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Research Article

Unified classification of pure metric geometries

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Abstract

Almost Norden, almost product Riemannian, almost Norden golden and almost golden Riemannian are pure metric geometries. We introduce α-metric and α-golden metric manifolds to unify the study of almost Norden manifolds and almost product Riemannian manifolds with null trace and almost Norden golden manifolds and almost golden Riemannian manifolds with null trace respectively. Then we can show the classifications of almost Norden manifolds and almost product Riemannian manifolds with null trace in a unified way. The bijection between α-metric and α-golden metric manifolds allows us to classify α-golden metric manifolds, i.e., we classify almost Norden golden manifolds and almost golden Riemannian manifolds with null trace simultaneously. Finally we characterize every class of the above four kind of pure metric manifolds by means of the first canonical and the well-adapted connections which are two distinguished connections shared by α-metric and α-golden metric manifolds.

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1. Introduction

The first and the second fundamental forms of a surface in the Euclidean space $\mathbb{R}^3$ are crucial issues in the study of the surface. As is well-known, they are related by means of the shape operator S as follows

$$II(X,Y) = I(S(X),Y) = I(X,S(Y)) = II(Y,X),$$

where I and II denote the first and the second fundamental form respectively. The first fundamental form is the metric of the surface. The shape operator is an endomorphism of each tangent plane to the surface, which is self-adjoint respect to the metric. The matrix of S is symmetric, thus being diagonalizable, and the eigenvalues are the principal curvatures of the surface.

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Taking into account these ideas one can consider a manifold M endowed with a pseudo-Riemannian metric g and a tensor field F of type $(1,1)$ which is self-adjoint respect to the metric, thus satisfying

$$g(FX,Y)=g(X,FY),\ \ \forall X,Y\in\mathfrak{X}(M). \tag{1.1}$$

According to [18] we also say that the metric g is pure respect to F. If one defines

$$\Phi(X,Y)=g(FX,Y)=g(X,FY)=\Phi(Y,X),\ \ \forall X,Y\in\mathfrak{X}(M), \tag{1.2}$$

then Φ is also a bilinear symmetric map and F has a symmetric matrix when local coordinates are given. This is the topic we are going to study in the present paper, showing in a unified way the classification of geometries admitting such a structure. We will consider the four following cases of polynomial structures of degree 2 (see; *e.g.*, [15]): almost complex, almost product, almost complex golden and almost golden. A pure metric respect one of the above polynomial structures defines a structure of almost Norden, almost product Riemannian, almost Norden golden and almost golden Riemannian manifold respectively.

A tensor field J of type $(1,1)$ on a manifold M is an almost complex or an almost product structure if it satisfies $J^2=-Id$ or $J^2=Id$ respectively, where Id denotes the identity tensor field of type $(1,1)$ on M. Both polynomial structures can be unified under the notion of α-structure. A tensor field J of type $(1,1)$ on M satisfying $J^2=\alpha Id$, where $\alpha\in\{-1,1\}$, is called an α-structure. One says that (M,J) is a $(J^2=\pm1)$-manifold (see [7]).

A metric g on a $(J^2=\pm1)$-manifold (M,J) satisfies (1.1) if and only if it also satisfies the equivalent condition $g(JX,JY)=\alpha g(X,Y)$, for all vector fields X,Y on M. Then, $(J^2=\pm1)$-manifolds endowed with a pure metric respect to the α-structure are called almost Norden and almost product Riemannian manifolds, according to $\alpha=-1$ or $\alpha=1$ and g being a Riemannian metric. They have been deeply studied in the Mathematical Literature (see; *e.g.*, [11, 12, 17, 19], which are articles focusing on the classification of these manifolds). Almost Norden manifolds necessarily have even dimension, but, in the case of almost product structures, one can introduce Riemannian metrics such that the polynomial structure being an isometry on odd dimensional manifolds. Then, to achieve a unified presentation using α-structures, we restrict our study in the case $\alpha=1$ to almost product Riemannian manifolds such that the trace of the polynomial structure vanishes, which assures that the manifold has even dimension. This kind of metric manifolds are called almost product Riemannian manifolds with null trace.

In this way, given an α-structure J on a even dimensional manifold M and given a pseudo-Riemannian metric g (in fact, Riemannian metric if $\alpha=1$), one says that (M,J,g) is an α-metric manifold if it satisfies one of the above two equivalent conditions

$$g(JX,JY)=\alpha g(X,Y),\ \ g(JX,Y)=g(X,JY),\ \ \forall X,Y\in\mathfrak{X}(M). \tag{1.3}$$

Thus, the notion of α-metric manifold covers almost Norden manifolds and almost product Riemannian manifolds with null trace simultaneously.

Almost complex golden and almost golden structures are also polynomial structures of degree 2 recently introduced by Crasmareanu and Hreţcanu in [2]. Since then, their study has been spread in several directions (see *e.g.*, [1, 3, 10, 13, 14, 16, 20]). The characteristic polynomials of almost complex golden and almost golden structures are $x^2-x-\frac{3}{2}$ and x^2-x-1 respectively, whose roots are $\phi=\frac{1+\sqrt{5}}{2}$ and $\bar{\phi}=\frac{1-\sqrt{5}}{2}$ in the golden case, and $\phi_c=\frac{1+\sqrt{5}i}{2}$ and $\bar{\phi}_c=\frac{1-\sqrt{5}i}{2}$ in the complex golden case. The numbers ϕ and ϕ_c are called the golden and the complex golden ratio respectively.

A tensor field φ of type $(1,1)$ on a manifold M is an almost complex golden or almost golden structure if it satisfies $\varphi^2=\varphi-\frac{3}{2}Id$ or $\varphi^2=\varphi+Id$ respectively. Like above, both kind of polynomial structures can be showed in a unified way as follows. A tensor field φ

of type $(1,1)$ on M satisfying

$$\varphi^2(X) = \varphi(X) + \frac{5\alpha - 1}{4}X, \ \ \forall X \in \mathfrak{X}(M), \tag{1.4}$$

is an almost complex golden or almost golden structure according to $\alpha = -1$ or $\alpha = 1$. A polynomial structure φ satisfying identity (1.4) will be generically referred as α-golden structure and (M, φ) will be named as α-golden manifold.

Many of the abovementioned references are devoted to the study of compatible metrics to α-golden structures. Following this unified presentation, one says that a pseudo-Riemannian metric g on an α-golden manifold (M, φ) is compatible with φ if it satisfies one of the two following equivalent conditions

$$g(\varphi X, \varphi Y) = g(\varphi X, Y) + \frac{5\alpha - 1}{4}g(X, Y), \ \ g(\varphi X, Y) = g(X, \varphi Y), \ \ \forall X, Y \in \mathfrak{X}(M). \tag{1.5}$$

If $\alpha = -1$ (resp. $\alpha = 1$ and g is a Riemannian metric) one recovers the notion of almost Norden golden (resp. almost golden Riemannian) manifold (see [2] and [14]).

Almost complex and almost product structures and almost complex golden and almost golden structures are closely related (see [2] and [14]). There exist two bijections between these sets of polynomial structures. One of them is a 1: 1 correspondence between almost complex and almost complex golden structures on a manifold, the other one, between almost product and almost golden structures. In a unified way, the bijection between the set of α-structures J and α-golden structures φ is given by

$$\varphi \mapsto J_\varphi = \frac{(-\alpha)}{\sqrt{5}}(Id - 2\varphi), \ \ J \mapsto \varphi_J = \frac{1}{2}(Id + \alpha\sqrt{5}J). \tag{1.6}$$

This bijection can be extended to the metric case without restraint because

$$g(\varphi X, Y) = g(X, \varphi Y) \Longleftrightarrow g(J_\varphi X, Y) = g(X, J_\varphi Y), \ \ \forall X, Y \in \mathfrak{X}(M). \tag{1.7}$$

If $\alpha = 1$, we restrict our study of α-golden structures φ and compatible pseudo-Riemannian metrics g to Riemannian metrics such that (M, J_φ, g) being an almost product Riemannian manifold with null trace. We say that (M, φ, g) is an α-golden metric manifold if φ and g satisfy the equivalent conditions (1.5) in the case $\alpha = -1$, and, in the case $\alpha = 1$, if φ and g also satisfy the above additional restriction. In this way, there exists a bijection between α-metric and α-golden metric manifolds in the following sense: (M, φ, g) is an α-golden metric manifold if and only if (M, J_φ, g) is an α-metric manifold. In these conditions we say that (J_φ, g) is the α-metric structure induced by the α-golden metric structure (φ, g).

Given an α-golden metric manifold (M, φ, g), if $\alpha = -1$ then (M, J_φ, g) is an almost Norden manifold and therefore g is a pseudo-Riemannian metric of signature (n, n), while if $\alpha = 1$ the trace of the almost product structure J_φ is null. Thus, an α-golden metric manifold always has even dimension.

It is interesting to highlight that equivalence (1.7), which allows to extend the bijection between α-structures and α-golden structures on a manifold to the metric case, it is also fully compatible with the purity condition of the (pseudo)-Riemannian metric in the following sense: a metric g on M is pure respect to an α-golden structure φ if and only if g is pure respect to the α-structure J_φ.

We are interested in the abovementioned four pure metric geometries: almost Norden manifolds, almost product Riemannian manifolds with null trace, almost Norden golden manifolds and almost golden Riemannian manifolds with null trace. We want to study these geometries, paired under the notions of α-metric and α-golden metric manifolds, highlighting common properties of all of them.

Our first goal is to classify α-metric manifolds in a unified way. The starting-points to achieve it are the classifications of almost Norden and almost product Riemannian manifolds with null trace obtained by Ganchev and Borisov and Staikova and Gribachev in [11] and [19] respectively. Given a manifold (M, J, g) in the above conditions, the fundamental tensor field Φ of type $(0, 2)$ defined as in identity (1.2)

$$\Phi(X, Y) = g(JX, Y), \ \forall X, Y \in \mathfrak{X}(M), \tag{1.8}$$

is one of the keys of both classifications, more precisely, its covariant derivative respect to the Levi-Civita connection of g, denoted by ∇^{g}. The next identity reveals its close relationship towards the tensor field $\nabla^{\mathrm{g}} J$ of type $(1, 2)$

$$(\nabla^{\mathrm{g}}_X \Phi)(Y, Z) = g((\nabla^{\mathrm{g}}_X J)Y, Z), \ \forall X, Y, Z \in \mathfrak{X}(M). \tag{1.9}$$

Thus, the tensor field $\nabla^{\mathrm{g}} J$ has also great interest to classify these kind of metric manifolds.

The class $\mathcal{W}_0$, named Kähler Norden manifolds in the complex case and locally product Riemannian manifolds in the product case, is characterized by one of the following equivalent conditions

$$\nabla^{\mathrm{g}} \Phi = 0, \ \nabla^{\mathrm{g}} J = 0.$$

In the case of α-metric manifolds, the class $\mathcal{W}_0$ will be generically referred as Kähler manifolds.

Nevertheless, Kähler manifolds are not the only class of both kind of manifolds which can be characterized by common defining conditions. Other class in the above situation is the class of integrable manifolds, almost Norden and almost product Riemannian manifolds, characterized in both cases by the vanishing of the Nijenhuis tensor of the polynomial structure J, which is defined as follows

$$N_J(X, Y) = J^2[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY], \ \forall X, Y \in \mathfrak{X}(M). \tag{1.10}$$

We will call generically integrable manifolds to this class of α-metric manifolds.

Not all of classes of the classifications of almost Norden manifolds and almost product Riemannian manifolds introduced in [11] and [19] share common defining conditions. For instance, given X, Y, Z vector fields on a $2n$-dimensional manifold M, the class $\mathcal{W}_1$ of both classifications is characterized by

$$\begin{aligned}(\nabla^{\mathrm{g}}_X \Phi)(Y, Z) = &\frac{1}{2n}(g(X, Y)\delta\Phi(Z) + g(X, Z)\delta\Phi(Y)) \\ &+ \frac{1}{2n}(g(X, JY)\delta\Phi(JZ) + g(X, JZ)\delta\Phi(JY)),\end{aligned}$$

in the complex case, while in the product case is characterized by

$$\begin{aligned}(\nabla^{\mathrm{g}}_X \Phi)(Y, Z) = &\frac{1}{2n}(g(X, Y)\delta\Phi(Z) + g(X, Z)\delta\Phi(Y)) \\ &- \frac{1}{2n}(g(X, JY)\delta\Phi(JZ) + g(X, JZ)\delta\Phi(JY)),\end{aligned}$$

The notion of α-metric manifold allows to obtain the following common defining condition of both classes

$$\begin{aligned}(\nabla^{\mathrm{g}}_X \Phi)(Y, Z) = &\frac{1}{2n}(g(X, Y)\delta\Phi(Z) + g(X, Z)\delta\Phi(Y)), \\ &- \frac{\alpha}{2n}(g(X, JY)\delta\Phi(JZ) + g(X, JZ)\delta\Phi(JY)),\end{aligned}$$

for all vector fields X, Y, Z on M (later, we will recall how the 1-form $\delta\Phi$ is defined).

