta(Sr).$$

(ii) Let r be in E_1^3.

$$\begin{aligned} s \wedge (s \wedge \xi) = \langle s, s\rangle \xi - \langle s, \xi\rangle s \\ = 1.\xi - 0.s \\ = \xi \end{aligned} \tag{2.1}$$

or

$$S^2\xi = \xi.$$

$r = ks + \xi$,(r is the starting position vector)

$$\begin{aligned} s \wedge \xi &= S\xi \\ s \wedge r &= Sr \\ s \wedge r &= S(ks + \xi) \\ s \wedge r &= k(Ss) + S\xi \\ s \wedge r &= S\xi. \end{aligned} \tag{2.2}$$

In the Lorentz space, if ξ rotates about a unit spacelike vector s through an angle θ, by using (2.1) and (2.2) we get

$$\begin{aligned} R(s,\theta)r &= R(s,\theta)(ks + \xi) \\ &= R(s,\theta)(ks) + R(s,\theta)\xi \\ &= ks + R(s,\theta)\xi \\ &= (r - \xi) + [(\cosh\theta)\xi + \sinh\theta(Sr)] \\ &= r - (1 - \cosh\theta)\xi + \sinh\theta(Sr) \\ &= r - (1 - \cosh\theta)S^2\xi + \sinh\theta(Sr) \\ &= r - (1 - \cosh\theta)S^2 r + \sinh\theta(Sr) \\ &= \left[I - (1 - \cosh\theta)S^2 + (\sinh\theta)S\right] r \\ &= \left[S^0 + (\sinh\theta)S^1 - (1 - \cosh\theta)S^2\right] r \\ &= Ar \end{aligned}$$

as required and

$$A = S^0 + (\sinh\theta)S^1 - (1 - \cosh\theta)S^2 = f(S).$$

It remains to show that A is proper semi-orthogonal (i.e., $A^{-1} = \epsilon A^T \epsilon$ and $\det A = +1$)

$$\begin{aligned}
A^T &= S^0 + (\sinh\theta)S^T - (1-\cosh\theta)(S^2)^T \\
\epsilon A^T \epsilon &= \epsilon\left[S^0 + (\sinh\theta)S^T - (1-\cosh\theta)(S^2)^T\right]\epsilon \\
&= \epsilon S^0 \epsilon + \epsilon(\sinh\theta)S^T\epsilon - \epsilon(1-\cosh\theta)(S^2)^T\epsilon \\
&= \epsilon^2 S^0 + (\sinh\theta)\epsilon S^T\epsilon - (1-\cosh\theta)\epsilon(S^T S^T)\epsilon \\
&= S^0 - (\sinh\theta)S - (1-\cosh\theta)\epsilon S^T I_3 S^T\epsilon \\
&= S^0 - (\sinh\theta)S - (1-\cosh\theta)\epsilon S^T\epsilon^2 S^T\epsilon \\
&= S^0 - (\sinh\theta)S - (1-\cosh\theta)(\epsilon S^T\epsilon)(\epsilon S^T\epsilon) \\
&= S^0 - (\sinh\theta)S - (1-\cosh\theta)(-S)(-S) \\
&= S^0 - (\sinh\theta)S - (1-\cosh\theta)S^2 \\
&= f(-S),
\end{aligned}$$

$$A^{-1} = f(-S).$$

Hence,

$$A^{-1} = f(-S) = R(-s,\theta) = R^{-1}(s,\theta).$$

Therefore, A is semi-orthogonal. (This also follows from the invariance of $\|r\|$).

Also the eigenvalues of the $S \leftrightarrow s$ (spacelike) is $0, 1, -1$ so from Lemma 1.2 we see that the eigenvalues of $f(S)$, i.e., of A, are i.e., $f(0), f(1), f(-1)$,i.e,

$$f(0) = 1 + (\sinh\theta).0 - (1-\cosh\theta).0^2 = 1$$

$$f(1) = 1 + (\sinh\theta).1 - (1-\cosh\theta).1^2 = e^\theta$$

$$f(-1) = 1 + (\sinh\theta).(-1) - (1-\cosh\theta).(-1)^2 = e^{-\theta}.$$

The product of these gives $\det A = +1$ i.e., $\det A = 1.e^\theta e^{-\theta} = 1$. Hence A is a proper semi-orthogonal matrix. □

3. Cayley Mapping

Theorem 3.1. *If T is a real semi skew-symmetric 3×3 matrix then the matrix*

$$A = (I-T)^{-1}(I+T)$$

represents a Lorentzian rotation and is a proper semi-orthogonal (This is called the Cayley mapping of T).