By means of α-metric structures we will show a unified classification of almost Norden manifolds and almost product Riemannian manifolds with null trace obtaining common characteristic conditions for all classes of both kind of even dimensional metric manifolds.

Given an α-golden metric manifold (M, φ, g), bearing in mind (1.6) and (1.10), one has the next identity that link the Nijenhuis tensors of φ and J_φ

$$N_{J_\varphi}(X, Y) = \frac{4}{5} N_\varphi(X, Y), \ \forall X, Y \in \mathfrak{X}(M), \tag{1.11}$$

(see [6, Lemma 22] or [16, Thm. 22] if $\alpha = -1$ and [2, Eq. (3.3)] or [10, Lemma 1.4] if $\alpha = 1$), which allows to claim that (M, φ, g) is an integrable manifold if and only if the α-metric manifold (M, J_φ, g) is integrable too.

Integrable manifolds are not the unique class of almost Norden golden or almost golden Riemannian manifolds that can be classified using the corresponding almost Norden or almost product Riemannian manifolds by the bijection (1.6). In [13] and [14], the authors studied analogous classes to locally product Riemannian and Kähler Norden manifolds previously recalled of almost golden Riemannian and almost Norden golden manifolds, named locally decomposable golden Riemannian and holomorphic Norden golden or Kähler Norden golden manifolds respectively, which are those manifolds such that the Levi-Civita connection of the (pseudo)-Riemannian metric parallelizes the polynomial structure. They proved that the α-golden metric manifold (M, φ, g) belongs to one of the abovementioned classes if and only if the α-metric manifold (M, J_φ, g) is a locally product Riemannian or a Kähler Norden manifold.

We extend the above situation: α-golden metric manifolds will be classified by using the classification of α-metric manifolds and the $1\colon 1$ correspondence (1.6). Thus, given an α-golden metric manifold (M, φ, g), we say that it belongs to certain class according to the classification of α-metric manifolds if the manifold (M, J_φ, g) belongs to this class of manifolds.

Our second goal, starting from the unified classification of α-metric manifolds previously obtained, is to classify α-golden metric manifolds as it is indicated above. In this way, we will obtain classifications of the four abovementioned pure metric geometries paired under the notions of α-metric and α-golden metric manifolds and compatible with the $1\colon 1$ correspondence between both kind of metric manifolds.

The comparison between classifications of α-metric and α-golden metric manifolds allows us to obtain significative results. For instance, in [1] the authors introduced a family of almost Norden golden manifolds named Kähler-Norden-Codazzi golden manifolds taking the class of Kähler-Norden-Codazzi manifolds of almost Norden manifolds as a model. In [5, Cor. 7], we proved that every Kähler-Norden-Codazzi manifold is in fact a Kähler Norden manifold. Thus, as Kähler Norden and Kähler Norden golden manifolds are corresponding classes we also proved that Kähler-Norden-Codazzi golden and Kähler Norden golden manifolds are the same class of almost Norden golden manifolds (see [5, Thm. 12]). This result was also achieved in [20, Cor. 3.4] using a different technique.

Observe that given an α-golden metric manifold (M, φ, g) and its corresponding α-metric manifold (M, J_φ, g), the Levi-Civita connection ∇^{g} only depends on the metric g. In [7] and [8], among all connections that parallelize the α-structure and the metric on an $(J^2 = \pm 1)$-metric manifold, two of them were distinguished, the first canonical and the well-adapted connections. Thus, in the particular case of the α-metric manifold (M, J_φ, g), there exist the two abovementioned connections attached to the induced α-metric structure (J_φ, g). Then, using the bijection between α-metric and α-golden metric structures, one can introduce two distinguished connections on (M, φ, g): the first canonical and the well-adapted connection of (M, J_φ, g). We will call them the first and the well-adapted connection of the manifold (M, φ, g) respectively. Therefore, the Levi-Civita connection, the first canonical and the well-adapted connection are three distinguished connections shared by an α-golden metric manifold and its corresponding α-metric manifold.

Now we will recall how the first canonical and the well-adapted connection of an α-metric (M, J, g) are introduced. The first canonical connection ∇^0 of (M, J, g) is given by

$$\nabla^0_X Y = \nabla^{\mathrm{g}}_X Y + \frac{(-\alpha)}{2}(\nabla^{\mathrm{g}}_X J)JY, \ \ \forall X, Y \in \mathfrak{X}(M). \tag{1.12}$$

It is easy to prove that ∇^0 satisfies $\nabla^0 J = 0$ and $\nabla^0 g = 0$ (see [7, Lemma 3.10]). The well-adapted connection ∇^{w} of (M, J, g) is the unique connection satisfying $\nabla^{\mathrm{w}} J = 0$,

$\nabla^{\mathrm{w}} g = 0$ and the equality

$$g(\mathrm{T}^{\mathrm{w}}(X,Y),Z) - g(\mathrm{T}^{\mathrm{w}}(Z,Y),X) = \alpha(g(\mathrm{T}^{\mathrm{w}}(JZ,Y),JX) - g(\mathrm{T}^{\mathrm{w}}(JX,Y),JZ)), \quad (1.13)$$

for all vector fields X, Y, Z on M, being T^{w} the torsion tensor of ∇^{w} (see [8, Thm. 4.4]).

Starting from identity (1.12) one can prove that the torsion tensor T^0 of ∇^0 satisfies

$$\mathrm{T}^0(X,Y) = \frac{(-\alpha)}{2}\left((\nabla^{\mathrm{g}}_X J)JY - (\nabla^{\mathrm{g}}_Y J)JX\right), \quad (1.14)$$

$$\mathrm{T}^0(JX,JY) + \alpha \mathrm{T}^0(X,Y) = -\frac{1}{2} N_J(X,Y) \ \ \forall X,Y \in \mathfrak{X}(M), \quad (1.15)$$

(see [7, Prop. 5.1]). In that paper, the following identities that link ∇^0 and ∇^{w} and their torsion tensors T^0 and T^{w} were also proven

$$g(\nabla^{\mathrm{w}}_X Y, Z) = g(\nabla^0_X Y, Z) + \frac{\alpha}{8} g(N_J(Y,Z),X), \quad (1.16)$$

$$g(\mathrm{T}^{\mathrm{w}}(X,Y),Z) = g(\mathrm{T}^0(X,Y),Z) + \frac{\alpha}{8}(g(N_J(Y,Z),X) - g(N_J(X,Z),Y)), \quad (1.17)$$

for all vector fields X, Y, Z on M (see [7, Prop. 6.5]). The above identities allow to claim that the first canonical and the well-adapted connection coincide if and only if the manifold (M, J, g) is integrable. Moreover, this class of α-metric manifolds is characterized by one of the following equivalent conditions

$$\mathrm{T}^0(JX,JY) + \alpha \mathrm{T}^0(X,Y) = 0, \ \ \mathrm{T}^{\mathrm{w}}(JX,JY) + \alpha \mathrm{T}^{\mathrm{w}}(X,Y) = 0,$$

for all vector fields X, Y on M (see also [7, Prop. 5.7]).

Integrable manifolds are not the only class of α-metric manifolds that can be characterized using the torsion tensor of one of these connections. In [4, Cor. 5.8], it is proved that quasi-Kähler manifolds, the corresponding class of α-metric manifolds to the class $\mathcal{W}_3$ of the classifications of almost Norden manifolds and almost product Riemannian manifolds with null trace showed in [11] and [19], can be characterized using the torsion tensor T^{w} as follows

$$\mathrm{T}^{\mathrm{w}}(JX,Y) + J\mathrm{T}^{\mathrm{w}}(X,Y) = 0, \ \ \forall X,Y \in \mathfrak{X}(M).$$

Both classes of α-metric manifolds, integrable and quasi-Kähler manifolds, illustrate the closely relationship that exists between the different classes of this kind of metrics and the torsion tensors of the first canonical and the well-adapted connection. Nevertheless, the most important representative of this fact can be found in [12]. In that paper, Ganchev and Mihova introduced the well-adapted connection of an almost Norden manifold in a different way from those used in [8] and characterized all classes of manifolds obtained by Ganchev and Borisov in [11] by means of tensors fields and 1-forms defined from the torsion tensor of the well-adapted connection. Following the ideas and procedures used in [12], first we will obtain defining conditions of all classes of α-metric manifolds using tensors fields and 1-forms defined from the torsion tensors of the first canonical and the well-adapted connections. Thereafter, we will characterize every class of α-golden metric manifolds obtaining new defining conditions from the characteristic conditions expressed using the torsion tensors of the first canonical and the well-adapted connection of their corresponding classes via the bijection between α-metric and α-golden metric manifolds. Summarizing, we will obtain two different classifications of α-metric and α-golden metric manifolds by means of the torsion tensors of the first canonical and the well-adapted connections of the manifolds, which are fully compatible with the 1: 1 correspondence between these two kind of metric manifolds.

Previously and separately, almost product Riemannian and almost golden Riemannian manifolds with null trace were classified by means of the torsion tensor of the first canonical connection, and almost Norden and almost Norden golden manifolds were classified using the torsion tensor of the well-adapted connection, in [9] and [6] respectively. The classifications of both quoted papers can be recovered from those obtained here. Thus, our present work broaden and unify the results showed there. It should be noted that the characterization of all classes of α-metric and α-metric golden by means of the torsion tensor of the first canonical connection follows the procedures used in [9], but the tools we use in the classification of α-golden metric manifolds by means of the torsion tensor of the well-adapted connection differ to those used in [6]. In that paper, the classification of almost Norden golden manifolds by means of the torsion tensor of the well-adapted connection were obtained starting from the classification of almost Norden manifolds showed in [12]. Here, we get defining conditions of all classes of α-golden metric manifolds by means of the the well-adapted connection starting from those previously obtained using the first canonical connection and identities (1.16) and (1.17) that link both connections.

The abovementioned results about α-metric and α-golden metric manifolds focus on the common features shared by both kind of pure metric manifolds. However there exist differences between them. We want to highlight one of them: the behaviour of the family of the so-called twin metrics. Given an α-metric manifold (M, J, g), the fundamental tensor Φ defined in (1.8) is a metric tensor called the twin metric of g. If one repeats this procedure introducing the twin metric of Φ one obtains the metric g itself in the case $\alpha = 1$. In the case $\alpha = -1$, if one repeats the above procedure over and over one obtains a periodic sequence generated by the cycle $(g, \Phi, -g, -\Phi)$. Thus, the family of twin metrics on α-metric manifolds is a periodic sequence. Analogously to the above procedure, one can define the family of twin metrics on an α-golden metric (M, φ, g) starting from the tensor field Φ_φ of type $(0, 2)$ defined as follows

$$\Phi_\varphi(X, Y) = g(\varphi X, Y), \ \ \forall X, Y \in \mathfrak{X}(M), \tag{1.18}$$

and the α-golden structure φ. In this case, one obtains a sequence of twin metrics defined starting from g and Φ_φ by the same recurrence relation that the (real and complex) Fibonacci numbers, which is the recurrence relation defined by the characteristic polynomial of the α-golden structure.

The organization of the paper is as follows:

Section 2 is devoted to the study of the sequence of twin metrics on α-metric and α-golden metric manifolds. First we will recall the cyclic behaviour of this sequence of metrics on α-metrics manifolds. Afterwards, in the case of α-golden metric manifolds, we will prove that it is a sequence with the same recurrence relation that the classic Fibonacci numbers (Theorem 2.1).

The challenge of Section 3 is to obtain a unified classification of almost Norden manifolds and almost product Riemannian with null trace in the framework of α-metric manifolds starting from the classifications of these manifolds showed in [11] and [19] (Theorem 3.8). Previously, we will recall well-known defining conditions of some classes of both kind of manifolds and we will introduce new characterizations of some other classes more useful than the original ones to obtain results in the next sections.

In Section 4, we will deepen on the relationship between the torsion tensors of the first canonical and the well-adapted connection of an α-metric manifold. Starting from these two torsion tensors, we will introduce two 1-forms, called the torsion form of the first canonical and the well-adapted connections, and we will prove that both 1-forms coincide (Lemmas 4.4 and 4.6). We will characterize all classes listed in Theorem 3.8 by means of the torsion tensors and the torsion forms of the two abovementioned connections (Theorems 4.7 and 4.8). We will finish this section showing explicit expressions of the well-adapted connection of integrable and quasi-Kähler α-metric manifolds (Remarks 4.9 and 4.10).

In Section 5 we will focus on α-golden metric manifolds. Given an α-golden metric manifold (M, φ, g), we will introduce the codifferential of the α-golden structure φ. We will also show useful relationships between tensors and forms defined from the α-golden metric structure (φ, g) with other tensors and forms defined from the induced α-metric structure (J_φ, g), which allow us to classify the α-golden metric manifolds according to Definition 5.1 (Theorem 5.4). Later, they also allow us to obtain defining conditions of all classes of these manifolds using the first canonical and the well adapted connections shared by the metric structures (φ, g) and (J_φ, g) (Theorems 5.9 and 5.12).

We will consider smooth manifolds and operators being of class C^∞. As in this Introduction, $\mathfrak{X}(M)$ denotes the module of vector fields of a manifold M and ∇^{g} denotes the Levi-Civita connection of a (pseudo)-Riemannian metric g.

2. Pure metric geometries

Given an $2n$-dimensional α-metric manifold (M, J, g), the fundamental tensor Φ of type $(0, 2)$ previously recalled in identity (1.8) is a pure metric respect to the α-structure J, also called the twin metric of g. To prove the above two claims it is enough to take account identity (1.3) and the definition of the fundamental tensor Φ itself

$$\Phi(X, Y) = g(JX, Y) = g(X, JY) = g(JY, X) = \Phi(Y, X),$$
$$\Phi(JX, Y) = \alpha g(X, Y) = g(JX, JY) = \Phi(X, JY), \ \ \forall X, Y \in \mathfrak{X}(M).$$

Observe that in the case $\alpha = 1$, the fundamental tensor Φ is a neutral metric of signature (n, n) instead of a Riemannian metric like the symmetric tensor field g.

Given an α-golden metric manifold (M, φ, g), bearing in mind (1.5), one easily proves that the tensor field Φ_φ is also a pure metric respect to φ

$$\Phi_\varphi(X, Y) = g(\varphi X, Y) = g(X, \varphi Y) = g(\varphi Y, X) = \Phi_\varphi(Y, X),$$
$$\Phi_\varphi(\varphi X, Y) = g(\varphi^2 X, Y) = g(\varphi X, \varphi Y) = \Phi_\varphi(X, \varphi Y), \ \ \forall X, Y \in \mathfrak{X}(M).$$

We will also call the twin metric of g to the tensor field defined in (1.18).

These facts allow to claim that Φ is a pure metric respect to J and Φ_φ is a pure metric respect to φ. Thus, one can define the twin metric of Φ and Φ_φ according to (1.8) and (1.18) as follows

$$\Phi^{(2)}(X, Y) = g(J^2 X, Y) = \alpha g(X, Y),$$
$$\Phi_\varphi^{(2)}(X, Y) = g(\varphi^2 X, Y) = g(\varphi X, Y) + \frac{5\alpha - 1}{4} g(X, Y), \ \ \forall X, Y \in \mathfrak{X}(M).$$

This procedure can be repeated indefinitely. In this way, one introduces the following families tensor fields of type $(0, 2)$ on (M, J, g) and (M, φ, g)

$$\begin{aligned} &\Phi^{(n)}(X, Y) = g(J^n X, Y), \\ &\Phi_\varphi^{(n)}(X, Y) = g(\varphi^n X, Y), \ \ \forall X, Y \in \mathfrak{X}(M), \ \ \forall n \in \mathbb{N}, \end{aligned} \tag{2.1}$$

respectively, where $\Phi^{(0)} = \Phi_\varphi^{(0)} = g$, $\Phi^{(1)} = \Phi$ and $\Phi_\varphi^{(1)} = \Phi_\varphi$. Obviously these sequences inherit the recurrence relations of α-structures and α-golden structures satisfying

$$\Phi^{(n)} = \alpha \Phi^{(n-2)}, \ \ \Phi_\varphi^{(n)} = \Phi_\varphi^{(n-1)} + \frac{5\alpha - 1}{4} \Phi_\varphi^{(n-2)}, \ \ \forall n \geq 2.$$

Note that in the case $\alpha = 1$ the sequence $\{\Phi_\varphi^{(n)}\}_{n \in \mathbb{N}}$ follows the same recurrence relation that the classic Fibonacci numbers.

It is easy to prove that $\Phi^{(n)}$ and $\Phi_\varphi^{(n)}$ are pure pseudo-Riemannian metrics respect to J and φ respectively, for all $n \in \mathbb{N}$. It is also true, following the above denomination, that $\Phi^{(n)}$ and $\Phi_\varphi^{(n)}$ are the twin metrics of $\Phi^{(n-1)}$ and $\Phi_\varphi^{(n-1)}$ respectively, for all $n \geq 1$. Thus, the pairs (J, g) and (φ, g) allow to define the sequences $\{\Phi^{(n)}\}_{n \in \mathbb{N}}$ and $\{\Phi_\varphi^{(n)}\}_{n \in \mathbb{N}}$ of pseudo-Riemannian metrics on M as it is indicated in (2.1).

The behaviour of $\{\Phi^{(n)}\}_{n \in \mathbb{N}}$ is characterized by the recurrence sequence $x_{n+2} - \alpha x_n = 0$, whose solution is given by $x_n = k_1(\sqrt{\alpha})^n + k_2(-\sqrt{\alpha})^n$, for all $n \in \mathbb{N}$, where k_1 and k_2 are constants. If $\alpha = 1$ and in addition $x_0 = 0$ and $x_1 = 1$, the unique solution of $x_{n+2} - x_n = 0$ is

$$x_n = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}$$

According to the above one it is easy to prove that $\{\Phi^{(n)}\}_{n\in\mathbb{N}}$ satisfies the following recurrence relation

$$\Phi^{(n)} = x_n \Phi + x_{n-1} g, \quad n \geq 1.$$

If $\alpha = -1$ and in addition $x_0 = 1$ and $x_1 = i$, where i denotes the imaginary unit, the unique solution of $x_{n+2} + x_n = 0$, is given by $x_n = i^n$, for all $n \in \mathbb{N}$. According to this sequence of complex numbers it is easy to prove that $\{\Phi^{(n)}\}_{n\in\mathbb{N}}$ satisfies the following recurrence relation

$$\Phi^{(n)} = \mathrm{Im}(x_n)\Phi + \mathrm{Re}(x_n) g, \quad \forall n \in \mathbb{N},$$

where Re and Im denotes the real and the imaginary part of a complex number respectively.

Summarizing, the sequence $\{\Phi^{(n)}\}_{n\in\mathbb{N}}$ of twin metrics defined on an α-metric manifold is a periodic sequence that satisfies the same recurrence relation that its α-structure. In the golden case, the sequence $\{\Phi_\varphi^{(n)}\}_{n\in\mathbb{N}}$ follows the same recurrence relation of the α-golden structure, but it is not a periodic sequence.

Theorem 2.1. *Let (M, φ, g) be an α-golden metric manifold. Let $\{\Phi_\varphi^{(n)}\}_{n\in\mathbb{N}}$ be the sequence of twin metrics defined as it is indicated in* (2.1).

i) *If $\alpha = 1$ it satisfies*

$$\Phi_\varphi^{(n)} = x_n \Phi_\varphi + x_{n-1} g, \quad n \geq 1, \tag{2.2}$$

where

$$\{x_n\}_{n\in\mathbb{N}} = \left\{ \frac{1}{\sqrt{5}} \left(\phi^n - \bar{\phi}^n \right) : n \in \mathbb{N} \right\} = \{0, 1, 1, 2, \ldots\}. \tag{2.3}$$

ii) *If $\alpha = -1$ it satisfies*

$$\Phi_\varphi^{(n)} = \mathrm{Im}(x_n)\Phi_\varphi + \mathrm{Re}(x_n) g, \quad \forall n \in \mathbb{N}, \tag{2.4}$$

where

$$\begin{aligned} \{x_n\}_{n\in\mathbb{N}} &= \left\{ \frac{\sqrt{5} + 2 + i}{10\sqrt{5}} \phi_c^n + \frac{\sqrt{5} - 2 - i}{10\sqrt{5}} \bar{\phi}_c^n : n \in \mathbb{N} \right\} \\ &= \left\{ 1, i, -\frac{3}{2} - i, -\frac{3}{2} - \frac{1}{2} i, \ldots \right\}. \end{aligned} \tag{2.5}$$

Proof. Given k_1 and k_2 constants, the solution of the recurrence sequence determined by identity (1.4), which is defined by the characteristic polynomial of the α-golden structure φ, is

$$x_n = k_1 \left(\frac{1 + \sqrt{5}\sqrt{\alpha}}{2} \right)^n + k_2 \left(\frac{1 - \sqrt{5}\sqrt{\alpha}}{2} \right)^n, \quad \forall n \in \mathbb{N}. \tag{2.6}$$

Taking into account the above identity, if $\alpha = 1$ and in addition $x_0 = 0$ and $x_1 = 1$ the unique solution of $x_{n+2} - x_{n+1} - x_n = 0$ is the sequence (2.3). Given X, Y vector fields on M, a direct calculus shows that

$$\begin{aligned} \Phi_\varphi^{(1)}(X, Y) &= x_1 \Phi_\varphi(X, Y) + x_0 g(X, Y), \\ \Phi_\varphi^{(2)}(X, Y) &= x_2 \Phi_\varphi(X, Y) + x_1 g(X, Y). \end{aligned}$$

Assuming identity (2.2), given X, Y vector fields on M one obtains

$$\begin{aligned}\Phi_\varphi^{(n+1)}(X,Y) &= \Phi_\varphi^{(n-1)}(\varphi^2 X,Y) = \Phi_\varphi^{(n-1)}(\varphi X,Y) + \Phi_\varphi^{(n-1)}(X,Y)\\ &= \Phi_\varphi^{(n)}(X,Y) + \Phi_\varphi^{(n-1)}(X,Y)\\ &= (x_n + x_{n-1})\Phi_\varphi(X,Y) + (x_{n-1} - x_{n-2})g(X,Y)\\ &= x_{n+1}\Phi_\varphi(X,Y) + x_n g(X,Y).\end{aligned}$$

Bearing in mind (2.6), if $\alpha = -1$ and in addition $x_0 = 1$ and $x_1 = i$ the unique solution of $x_{n+2} - x_{n+1} + \frac{3}{2}x_n = 0$ is the sequence (2.5). Given X, Y vector fields on M, trivially one has

$$\begin{aligned}\Phi_\varphi^{(0)}(X,Y) &= \mathrm{Im}(x_0)\Phi_\varphi(X,Y) + \mathrm{Re}(x_0)g(X,Y),\\ \Phi_\varphi^{(1)}(X,Y) &= \mathrm{Im}(x_1)\Phi_\varphi(X,Y) + \mathrm{Re}(x_1)g(X,Y).\end{aligned}$$

Assuming identity (2.4), given X, Y vector fields on M one obtains

$$\begin{aligned}\Phi_\varphi^{(n+1)}(X,Y) &= \Phi_\varphi^{(n-1)}(\varphi^2 X,Y) = \Phi_\varphi^{(n-1)}(\varphi X,Y) - \frac{3}{2}\Phi_\varphi^{(n-1)}(X,Y)\\ &= \Phi_\varphi^{(n)}(X,Y) - \frac{3}{2}\Phi_\varphi^{(n-1)}(X,Y)\\ &= \mathrm{Im}\left(x_n - \frac{3}{2}x_{n-1}\right)\Phi_\varphi(X,Y) + \mathrm{Re}\left(x_n - \frac{3}{2}x_{n-1}\right)g(X,Y)\\ &= \mathrm{Im}(x_{n+1})\Phi_\varphi(X,Y) + \mathrm{Re}(x_{n+1})g(X,Y).\end{aligned}$$

□

The above result reveals the golden properties of an α-golden metric structure: the sequence of twin metrics of such a structure are parametrized by means of the powers of ϕ and $\bar{\phi}$ or ϕ_c and $\bar{\phi}_c$, having among them the same relationship that the (real or complex) Fibonacci numbers.