For $T = \begin{bmatrix} 0 & c & -b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \leftrightarrow$ *spacelike vector $t = (a, b, c)$, we shall show that A represents the rotation $R(\widehat{t}, \theta)$, and t represents spacelike vector, where $\|t\| = \sqrt{|-a^2+b^2+c^2|}$ and $\widehat{t} = \frac{t}{\|t\|}$.*

Proof. The characteristic equation of T is $\lambda^3 - t^2\lambda = 0$, and so from the Cayley-Hamilton theorem, $T^3 = t^2 T$. Since t is a spacelike vector, $\det(I - T)$ is always different from zero. Now $(I - T)^{-1}$ is expressible in the form of $(I - T)^{-1} = I + aT + bT^2$ if and only if

$$(I-T)^{-1}(I-T) = I + (a - bt^2 - 1)T + (b - a)T^2,$$

i.e., if and only if $a - bt^2 = 1$ and $a - b = 0$, i.e., $a = b = \frac{1}{1-t^2}$. So

$$(I-T)^{-1} = I + \frac{1}{1-t^2}T + \frac{1}{1-t^2}T^2$$

and

$$A = (I-T)^{-1}(I+T).$$

Then we get

$$\begin{aligned} A &= \left(I + \frac{1}{1-t^2}T + \frac{1}{1-t^2}T^2\right)(I+T) \\ &= I + \frac{2}{1-t^2}T + \frac{2}{1-t^2}T^2 \end{aligned}$$

Let $\widehat{T} = \frac{T}{t}$, then

$$A = I + \left(\frac{2t}{1-t^2}\right)\widehat{T} + \left(\frac{2t^2}{1-t^2}\right)\widehat{T}^2$$

$$\begin{aligned} A &= I + (\sinh\theta)\,\widehat{T} - (1 - \cosh\theta)\,\widehat{T}^2 \\ &= f(\widehat{T}). \end{aligned}$$

□

It follows from Theorem 2.1 that A represents $R(\overrightarrow{s}, \theta)$, and is proper semi-orthogonal. The case $\theta = 0$ corresponds to the $R = I$ but $T \neq 0$ since $\|\widehat{t}\| = 1$.

Theorem 3.2. *If $A_{3\times3}$ is a real proper semi-orthogonal matrix and $T = (A - I)(A + I)^{-1}$ then T is semi skew-symmetric and its Cayley mapping is A.*

Proof. From the definition of T we have

$$\begin{aligned} (A+I)T &= (A+I)(A-I)(A+I)^{-1} \\ &= (A-I)(A+I)(A+I)^{-1} \\ &= A - I) \end{aligned}$$

Hence

$$T^T(A^T + I) = (A^T - I)$$

and

$$\begin{aligned} T^T(A^T+I)(\epsilon A\epsilon) &= (A^T-I)(\epsilon A\epsilon) \\ T^T\left(A^T\epsilon A\epsilon+\epsilon A\epsilon\right) &= A^T(\epsilon A\epsilon)-\epsilon A\epsilon \\ T^T\left(\epsilon\epsilon A^T\epsilon A\epsilon+\epsilon A\epsilon\right) &= \epsilon\epsilon A^T(\epsilon A\epsilon)-\epsilon A\epsilon \\ T^T\left[\epsilon(\epsilon A^T\epsilon)A\epsilon+\epsilon A\epsilon\right] &= \epsilon(\epsilon A^T\epsilon)A\epsilon-\epsilon A\epsilon \\ T^T\left(\epsilon A^{-1}A\epsilon+\epsilon A\epsilon\right) &= \epsilon A^{-1}A\epsilon-\epsilon A\epsilon \\ T^T\left(\epsilon I\epsilon+\epsilon A\epsilon\right) &= \epsilon I\epsilon-\epsilon A\epsilon \\ T^T\left(I+\epsilon A\epsilon\right) &= -\epsilon A\epsilon. \end{aligned}$$

Since A is proper semi-orthogonal and $(I+A)$ is regular, there exists $(I+\epsilon A\epsilon)^{-1}$ where

$$\begin{aligned} \det\left[I+\epsilon A\epsilon\right]^{-1} &= \det\left[\epsilon\epsilon+\epsilon A\epsilon\right]^{-1} \\ &= \det\left[\epsilon(I+A)\epsilon\right]^{-1} \\ &= \det(I+A)^{-1}. \end{aligned}$$

So

$$\begin{aligned} T^T\left(I+\epsilon A\epsilon\right)\left(I+\epsilon A\epsilon\right)^{-1} &= \left(I-\epsilon A\epsilon.\right)\left(I+\epsilon A\epsilon\right)^{-1} \\ T^T &= \left(I-\epsilon A\epsilon\right)\left(I+\epsilon A\epsilon\right)^{-1} \\ &= \left(\epsilon\epsilon-\epsilon A\epsilon\right)\left(I+\epsilon A\epsilon\right)^{-1} \\ &= \epsilon(I-A)\epsilon\epsilon^{-1}(I+A)^{-1}\epsilon \\ &= \epsilon(I-A)I(I+A)^{-1}\epsilon \\ &= -\epsilon\left[(A-I)(A+I)^{-1}\right]\epsilon \\ &= -\epsilon T\epsilon, \end{aligned}$$

that is T is semi skew-symmetric. Also from the definition

$$\begin{aligned} T(A+I) &= (A-I)(A+I)^{-1}(A+I) \\ T(A+I) &= A-I \\ TA+T) &= A-I \\ (I-T)A &= I+T \\ A &= (I-T)^{-1}(I+T) \end{aligned}$$

and this is the Cayley mapping of T. □

Corollary 1. *a)* $A-A^{-1}=(2\sinh\theta)S$
b) $TraceA=1+2cosh\theta$ *(which is the sum of the eigenvalues of A)*

ÖZET: Chong ve Andrews E^3' de dönme matrisleri üzerine çalışmışlardır[2]. Bu çalışmada, E_1^3 de Cayley formülü, Andrews' un sonuçlarında bazı değişiklikler yapılarak genelleştirilip, yarı- antisimetrik bir matristen elde ediliyor. Ayrıca Cayley formülü üzerine bazı sonuçlar veriliyor.

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