Remark 2.2. One can consider an anti-pure metric g on an $(J^2 = \pm 1)$-manifold (M, J) by the relation $g(JX, Y) = -g(X, JY)$, for all vector X, Y on M. In this case, (M, J, g) is $(J^2 = \pm 1)$-metric manifold such that $\alpha\varepsilon = -1$ (see [7]). In the golden case such metrics are not studied in this work because the corresponding metrics structures are not in bijection by means of (1.6). Indeed, given X, Y vector fields on M one has

$$\begin{aligned}g(\varphi_J X, Y) &= g\left(\frac{1}{2}(Id + \alpha\sqrt{5}J)X, Y\right) = \frac{1}{2}g(X,Y) + \frac{\alpha\sqrt{5}}{2}g(JX,Y)\\ &= \frac{1}{2}g(X,Y) - \frac{\alpha\sqrt{5}}{2}g(X,JY) = g\left(X, \frac{1}{2}(Id - \alpha\sqrt{5}J)Y\right)\\ &\neq -g(X, \varphi_J Y).\end{aligned}$$

3. Classification of α-metric manifolds

Almost Norden manifolds and almost product Riemannian manifolds with null trace were classified by Ganchev and Borisov and Staikova and Gribachev in [11] and [19] respectively. All classes of both classifications were defined by imposing conditions that involve tensors and forms introduced starting from the tensor field $\nabla^{\mathrm{g}} J$. Now, we will recall the definition and main properties of these tensors and forms that allow to characterize all classes of α-metric manifolds starting with the tensor $\nabla^{\mathrm{g}} J$ itself.

Let (M, J, g) be an α-metric manifold. The tensor field $\nabla^{\mathrm{g}} J$ of type $(1, 2)$ is defined as follows

$$(\nabla^{\mathrm{g}}_X)JY = \nabla^{\mathrm{g}}_X JY - J\nabla^{\mathrm{g}}_X Y, \ \ \forall X, Y \in \mathfrak{X}(M).$$

As direct consequence of the above definition it is easy to prove the next result.

Lemma 3.1. *Let (M, J, g) be an α-metric manifold. The following relations hold:*

$$(\nabla^{g}_{X}J)JY = -J(\nabla^{g}_{X}J)Y, \tag{3.1}$$

$$g((\nabla^{g}_{X}J)Y, Z) = g((\nabla^{g}_{X}J)Z, Y), \tag{3.2}$$

$$g((\nabla^{g}_{X}J)JY, Z) = -g((\nabla^{g}_{X}J)Y, JZ), \tag{3.3}$$

$$g((\nabla^{g}_{X}J)JY, Z) = -g((\nabla^{g}_{X}J)JZ, Y), \ \ \forall X, Y, Z \in \mathfrak{X}(M). \tag{3.4}$$

As direct consequence of (3.4), the fundamental tensor field Φ satisfies

$$(\nabla^{g}_{X}\Phi)(JY, Y) = g((\nabla^{g}_{X}J)JY, Y) = 0, \ \ \forall X, Y \in \mathfrak{X}(M). \tag{3.5}$$

Integrable and quasi-Kähler manifolds are characterized by the vanishing of the Nijenhuis tensor of J and the so-called second Nijenhuis tensor of J (see [11] and [19]), which can be expressed using the tensor field $\nabla^{g}J$ as follows

$$N_J(X, Y) = (\nabla^{g}_{X}J)JY + (\nabla^{g}_{JX}J)Y - (\nabla^{g}_{Y}J)JX - (\nabla^{g}_{JY}J)X, \tag{3.6}$$

$$\widetilde{N}_J(X, Y) = (\nabla^{g}_{X}J)JY + (\nabla^{g}_{JX}J)Y + (\nabla^{g}_{Y}J)JX + (\nabla^{g}_{JY}J)X, \ \ \forall X, Y \in \mathfrak{X}(M). \tag{3.7}$$

Lemma 3.1 allows to prove the next identities properties of the Nijenhuis type tensors.

Lemma 3.2. *Let (M, J, g) be an α-metric manifold. The following relations hold:*

$$N_J(JX, JY) = \alpha N_J(X, Y), \tag{3.8}$$

$$g(N_J(JX, Y), JZ) = -\alpha g(N_J(X, Y), Z), \tag{3.9}$$

$$\underset{XYZ}{\mathfrak{S}} g(N_J(X, Y), JZ) = -2 \underset{XYZ}{\mathfrak{S}} g(\nabla^{g}_{JX}J)JY, Z), \tag{3.10}$$

$$\underset{XYZ}{\mathfrak{S}} g(\widetilde{N}_J(X, Y), JZ) = -2\alpha \underset{XYZ}{\mathfrak{S}} g((\nabla^{g}_{X}J)Y, Z), \tag{3.11}$$

$$\begin{aligned} \widetilde{N}_J(X, Y) = {} & (\nabla^{g}_{X+Y}J)(JX + JY) + (\nabla^{g}_{JX+JY}J)(X + Y) \\ & - ((\nabla^{g}_{X}J)JX + (\nabla^{g}_{JX}X) + (\nabla^{g}_{Y}J)JY + (\nabla^{g}_{JY}J)Y), \end{aligned} \tag{3.12}$$

for all vector fields X, Y, Z on M, where $\underset{XYZ}{\mathfrak{S}}$ denotes the cyclic sum by X, Y, Z.

Proof. Given X, Y, Z vector fields on M, bearing in mind (3.3) and (3.6) one has

$$\begin{aligned} N_J(JX, JY) &= \alpha(\nabla^{g}_{JX}J)Y + \alpha(\nabla^{g}_{X}J)JY - \alpha(\nabla^{g}_{JY}J)X - \alpha(\nabla^{g}_{Y}J)JX \\ &= \alpha N_J(X, Y), \\ g(N_J(JX, Y), JZ) &= -\alpha g((\nabla^{g}_{JX}J)Y, Z) - \alpha g((\nabla^{g}_{X}J)JY, Z) \\ &\quad + \alpha g((\nabla^{g}_{Y}J)JX, Z) + \alpha g((\nabla^{g}_{JY}J)X, Z) \\ &= -\alpha g(N_J(X, Y), Z). \end{aligned}$$

Given X, Y, Z vector fields on M, bearing in mind (3.2), (3.3) and (3.6), one has

$$\begin{aligned} g(N_J(X, Y), JZ) &= -\alpha g((\nabla^{g}_{X}J)Y, Z) - g((\nabla^{g}_{JX}J)JY, Z) \\ &\quad + \alpha g((\nabla^{g}_{Y}J)Z, X) - g((\nabla^{g}_{JY}J)JZ, X), \\ g(N_J(Y, Z), JX) &= -\alpha g((\nabla^{g}_{Y}J)Z, X) - g((\nabla^{g}_{JY}J)JZ, X) \\ &\quad + \alpha g((\nabla^{g}_{Z}J)X, Y) - g((\nabla^{g}_{JZ}J)JX, Y), \\ g(N_J(Z, X), JY) &= -\alpha g((\nabla^{g}_{Z}J)X, Y) - g((\nabla^{g}_{JZ}J)JX, Y) \\ &\quad + \alpha g((\nabla^{g}_{X}J)Y, Z) - g((\nabla^{g}_{JX}J)JY, Z), \end{aligned}$$

then, summing up the above equalities one obtains (3.10).

Given X, Y, Z vector fields on M, taking into account (3.2), (3.3) and (3.7), one has

$$\begin{aligned} g(\widetilde{N}_J(X,Y), JZ) &= -\alpha g((\nabla^{\mathrm{g}}_X J)Y, Z) - g((\nabla^{\mathrm{g}}_{JX} J)JY, Z) \\ &\quad - \alpha g((\nabla^{\mathrm{g}}_Y J)Z, X)) + g((\nabla^{\mathrm{g}}_{JY} J)JZ, X) \\ g(\widetilde{N}_J(Y,Z), JX) &= -\alpha g((\nabla^{\mathrm{g}}_Y J)Z, X) - g((\nabla^{\mathrm{g}}_{JY} J)JZ, X) \\ &\quad - \alpha g((\nabla^{\mathrm{g}}_Z J)X, Y) + g((\nabla^{\mathrm{g}}_{JZ} J)JX, Y), \\ g(\widetilde{N}_J(Z,X), JY) &= -\alpha g((\nabla^{\mathrm{g}}_Z J)X, Y) - g((\nabla^{\mathrm{g}}_{JZ} J)JX, Y) \\ &\quad - \alpha g((\nabla^{\mathrm{g}}_X J)Y, Z) + g((\nabla^{\mathrm{g}}_{JX} J)JY, Z), \end{aligned}$$

then, summing up the above equalities one obtains (3.11).

Given X, Y vector fields on M, as $\nabla^{\mathrm{g}} J$ is a tensor field of type $(1,1)$ one has

$$\begin{aligned} (\nabla^{\mathrm{g}}_{X+Y} J)(J(X+Y)) &= (\nabla^{\mathrm{g}}_X J)JX + (\nabla^{\mathrm{g}}_Y J)JY + (\nabla^{\mathrm{g}}_X J)JY + (\nabla^{\mathrm{g}}_Y J)JX, \\ (\nabla^{\mathrm{g}}_{J(X+Y)} J)(X+Y) &= (\nabla^{\mathrm{g}}_{JX} J)X + (\nabla^{\mathrm{g}}_{JY} J)Y + (\nabla^{\mathrm{g}}_{JX} J)Y + (\nabla^{\mathrm{g}}_{JY} J)X, \end{aligned}$$

then, summing up the above equalities one easily obtains (3.12). □

The tensor field $\nabla^{\mathrm{g}}\Phi$ of type $(0,3)$ allows to introduce a 1-form which is the last key to complete the classification of almost Norden manifolds and almost product Riemannian manifolds with null trace.

Lemma 3.3 ([11, Eq. (3)] and [19])**.** *Let (M, J, g) be a $2n$-dimensional α-metric manifold. Let $p \in M$ and let $(v_1, \ldots, v_{2n})$ be a basis of $T_p(M)$. The quantity*

$$\delta\Phi(v) = \sum_{i,j}^{2n} g^{ij} (\nabla^{\mathrm{g}}_{v_i}\Phi)(v_j, v) = \sum_{i,j}^{2n} g^{ij} g((\nabla^{\mathrm{g}}_{v_i} J)v_j, v), \ \ v \in T_p(M), \tag{3.13}$$

where the matrix $(g^{ij})_{i,j=1}^{2n}$ is the inverse matrix of $(g_{ij})_{i=1}^{2n} = (g_p(v_i, v_j))_{i,j=1}^{2n}$, does not depend on the chosen basis. Thus, $\delta\Phi$ defined as (3.13) is a 1-form called the codifferential of Φ.

The codifferential of Φ can be locally defined as follows

$$\delta\Phi(X) = \sum_{i,j=1}^{2n} g^{ij} g((\nabla^{\mathrm{g}}_{X_i} J)X_j, X), \ \ \forall X \in \mathfrak{X}(M), \tag{3.14}$$

where $(X_1, \ldots, X_{2n})$ is a local basis of TM and the matrix $(g^{ij})_{i,j=1}^{2n}$ is the inverse matrix of $(g(X_i, X_j))_{i,j=1}^{2n}$.

The G-structure defined by (J, g) over M allows to choose local basis that simplify the above expression of the 1-form $\delta\Phi$.

Lemma 3.4 ([8, Prop. 3.3 and 3.4])**.** *Let (M, J, g) be a $2n$-dimensional α-metric manifold. For every $p \in M$ there exist a local basis $(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ of TM such that*

i) *if $\alpha = -1$ it satisfies*

$$JX_i = Y_i, \ \ g(X_i, X_j) = g(Y_i, Y_j) = 0, \ \ g(X_i, Y_j) = \delta_{ij}, \tag{3.15}$$

for all $i, j = 1, \ldots, n$.

ii) *if $\alpha = 1$ it satisfies*

$$JX_i = X_i, \ \ JY_i = -Y_i, \ \ g(X_i, X_j) = g(Y_i, Y_j) = \delta_{ij}, \ \ g(X_i, Y_j) = 0, \tag{3.16}$$

for all $i, j = 1, \ldots, n$.

A local basis satisfying the above conditions is called an adapted local basis.

Lemma 3.5. *Let (M, J, g) be a $2n$-dimensional α-metric manifold. The 1-form $\delta\Phi$ can be locally expressed by means of an adapted local basis $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ as follows:*

$$\delta\Phi(X) = \sum_{i=1}^{n}(g((\nabla^{\mathrm{g}}_{X_i}J)Y_i, X) + g((\nabla^{\mathrm{g}}_{Y_i}J)X_i, X)), \quad \forall X \in \mathfrak{X}(M), \quad \text{if } \alpha = -1. \tag{3.17}$$

$$\delta\Phi(X) = \sum_{i=1}^{n}(g((\nabla^{\mathrm{g}}_{X_i}J)X_i, X) + g((\nabla^{\mathrm{g}}_{Y_i}J)Y_i, X)), \quad \forall X \in \mathfrak{X}(M), \quad \text{if } \alpha = 1. \tag{3.18}$$

Here we will obtain new characteristic conditions of the classes $\mathcal{W}_3$ and $\mathcal{W}_1 \oplus \mathcal{W}_3$ different from the original ones showed in [11] and [19], which are identities (3.19) and (3.22).

Lemma 3.6. *Let (M, J, g) be an α-metric manifold. The following conditions are equivalent:*

$$\widetilde{N}_J(X, Y) = 0, \tag{3.19}$$

$$N_J(X, Y) = 2((\nabla^{\mathrm{g}}_X J)JY + (\nabla^{\mathrm{g}}_{JX} J)Y), \tag{3.20}$$

$$(\nabla^{\mathrm{g}}_X J)JX + (\nabla^{\mathrm{g}}_{JX} J)X = 0, \tag{3.21}$$

for all X, Y vector fields on M.

Proof. If the second Nijenhuis tensor vanishes, bearing in mind (3.6) and (3.7) it is obvious that the Nijenhuis tensor satisfies (3.20).

If the Nijenhuis tensor satisfies (3.20), as it is a skew-symmetric tensor field, one gets identity (3.21).

Finally, if the tensor field $\nabla^{\mathrm{g}} J$ satisfies (3.21) then

$$(\nabla^{\mathrm{g}}_X J)JX + (\nabla^{\mathrm{g}}_{JX} J)X = (\nabla^{\mathrm{g}}_Y J)JY + (\nabla^{\mathrm{g}}_{JY} J)Y = 0,$$
$$(\nabla^{\mathrm{g}}_{X+Y} J)(J(X+Y)) + (\nabla^{\mathrm{g}}_{J(X+Y)} J)(X+Y) = 0, \quad \forall X, Y \in \mathfrak{X}(M),$$

therefore, bearing in mind (3.12) one gets that the second Nijenhuis tensor vanishes. □

Lemma 3.7. *Let (M, J, g) be a $2n$-dimensional α-metric manifold. The following conditions are equivalent:*

$$\underset{XYZ}{\mathfrak{S}}(\nabla^{\mathrm{g}}_X \Phi)(Y, Z) = \frac{1}{n}\underset{XYZ}{\mathfrak{S}}(g(X, Y)\delta\Phi(Z) - \alpha g(X, JY)\delta\Phi(JZ)), \tag{3.22}$$

$$g((\nabla^{\mathrm{g}}_X J)JX, Y) + g((\nabla^{\mathrm{g}}_{JX} J)X, Y) = \frac{1}{n}(g(JX, X)\delta\Phi(Y) - g(X, X)\delta\Phi(JY)), \tag{3.23}$$

for all X, Y, Z vector fields on M.

Proof. As direct consequence of (1.3), (1.9), (3.2) and (3.5), if one evaluates identity (3.22) on (X, Y, JX) then one obtains (3.23).

Given X, Y, Z vector fields on M, identities (1.3), (3.12) and (3.23) lead to the next one

$$\begin{aligned} g(\widetilde{N}_J(X, Y), Z) &= \frac{1}{n}(g(JX + JY, X + Y)\delta\Phi(Z) - g(X + Y, X + Y)\delta\Phi(JZ)) \\ &\quad - \frac{1}{n}(g(JX, X)\delta\Phi(Z) - g(X, X)\delta\Phi(JZ)) \\ &\quad - \frac{1}{n}(g(JY, Y)\delta\Phi(Z) - g(Y, Y)\delta\Phi(JZ)) \\ &= \frac{2}{n}(g(JX, Y)\delta\Phi(Z) - g(X, Y)\delta\Phi(JZ)), \end{aligned} \tag{3.24}$$

therefore

$$\underset{XYZ}{\mathfrak{S}} g(\widetilde{N}_J(X, Y), JZ) = \frac{2}{n}\underset{XYZ}{\mathfrak{S}}(g(JX, Y)\delta\Phi(JZ) - \alpha g(X, Y)\delta\Phi(Z)),$$

then, taking into account (1.3), (1.9) and (3.11), one obtains easily identity (3.22) from the above one. □

The above results and the classifications of almost Norden manifolds and almost product Riemannian manifolds with null trace allow to classify both kind of metric manifolds in a unified way as follows. Note that only the characteristic conditions of the classes $\mathcal{W}_3$ and $\mathcal{W}_1 \oplus \mathcal{W}_3$ have been changed with respect to the abovementioned classifications replacing the original defining conditions (3.19) and (3.22) by (3.21) and (3.23) respectively.

Theorem 3.8. *Let (M, J, g) be a $2n$-dimensional α-metric manifold. Then one has the following classes of this kind of manifolds:*

i) *The class $\mathcal{W}_0$ or Kähler manifolds characterized by one of the following equivalent conditions*

$$\nabla^{\mathrm{g}}\Phi = 0, \quad \nabla^{\mathrm{g}}J = 0.$$

ii) *The class $\mathcal{W}_1$ characterized by the condition*

$$\begin{aligned}(\nabla^{\mathrm{g}}_X \Phi)(Y, Z) = &\frac{1}{2n}(g(X, Y)\delta\Phi(Z) + g(X, Z)\delta\Phi(Y)) \\ &+ \frac{(-\alpha)}{2n}(g(X, JY)\delta\Phi(JZ) + g(X, JZ)\delta\Phi(JY)),\end{aligned} \tag{3.25}$$

for all X, Y, Z vector fields on M.

iii) *The class $\mathcal{W}_2$ characterized by the conditions*

$$\delta\Phi = 0, \quad N_J = 0. \tag{3.26}$$

iv) *The class $\mathcal{W}_3$ or quasi-Kähler manifolds characterized by the condition*

$$(\nabla^{\mathrm{g}}_X J)JX + (\nabla^{\mathrm{g}}_{JX} J)X = 0, \quad \forall X \in \mathfrak{X}(M).$$

v) *The class $\mathcal{W}_1 \oplus \mathcal{W}_2$ or integrable manifolds characterized by the condition*

$$N_J = 0.$$

vi) *The class $\mathcal{W}_2 \oplus \mathcal{W}_3$ characterized by the condition*

$$\delta\Phi = 0. \tag{3.27}$$

vii) *The class $\mathcal{W}_1 \oplus \mathcal{W}_3$ characterized by the condition*

$$g((\nabla^{\mathrm{g}}_X J)JX, Y) + g((\nabla^{\mathrm{g}}_{JX} J)X, Y) = \frac{1}{n}(g(X, JX)\delta\Phi(Y) - g(X, X)\delta\Phi(JY)),$$

for all X, Y vector fields on M.

viii) *The class $\mathcal{W}$ or the whole class of α-metric manifolds.*

4. Classification of α-metric manifolds using canonical connections

The main goal of this section is to characterize all classes of α-metric manifold showed Theorem 3.8 by means of the torsion tensors of the first canonical and the well adapted connections.

First of all, we will link the tensor field $\nabla^{\mathrm{g}}J$ and the torsion tensor of the first canonical connection.

Lemma 4.1. *Let (M, J, g) be an α-metric manifold. The following relation holds:*

$$g((\nabla^{\mathrm{g}}_X J)Y, Z) = -g(\mathrm{T}^0(X, JY), Z) + g(\mathrm{T}^0(JY, Z), X) - g(\mathrm{T}^0(Z, X), JY), \tag{4.1}$$

for all vector fields X, Y, Z on M.

Proof. Given X, Y, Z vector fields on M, as ∇^0 is adapted to (J, g), according to [7, Prop. 3.6], its potential tensor

$$S^0(X, Y) = \nabla^0_X Y - \nabla^{\mathrm{g}}_X Y = \frac{(-\alpha)}{2}(\nabla^{\mathrm{g}}_X J)JY,$$

satisfies

$$g(S^0(X,Y),Z) = \frac{(-\alpha)}{2} g((\nabla^{\mathrm{g}}_X J)JY,Z)$$
$$= \frac{1}{2}(g(\mathrm{T}^0(X,Y),Z) - g(\mathrm{T}^0(Y,Z),X) + g(\mathrm{T}^0(Z,X),Y)),$$

thus, (4.1) is a direct consequence of the above equality. □

Now, starting from identities (1.16) and (1.17), we will deepen on the relationship between the first canonical connection and the well-adapted connection and their corresponding torsion tensors.

As direct consequence of identity (1.16) that links both connections by means of the Nijenhuis tensor one can claim that ∇^0 and ∇^{w} coincide when this tensor vanishes.

Proposition 4.2. *Let (M,J,g) be an α-metric manifold. If (M,J,g) belongs to the classes $\mathcal{W}_0$, $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_1 \oplus \mathcal{W}_2$ then ∇^0 and ∇^{w} are the same connection.*

The next result establishes new identities that involve both torsion tensors.

Lemma 4.3. *Let (M,J,g) be an α-metric manifold. The following relations hold:*

$$g(\mathrm{T}^{\mathrm{w}}(JX,Y),JX) + \alpha g(\mathrm{T}^{\mathrm{w}}(X,Y),X) = g(\mathrm{T}^0(JX,Y),JX) + \alpha g(\mathrm{T}^0(X,Y),X), \quad (4.2)$$

$$\mathrm{T}^{\mathrm{w}}(JX,JY) + \alpha \mathrm{T}^{\mathrm{w}}(X,Y) = \mathrm{T}^0(JX,JY) + \alpha \mathrm{T}^0(X,Y), \quad (4.3)$$

$$g(\mathrm{T}^{\mathrm{w}}(JX,Y),Z) + g(\mathrm{T}^{\mathrm{w}}(X,Y),JZ) = \frac{\alpha}{4} g(\widetilde{N}_J(X,JZ),Y), \quad (4.4)$$

for all X,Y vector fields on M.

Proof. Given X,Y vector fields on M, identities (1.17) and (3.9) and the skew-symmetric condition of the Nijenhuis tensor lead to the next two

$$g(\mathrm{T}^{\mathrm{w}}(X,Y),X) = g(\mathrm{T}^0(X,Y),X) + \frac{\alpha}{8} g(N_J(Y,X),X),$$
$$\alpha g(\mathrm{T}^{\mathrm{w}}(JX,Y),JX) = \alpha g(\mathrm{T}^0(JX,Y),JX) - \frac{\alpha}{8} g(N_J(Y,X),X),$$

then, summing up the above equalities one obtains (4.2).

Given X,Y,Z vector fields on M, bearing in mind (1.17) and (3.9) one obtains

$$\alpha g(\mathrm{T}^{\mathrm{w}}(X,Y),Z) = \alpha g(\mathrm{T}^0(X,Y),Z) + \frac{1}{8}(g(N_J(Y,Z),X) - g(N_J(X,Z),Y)),$$
$$g(\mathrm{T}^{\mathrm{w}}(JX,JY),Z) = g(\mathrm{T}^0(JX,JY),Z) - \frac{1}{8}(g(N_J(Y,Z),X) - g(N_J(X,Z),Y)),$$

then, summing up the above equalities one gets the next equivalent identity to (4.3)

$$g(\mathrm{T}^{\mathrm{w}}(JX,JY) + \alpha \mathrm{T}^{\mathrm{w}}(X,Y),Z) = g(\mathrm{T}^0(JX,JY) + \alpha \mathrm{T}^0(X,Y),Z).$$

Given X,Y,Z vector fields on M, bearing in mind (1.17), (3.8) and (3.9) one obtains

$$g(\mathrm{T}^{\mathrm{w}}(JX,Y),Z) = g(\mathrm{T}^0(JX,Y),Z) + \frac{\alpha}{8}(g(N_J(Y,Z),JX) - g(N_J(JX,Z),Y)),$$
$$g(\mathrm{T}^{\mathrm{w}}(X,Y),JZ) = g(\mathrm{T}^0(X,Y),JZ) + \frac{\alpha}{8}(-g(N_J(Y,Z),JX) - g(N_J(JX,Z),Y)),$$

then, bearing in mind (1.14), (3.2), (3.3), (3.6), (3.7) and summing up the above equalities one obtains

$$g(\mathrm{T}^{\mathrm{w}}(JX,Y),Z) + g(\mathrm{T}^{\mathrm{w}}(X,Y),JZ) = \frac{1}{4} g((\nabla^{\mathrm{g}}_X J)Z + \alpha(\nabla^{\mathrm{g}}_{JX} J)JZ,Y)$$
$$+ \frac{1}{4} g(\alpha(\nabla^{\mathrm{g}}_{JZ} J)JX + (\nabla^{\mathrm{g}}_Z J)JX,Y))$$
$$= \frac{\alpha}{4} g(\widetilde{N}_J(X,JZ),Y).$$

□

To obtain the classification of almost Norden manifolds using the well-adapted connection, in [12, Sect. 5], Ganchev and Mihova introduced a 1-form defined from its torsion tensor. In [9, Lemma 3.1], this definition was generalized to almost product Riemannian manifolds with null trace defining a 1-form from the first canonical connection. Below we will introduce both 1-forms on α-metric manifolds.

Lemma 4.4. *Let (M, J, g) be a $2n$-dimensional α-metric manifold. Let $p \in M$ and let $(v_1, \ldots, v_{2n})$ be a basis of $T_p(M)$. The quantities*

$$t^0(v) = \sum_{i,j}^{2n} g^{ij} g(\mathrm{T}^0(v, v_i), v_j), \ \ t^{\mathrm{w}}(v) = \sum_{i,j}^{2n} g^{ij} g(\mathrm{T}^{\mathrm{w}}(v, v_i), v_j), \ \ v \in T_p(M),$$

where the matrix $(g^{ij})_{i,j=1}^{2n}$ is the inverse matrix of $(g_{ij})_{i=1}^{2n} = (g_p(v_i, v_j))_{i,j=1}^{2n}$, do not depend on the chosen basis. Thus, t^0 and t^{w} defined as above are two 1-forms called the torsion form of ∇^0 and ∇^{w} respectively.

Analogously to the 1-form $\delta\Phi$, the torsion forms t^0 and t^{w} can be locally defined as follows:

$$\begin{aligned} t^0(X) &= \sum_{i,j=1}^{2n} g^{ij} g(\mathrm{T}^0(X, X_i), X_j), \\ t^{\mathrm{w}}(X) &= \sum_{i,j=1}^{2n} g^{ij} g(\mathrm{T}^{\mathrm{w}}(X, X_i), X_j), \ \ \forall X \in \mathfrak{X}(M), \end{aligned}$$

where $(X_1, \ldots, X_{2n})$ is a local basis of TM and the matrix $(g^{ij})_{i,j=1}^{2n}$ is the inverse matrix of $(g(X_i, X_j))_{i,j=1}^{2n}$.

Adapted local basis introduced in Lemma 3.4 allow to simplify the above expressions.

Lemma 4.5. *Let (M, J, g) be a $2n$-dimensional α-metric manifold. The torsion forms t^0 and t^{w} can be locally expressed by means of an adapted local basis $(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ as follows:*

i) *if $\alpha = -1$, according to (3.15), one has*

$$\begin{aligned} t^0(X) &= \sum_{i=1}^{n} (g(\mathrm{T}^0(X, X_i), Y_i) + g(\mathrm{T}^0(X, Y_i), X_i)), \\ t^{\mathrm{w}}(X) &= \sum_{i=1}^{n} (g(\mathrm{T}^{\mathrm{w}}(X, X_i), Y_i) + g(\mathrm{T}^{\mathrm{w}}(X, Y_i), X_i)), \ \ \forall X \in \mathfrak{X}(M). \end{aligned} \tag{4.5}$$

ii) *if $\alpha = 1$, according to (3.16), one has*

$$\begin{aligned} t^0(X) &= \sum_{i=1}^{n} (g(\mathrm{T}^0(X, X_i), X_i) + g(\mathrm{T}^0(X, Y_i), Y_i)), \\ t^{\mathrm{w}}(X) &= \sum_{i=1}^{n} (g(\mathrm{T}^{\mathrm{w}}(X, X_i), X_i) + g(\mathrm{T}^{\mathrm{w}}(X, Y_i), Y_i)), \ \ \forall X \in \mathfrak{X}(M). \end{aligned} \tag{4.6}$$

The next result links the 1-forms $\delta\Phi$, t^0 and t^{w} on α-metric manifolds.

Lemma 4.6. *Let (M, J, g) be a $2n$-dimensional α-metric manifold. The following relations hold:*

$$t^{\mathrm{w}}(X) = t^0(X), \tag{4.7}$$

$$\delta\Phi(X) = 2t^0(JX), \ \ \forall X \in \mathfrak{X}(M). \tag{4.8}$$

Proof. We distinguish cases according to $\alpha = -1$ or $\alpha = 1$ to prove identity (4.7). If $\alpha = -1$, then given a vector field X and given an adapted local basis $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ satisfying (3.15), taking into account (3.8) and (3.9) one has

$$g(N_J(X, X_i), Y_i) = -g(N_J(JX, JX_i), JX_i) = -g(N_J(X, Y_i), X_i), \tag{4.9}$$

for $i = 1, \dots, n$, then, starting from identity (1.17) one has

$$g(\mathrm{T^w}(X, X_i), Y_i) = g(\mathrm{T}^0(X, X_i), Y_i) - \frac{1}{8}(g(N_J(X_i, Y_i), X) - g(N_J(X, Y_i), X_i)),$$
$$g(\mathrm{T^w}(X, Y_i), X_i) = g(\mathrm{T}^0(X, Y_i), X_i) - \frac{1}{8}(g(N_J(Y_i, X_i), X) - g(N_J(X, X_i), Y_i)),$$

for $i = 1, \dots, n$, then bearing in mind that the Nijenhuis tensor is a skew-symmetric tensor and identities (4.9) and summing up the above equalities one gets

$$g(\mathrm{T^w}(X, X_i), Y_i) + g(\mathrm{T^w}(X, Y_i), X_i) = g(\mathrm{T}^0(X, X_i), Y_i) + g(\mathrm{T}^0(X, Y_i), X_i), \;\; i = 1, \dots, n,$$

thus proving (4.7) in the complex case. If $\alpha = 1$, given a vector field X and given an adapted local basis $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ satisfying (3.16), taking into account (3.8) and (3.9) one has

$$g(N_J(X, X_i), X_i) = g(N_J(JX, JX_i), JX_i) = -g(N_J(X, X_i), X_i),$$
$$g(N_J(X, Y_i), Y_i) = -g(N_J(JX, JY_i), JY_i) = -g(N_J(X, Y_i), Y_i), \;\; i = 1, \dots, n,$$

therefore

$$g(N_J(X_i, X_i), X) = g(N_J(Y_i, Y_i), X) = 0,$$
$$g(N_J(X, X_i), X_i) = g(N_J(X, Y_i), Y_i) = 0, \;\; i = 1, \dots, n,$$

then, identity (1.17) and the above equalities lead to next ones

$$g(\mathrm{T^w}(X, X_i), X_i) = g(\mathrm{T}^0(X, X_i), X_i),$$
$$g(\mathrm{T^w}(X, Y_i), Y_i) = g(\mathrm{T}^0(X, Y_i), Y_i), \;\; i = 1, \dots, n,$$

thus proving (4.7) in the product case.

To prove identity (4.8) we distinguish cases once again. If $\alpha = -1$, given a vector field X and given an adapted local basis $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ satisfying (3.15), bearing in mind (1.14), (3.2), (3.4), (3.17) and (4.5) one obtains

$$\begin{aligned} t^0(JX) &= \frac{1}{2}\sum_{i=1}^{n}(g((\nabla^{\mathrm{g}}_{JX}J)JX_i, Y_i) + g((\nabla^{\mathrm{g}}_{X_i}J)X, Y_i)) \\ &\quad + \frac{1}{2}\sum_{i=1}^{n}(g((\nabla^{\mathrm{g}}_{JX}J)JY_i, X_i) + g((\nabla^{\mathrm{g}}_{Y_i}J)X, X_i)) \\ &= \frac{1}{2}\sum_{i=1}^{n}(g((\nabla^{\mathrm{g}}_{X_i}J)Y_i, X) + g((\nabla^{\mathrm{g}}_{Y_i}J)X_i, X)) = \frac{1}{2}\delta\Phi(X). \end{aligned}$$

If $\alpha = 1$, given a vector field X and given an adapted local basis $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ satisfying (3.16), starting from (1.14), (3.18) and (4.6), and taking into account (3.2) and (3.5) one obtains

$$\begin{aligned} t^0(JX) &= -\frac{1}{2}\sum_{i=1}^{n}(g((\nabla^{\mathrm{g}}_{JX}J)JX_i, X_i) - g((\nabla^{\mathrm{g}}_{X_i}J)X, X_i)) \\ &\quad - \frac{1}{2}\sum_{i=1}^{n}(g((\nabla^{\mathrm{g}}_{JX}J)JY_i, Y_i) - g((\nabla^{\mathrm{g}}_{Y_i}J)X, Y_i)) \\ &= \frac{1}{2}\sum_{i=1}^{n}(g((\nabla^{\mathrm{g}}_{X_i}J)X_i, X) + g((\nabla^{\mathrm{g}}_{Y_i}J)Y_i, X)) = \frac{1}{2}\delta\Phi(X). \end{aligned}$$

Last two equalities allow to claim that identity (4.8) is true. □

The next two theorems provide characteristic conditions of all classes of α-metric manifolds using the first canonical and the well adapted connections instead of the Levi-Civita connection like in Theorem 3.8.

Theorem 4.7. *Let (M, J, g) be a $2n$-dimensional α-metric manifold. The classes given in Theorem 3.8 can be characterized by means of the torsion tensor and the torsion form of the first canonical connection as follows:*

i) *The class $\mathcal{W}_0$ or Kähler manifolds characterized by the condition*
$$\mathrm{T}^0(X,Y) = 0, \ \forall X, Y \in \mathfrak{X}(M). \tag{4.10}$$
ii) *The class $\mathcal{W}_1$ characterized by the condition*
$$\mathrm{T}^0(X,Y) = \frac{1}{2n}(t^0(X)Y - t^0(Y)X - \alpha t^0(JX)JY + \alpha t^0(JY)JX), \ \forall X, Y \in \mathfrak{X}(M). \tag{4.11}$$
iii) *The class $\mathcal{W}_2$ characterized by the conditions*
$$t^0(X) = 0, \ \mathrm{T}^0(JX, JY) + \alpha \mathrm{T}^0(X,Y) = 0, \ \forall X, Y \in \mathfrak{X}(M). \tag{4.12}$$
iv) *The class $\mathcal{W}_3$ or quasi-Kähler manifolds characterized by the condition*
$$g(\mathrm{T}^0(JX,Y), JX) + \alpha g(\mathrm{T}^0(X,Y), X) = 0, \ \forall X, Y \in \mathfrak{X}(M). \tag{4.13}$$
v) *The class $\mathcal{W}_1 \oplus \mathcal{W}_2$ or integrable manifolds characterized by the condition*
$$\mathrm{T}^0(JX, JY) + \alpha \mathrm{T}^0(X,Y) = 0, \ \forall X, Y \in \mathfrak{X}(M). \tag{4.14}$$
vi) *The class $\mathcal{W}_2 \oplus \mathcal{W}_3$ characterized by the condition*
$$t^0(X) = 0, \ \forall X \in \mathfrak{X}(M).$$
vii) *The class $\mathcal{W}_1 \oplus \mathcal{W}_3$ characterized by the condition*
$$g(\mathrm{T}^0(JX,Y), JX) + \alpha g(\mathrm{T}^0(X,Y), X) = \frac{1}{n}(g(X, JX)t^0(JY) - \alpha g(X,X)t^0(Y)), \tag{4.15}$$
for all X, Y vector fields on M.

viii) *The class $\mathcal{W}$ or the whole class of α-metric manifolds.*

Proof.

i) The manifold (M, J, g) is a Kähler manifold if and only if $\nabla^{\mathrm{g}} J = 0$, which is equivalent to the Levi-Civita and the first canonical connection coincide; i.e., $\nabla^0 = \nabla^{\mathrm{g}}$. Then, (M, J, g) belongs to the class $\mathcal{W}_0$ if and only if the first canonical connection is a torsion-free connection.

ii) Given X, Y, Z vector fields on M, if (M, J, g) belongs to the class $\mathcal{W}_1$, then taking into account (1.9) and (3.25), one gets the following equalities
$$\begin{aligned}\frac{1}{2}g((\nabla^{\mathrm{g}}_X J)JY, Z) &= \frac{1}{4n}(g(X, JY)\delta\Phi(Z) + g(X,Z)\delta\Phi(JY)) \\ &\quad - \frac{1}{4n}(g(X,Y)\delta\Phi(JZ) + g(X, JZ)\delta\Phi(Y)), \\ \frac{1}{2}g((\nabla^{\mathrm{g}}_Y J)JX, Z) &= \frac{1}{4n}(g(Y, JX)\delta\Phi(Z) + g(Y,Z)\delta\Phi(JX)) \\ &\quad - \frac{1}{4n}(g(Y,X)\delta\Phi(JZ) + g(Y, JZ)\delta\Phi(X)),\end{aligned}$$
then, taking into account (1.3), (1.14) and (4.8), one concludes
$$\begin{aligned}g(\mathrm{T}^0(X,Y), Z) &= \frac{\alpha}{4n}g(\delta\Phi(JX)Y - \delta\Phi(JY)X - \delta\Phi(X)JY + \delta\Phi(Y)JX, Z) \\ &= \frac{1}{2n}g(t^0(X)Y - t^0(Y)X - \alpha t^0(JX)JY + \alpha t^0(JY)JX, Z),\end{aligned} \tag{4.16}$$

which is an equivalent condition to those showed in (4.11).

Given X, Y, Z vector fields on M and bearing in mind (1.3), (4.1), (4.8) and (4.16), one obtains that $g((\nabla^{\mathrm{g}}_X J)Y, Z)$ is equal to the following expressions

$$\begin{aligned}
&-\frac{1}{2n}(g(JY,Z)t^0(X) - g(X,Z)t^0(JY) - g(Y,Z)t^0(JX) + g(JX,Z)t^0(Y))\\
&+\frac{1}{2n}(g(Z,X)t^0(JY) - g(JY,X)t^0(Z) - g(JZ,X)t^0(Y) + g(Y,X)t^0(JZ))\\
&-\frac{1}{2n}(g(X,JY)t^0(Z) - g(Z,JY)t^0(X)) - g(X,Y)t^0(JZ) + g(Z,Y)t^0(JX))\\
&=\frac{1}{n}(g(X,Y)t^0(JZ) + g(X,Z)t^0(JY) - g(X,JY)t^0(Z) - g(X,JZ)t^0(Y))\\
&=\frac{1}{2n}(g(X,Y)\delta\Phi(Z) + g(X,Z)\delta\Phi(Y)) + \frac{(-\alpha)}{2n}(g(X,JY)\delta\Phi(JZ) + g(X,JZ)\delta\Phi(JY)),
\end{aligned}$$

which is the defining condition (3.25) of the class $\mathcal{W}_1$ (see (1.9)).

iii) The equivalence between (3.26) and (4.12) is due to (1.15) and (4.8).

iv) Given X, Y vector fields on M, identities (1.14), (3.4) and (3.5) lead to the following ones

$$\begin{aligned}
g(\mathrm{T}^0(X,Y),X) &= \frac{\alpha}{2}g((\nabla^{\mathrm{g}}_X J)JX, Y),\\
g(\mathrm{T}^0(JX,Y),JX) &= \frac{1}{2}g((\nabla^{\mathrm{g}}_{JX} J)X, Y),\\
g(\mathrm{T}^0(JX,Y),JX) + \alpha g(\mathrm{T}^0(X,Y),X) &= \frac{1}{2}g((\nabla^{\mathrm{g}}_X J)JX + (\nabla^{\mathrm{g}}_{JX} J)X, Y). \qquad (4.17)
\end{aligned}$$

Thus, the characterization (4.13) of quasi-Kähler manifolds is a direct consequence of identity (3.21) and the above one.

v) By virtue of identity (1.15) one gets the characterization of integrable manifolds by means of (4.14).

vi) The class $\mathcal{W}_2 \oplus \mathcal{W}_3$ is characterized by the vanishing of $\delta\Phi$ (see (3.27)), then identity (4.8) allows to claim that this class of α-metric manifolds is also characterized by the vanishing of t^0.

vii) Given X, Y vector fields on M, bearing in mind (3.23), (4.8) and (4.17), one obtains the defining condition of the manifolds of the class $\mathcal{W}_1 \oplus \mathcal{W}_3$ showed in (4.15)

$$\begin{aligned}
g(\mathrm{T}^0(JX,Y),JX) + \alpha g(\mathrm{T}^0(X,Y),X) &= \frac{1}{2n}(g(X,JX)\delta\Phi(Y) - g(X,X)\delta\Phi(JY))\\
&= \frac{1}{n}(g(X,JX)t^0(JY) - \alpha g(X,X)t^0(Y)).
\end{aligned}$$

□

Theorem 4.8. *Let (M, J, g) be a $2n$-dimensional α-metric manifold. The classes given in Theorem 3.8 can be characterized by means of the torsion tensor and the torsion form of the well-adapted connection as follows:*

i) *The class $\mathcal{W}_0$ or Kähler manifolds characterized by the condition*

$$\mathrm{T}^{\mathrm{w}}(X,Y) = 0, \;\; \forall X, Y \in \mathfrak{X}(M).$$

ii) *The class $\mathcal{W}_1$ characterized by the condition*

$$\mathrm{T}^{\mathrm{w}}(X,Y) = \frac{1}{2n}(t^{\mathrm{w}}(X)Y - t^{\mathrm{w}}(Y)X - \alpha t^{\mathrm{w}}(JX)JY + \alpha t^{\mathrm{w}}(JY)JX), \;\; \forall X, Y \in \mathfrak{X}(M). \quad (4.18)$$

iii) *The class $\mathcal{W}_2$ characterized by the conditions*

$$t^{\mathrm{w}}(X) = 0, \;\; \mathrm{T}^{\mathrm{w}}(JX, JY) + \alpha \mathrm{T}^{\mathrm{w}}(X,Y) = 0, \;\; \forall X, Y \in \mathfrak{X}(M). \qquad (4.19)$$

iv) *The class* $\mathcal{W}_3$ *or quasi-Kähler manifolds characterized by the condition*

$$\mathrm{T}^{\mathrm{w}}(JX,Y)+J\mathrm{T}^{\mathrm{w}}(X,Y)=0,\ \ \forall X,Y\in\mathfrak{X}(M). \tag{4.20}$$

v) *The class* $\mathcal{W}_1\oplus\mathcal{W}_2$ *or integrable manifolds characterized by the condition*

$$\mathrm{T}^{\mathrm{w}}(JX,JY)+\alpha\mathrm{T}^{\mathrm{w}}(X,Y)=0,\ \ \forall X,Y\in\mathfrak{X}(M). \tag{4.21}$$

vi) *The class* $\mathcal{W}_2\oplus\mathcal{W}_3$ *characterized by the condition*

$$t^{\mathrm{w}}(X)=0,\ \ \forall X\in\mathfrak{X}(M).$$

vii) *The class* $\mathcal{W}_1\oplus\mathcal{W}_3$ *characterized by the condition*

$$\mathrm{T}^{\mathrm{w}}(JX,Y)+J\mathrm{T}^{\mathrm{w}}(X,Y)=\frac{1}{n}(t^{\mathrm{w}}(JY)X-t^{\mathrm{w}}(Y)JX),\ \ \forall X,Y,Z\in\mathfrak{X}(M). \tag{4.22}$$

viii) *The class* $\mathcal{W}$ *or the whole class of* α*-metric manifolds.*

Proof.

i) If the manifold (M,J,g) belongs to the class $\mathcal{W}_0$ then $\nabla^{\mathrm{w}}=\nabla^0=\nabla^{\mathrm{g}}$ (see Proposition 4.2 and identity (4.10)), hence the well-adapted connection is a torsion-free connection. Reciprocally, recalling that the well-adapted connection parallelizes the metric g, if in addition it is a symmetric connection then $\nabla^{\mathrm{w}}=\nabla^{\mathrm{g}}$ and therefore $\nabla^{\mathrm{g}}J=\nabla^{\mathrm{w}}J=0$, thus proving that (M,J,g) is a Kähler manifold.

ii) If (M,J,g) is a manifold of class $\mathcal{W}_1$ then $\nabla^{\mathrm{w}}=\nabla^0$ (see Proposition 4.2), therefore identities (4.11) and (4.18) coincide, which allows to conclude that (4.18) is true. Reciprocally, given X,Y vector fields on M, if (4.18) is true then one has

$$\mathrm{T}^{\mathrm{w}}(JX,JY)=\frac{1}{2n}(t^{\mathrm{w}}(JX)JY-t^{\mathrm{w}}(JY)JX-\alpha t^{\mathrm{w}}(X)Y+\alpha t^{\mathrm{w}}(Y)X)=-\alpha\mathrm{T}^{\mathrm{w}}(X,Y),$$

thus, bearing in mind (4.3) one obtains

$$\mathrm{T}^0(JX,JY)+\alpha\mathrm{T}^0(X,Y)=\mathrm{T}^{\mathrm{w}}(JX,JY)+\alpha\mathrm{T}^{\mathrm{w}}(X,Y)=0,$$

therefore one concludes that (M,J,g) is an integrable manifold and hence $\nabla^{\mathrm{w}}=\nabla^0$ (see (1.16)). Then identities (4.11) and (4.18) coincide, which allows to claim that (M,J,g) belongs to the class $\mathcal{W}_1$.

iii) As direct consequence of (4.3) and (4.7) one concludes that identities (4.12) and (4.19) are equivalent, thus, as the first ones assure that (M,J,g) belongs to the class $\mathcal{W}_2$, the second ones also assure it.

iv) As direct consequence of (4.4) one concludes that the second Nijenhuis tensor vanishes if and only if the torsion tensor of the well-adapted connection satisfies identity (4.20).

v) Identity (4.3) and the characterization of integrable manifolds using (4.14) allow to claim that (4.21) is also a defining condition of the class $\mathcal{W}_1\oplus\mathcal{W}_2$.

vi) Bearing in mind (4.7) and (4.8), the 1-forms $\delta\Phi$ and t^{w} vanish at once, then identity (3.27) allows to claim that (M,J,g) is of class $\mathcal{W}_2\oplus\mathcal{W}_3$ if and only if the torsion form t^{w} vanishes.

vii) If (M,J,g) is of class $\mathcal{W}_1\oplus\mathcal{W}_3$ bearing in mind (3.24) and (4.4) one has

$$g(\widetilde{N}_J(X,Z),Y)=\frac{2}{n}(g(JX,Z)\delta\Phi(Y)-g(X,Z)\delta\Phi(JY)),$$
$$g(\widetilde{N}_J(X,Z),Y)=4(g(\mathrm{T}^{\mathrm{w}}(JX,Y),JZ)+\alpha g(\mathrm{T}^{\mathrm{w}}(X,Y),Z)),\ \ \forall X,Y,Z\in\mathfrak{X}(M),$$

thus, taking into account (1.3), (4.7) and (4.8) one obtains the characteristic condition (4.22)

$$\begin{aligned} J\mathrm{T}^{\mathrm{w}}(JX,Y)+\alpha\mathrm{T}^{\mathrm{w}}(X,Y) &= \frac{1}{2n}(\delta\Phi(Y)JX-\delta\Phi(JY)X),\\ \mathrm{T}^{\mathrm{w}}(JX,Y)+J\mathrm{T}^{\mathrm{w}}(X,Y) &= \frac{1}{n}(t^{\mathrm{w}}(JY)X-t^{\mathrm{w}}(Y)JX),\ \ \forall X,Y\in\mathfrak{X}(M). \end{aligned}$$

Reciprocally, given X,Y,Z vector fields on M, identities (1.3) and (4.22) lead to next ones

$$\begin{aligned} g(J\mathrm{T}^{\mathrm{w}}(JX,Y),Z)+\alpha g(\mathrm{T}^{\mathrm{w}}(X,Y),Z) &= \frac{1}{n}(g(JX,Z)t^{\mathrm{w}}(JY)-\alpha g(X,Z)t^{\mathrm{w}}(Y))\\ g(\mathrm{T}^{\mathrm{w}}(JX,Y),JZ)+\alpha g(\mathrm{T}^{\mathrm{w}}(X,Y),Z) &= \frac{1}{n}(g(JX,Z)t^{\mathrm{w}}(JY)-\alpha g(X,Z)t^{\mathrm{w}}(Y)), \end{aligned}$$

then, if one takes $Z=X$ in the last identity, taking into account (4.2) and (4.7), one obtains the defining condition (4.15) of the class $\mathcal{W}_1\oplus\mathcal{W}_3$

$$g(\mathrm{T}^0(JX,Y),JX)+\alpha g(\mathrm{T}^0(X,Y),X)=\frac{1}{n}(g(X,JX)t^0(JY)-\alpha g(X,X)t^0(Y)).$$

□

Remark 4.9. As direct consequence of identity (1.16) one obtains that an α-metric manifold is integrable if and only if the first canonical and the well-adapted coincide. This fact allows to claim that (M,J,g) is an integrable α-metric manifold if and only if

$$\nabla^{\mathrm{w}}_X Y=\nabla^{\mathrm{g}}_X Y+\frac{(-\alpha)}{2}(\nabla^{\mathrm{g}}_X J)JY,\ \ \forall X,Y,\in\mathfrak{X}(M). \tag{4.23}$$

In this case, its torsion tensor satisfies

$$\mathrm{T}^{\mathrm{w}}(X,Y)=\frac{(-\alpha)}{2}\left((\nabla^{\mathrm{g}}_X J)JY-(\nabla^{\mathrm{g}}_Y J)JX\right),\ \ \forall X,Y\in\mathfrak{X}(M).$$

Thus, identity (4.23) is an explicit expression of the well-adapted connection in the case of non-Kähler α-metric manifolds of the classes $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_1\oplus\mathcal{W}_2$.

Remark 4.10. Recalling that α-metric manifolds are a particular case of $(J^2=\pm1)$-metric manifolds, the characterization (4.20) of the class $\mathcal{W}_3$ is consequence of [4, Cor. 5.8]. Thus, an α-metric manifold (M,J,g) is of class $\mathcal{W}_3$ if and only the torsion tensor of the well-adapted connection satisfies

$$\mathrm{T}^{\mathrm{w}}(X,Y)=\frac{(-\alpha)}{4}N_J(X,Y),\ \ \forall X,Y\in\mathfrak{X}(M). \tag{4.24}$$

Indeed, if (M,J,g) is a quasi-Kähler manifold, bearing in mind (1.14), (1.17), (3.2), (3.4) and the characterization (3.20) one gets

$$\begin{aligned} g(\mathrm{T}^{\mathrm{w}}(X,Y),Z) &= \frac{(-\alpha)}{2}\left(g((\nabla^{\mathrm{g}}_X J)JY,Z)-g((\nabla^{\mathrm{g}}_Y J)JX,Z)\right)\\ &\quad+\frac{\alpha}{4}(g((\nabla^{\mathrm{g}}_Y J)JZ,X)+g(\nabla^{\mathrm{g}}_{JY}J)Z,X))\\ &\quad-\frac{\alpha}{4}(g((\nabla^{\mathrm{g}}_X J)JZ,Y)+g(\nabla^{\mathrm{g}}_{JX}J)Z,Y))\\ &= \frac{(-\alpha)}{4}g(N_J(X,Y),Z),\ \ \forall X,Y,Z\in\mathfrak{X}(M), \end{aligned}$$

then one obtains identity (4.24). Reciprocally, given X, Y vector fields on M, starting from (4.24) and bearing in mind (3.9), one has

$$g(\mathrm{T}^{\mathrm{w}}(X,Y),X) = \frac{(-\alpha)}{4} g(N_J(X,Y),X),$$
$$g(\mathrm{T}^{\mathrm{w}}(JX,Y),JX) = \frac{(-\alpha)}{4} g(N_J(JX,Y),JX) = \frac{1}{4} g(N_J(X,Y),X),$$

then, taking into account the above equalities and identity (4.2), one gets

$$g(\mathrm{T}^{\mathrm{w}}(JX,Y),JX) + \alpha g(\mathrm{T}^{\mathrm{w}}(X,Y),X) = 0,$$
$$g(\mathrm{T}^{0}(JX,Y),JX) + \alpha g(\mathrm{T}^{0}(X,Y),X) = 0,$$

therefore one concludes that (M, J, g) is a quasi-Kähler manifold (see (4.13)).

The new defining condition of quasi-Kähler manifolds introduced in (4.24) allows to show the explicit expression (4.25) of the well-adapted connection on this class of α-metric manifolds. But first, we need to recall the relation between the potential and the torsion tensor of the well-adapted connection. As ∇^{w} is adapted to (J, g) according to [7, Prop. 3.6], its potential tensor

$$S^{\mathrm{w}}(X,Y) = \nabla^{\mathrm{w}}_X Y - \nabla^{\mathrm{g}}_X Y, \ \ \forall X, Y \in \mathfrak{X}(M),$$

satisfies

$$g(S^{\mathrm{w}}(X,Y),Z) = \frac{1}{2}(g(\mathrm{T}^{\mathrm{w}}(X,Y),Z) - g(\mathrm{T}^{\mathrm{w}}(Y,Z),X) + g(\mathrm{T}^{\mathrm{w}}(Z,X),Y),$$

for all vector fields X, Y, Z on M.

Given a quasi-Kähler manifold (M, J, g), bearing in mind (3.20) and (4.24), identities (3.2), (3.4) and the above one lead to the following equality

$$g(S^{\mathrm{w}}(X,Y),Z) = g\left(\frac{(-\alpha)}{2}(\nabla^{\mathrm{g}}_X J)JY + \frac{(-\alpha)}{4}((\nabla^{\mathrm{g}}_Y J)JX - (\nabla^{\mathrm{g}}_{JY} J)X), Z\right),$$

for all X, Y, Z vector fields on M, thus the well-adapted connection is given by

$$\nabla^{\mathrm{w}}_X Y = \nabla^{\mathrm{g}}_X Y + \frac{(-\alpha)}{2}(\nabla^{\mathrm{g}}_X J)JY + \frac{(-\alpha)}{4}((\nabla^{\mathrm{g}}_Y J)JX - (\nabla^{\mathrm{g}}_{JY} J)X), \ \ \forall X, Y \in \mathfrak{X}(M). \tag{4.25}$$

Reciprocally, if the well-adapted connection satisfies the above identity then

$$\begin{aligned}\mathrm{T}^{\mathrm{w}}(X,Y) &= \frac{(-\alpha)}{2}((\nabla^{\mathrm{g}}_X J)JY - (\nabla^{\mathrm{g}}_Y J)JX) \\ &\quad + \frac{(-\alpha)}{4}((\nabla^{\mathrm{g}}_Y J)JX - (\nabla^{\mathrm{g}}_{JY} J)X - (\nabla^{\mathrm{g}}_X J)JY + (\nabla^{\mathrm{g}}_{JX} J)Y) \\ &= \frac{(-\alpha)}{4} N_J(X,Y), \ \ \forall X, Y \in \mathfrak{X}(M).\end{aligned}$$

Remark 4.11. Given a quasi-Kähler manifold (M, J, g), starting from (4.20), it is easy to prove that the torsion tensor of the well-adapted connection satisfies

$$\mathrm{T}^{\mathrm{w}}(JX,JY) = \alpha \mathrm{T}^{\mathrm{w}}(X,Y), \ \ \forall X, Y, \in \mathfrak{X}(M). \tag{4.26}$$

Moreover, if the above condition is satisfied then (M, J, g) is a quasi-Kähler manifold (see [4, Prop. 5.3]). Thus, identity (4.26), which is another characterization of the class $\mathcal{W}_3$, makes that the well-adapted connection of an almost Norden manifold or an almost product Riemannian manifold with null trace of the class $\mathcal{W}_3$ being the analogous to the Chern connection of an almost Hermitian or an almost para-Hermitian manifold (see [4]).

5. Classification of α-golden metric manifolds

As we said in the Introduction, one can classify the α-golden metric manifolds using the list of classes of α-metric manifolds showed in Theorem 3.8 and the bijection between both kind of manifolds as follows.

Definition 5.1. Let (M, φ, g) be an α-golden manifold. We say that (M, φ, g) belongs to certain class according to the classification given in Theorem 3.8 if the α-metric manifold (M, J_φ, g) belongs to this class of manifolds.

Now we will carry on the defining conditions of all classes of α-metric manifolds to α-golden metric manifolds. For that purpose, we need link the tensors fields and 1-forms related with (J_φ, g) involved in the abovementioned classification with other ones defined from (φ, g).

Lemma 5.2. *Let (M, φ, g) be an α-golden metric manifold and let (J_φ, g) be the α-metric structure induced by (φ, g). The following relations hold:*

$$(\nabla^{\mathrm{g}}_X J_\varphi)Y = \frac{2\alpha}{\sqrt{5}}(\nabla^{\mathrm{g}}_X \varphi)Y, \tag{5.1}$$

$$(\nabla^{\mathrm{g}}_X J_\varphi)J_\varphi X + (\nabla^{\mathrm{g}}_{J_\varphi X} J_\varphi)X = \frac{4}{5}((\nabla^{\mathrm{g}}_X \varphi)\varphi X + (\nabla^{\mathrm{g}}_{\varphi X} \varphi)X - (\nabla^{\mathrm{g}}_X \varphi)X), \tag{5.2}$$

for all X, Y vector fields on M.

Proof. Given X, Y vector fields on M, starting from identity (1.6) it is easy to prove the first equality

$$(\nabla^{\mathrm{g}}_X J_\varphi)Y = \nabla^{\mathrm{g}}_X J_\varphi Y - J_\varphi(\nabla^{\mathrm{g}}_X Y) = \frac{2\alpha}{\sqrt{5}}(\nabla^{\mathrm{g}}_X \varphi Y - \varphi(\nabla^{\mathrm{g}}_X Y)) = \frac{2\alpha}{\sqrt{5}}(\nabla^{\mathrm{g}}_X \varphi)Y.$$

Given a vector field X on M, identities (1.6) and (5.1) lead to the following ones

$$(\nabla^{\mathrm{g}}_X J_\varphi)J_\varphi X = \frac{4}{5}(\nabla^{\mathrm{g}}_X \varphi)\varphi X - \frac{2}{5}(\nabla^{\mathrm{g}}_X \varphi)X,$$
$$(\nabla^{\mathrm{g}}_{J_\varphi X} J_\varphi)X = \frac{4}{5}(\nabla^{\mathrm{g}}_{\varphi X} \varphi)X - \frac{2}{5}(\nabla^{\mathrm{g}}_X \varphi)X,$$

then, summing up the above equalities one gets identity (5.2). □

Analogously to the 1-form $\delta\Phi$ introduced in Lemma 3.3, we directly introduce the codifferential of an α-golden structure φ using local basis in a similar way to identity (3.14).

Definition 5.3. Let (M, φ, g) be a $2n$-dimensional α-golden metric manifold. There exists a 1-form $\delta\varphi$, called the codifferential of φ, which locally can be defined as follows

$$\delta\varphi(X) = \sum_{i,j=1}^{2n} g^{ij} g((\nabla^{\mathrm{g}}_{X_i} \varphi)X_j, X) \;\; \forall X \in \mathfrak{X}(M), \tag{5.3}$$

where $(X_1, \ldots, X_{2n})$ is a local basis of TM and the matrix $(g^{ij})_{i,j=1}^{2n}$ is the inverse matrix of $(g(X_i, X_j))_{i,j=1}^{2n}$.

Identity (5.1) allows to establish easily the relation between the 1-forms $\delta\Phi$ and $\delta\varphi$

$$\delta\Phi(X) = \frac{2\alpha}{\sqrt{5}}\delta\varphi(X), \;\; \forall X \in \mathfrak{X}(M).$$

In [20, Eq. (8)], the authors introduce the class of quasi-Kähler Norden golden manifolds using the twin metric of the pseudo-Riemannian metric as follows: an almost Norden golden manifold (M,φ,g) is a quasi-Kähler Norden golden manifold if and only if the twin metric Φ_φ satisfies

$$(\nabla^{\mathrm{g}}_X\Phi_\varphi)(Y,Z)+(\nabla^{\mathrm{g}}_Y\Phi_\varphi)(Z,X)+(\nabla^{\mathrm{g}}_Z\Phi_\varphi)(X,Y)=0,\ \ \forall X,Y,Z\in\mathfrak{X}(M). \tag{5.4}$$

Given X,Y,Z vector fields on M, bearing in mind (1.18), one has

$$\begin{aligned}(\nabla^{\mathrm{g}}_X\Phi_\varphi)(Y,Z)&=X(\Phi_\varphi(Y,Z))-\Phi_\varphi(\nabla^{\mathrm{g}}_XY,Z)-\Phi_\varphi(Y,\nabla^{\mathrm{g}}_XZ)\\&=X(g(\varphi Y,Z))-g(\varphi\nabla^{\mathrm{g}}_XY,Z)-g(\varphi Y,\nabla^{\mathrm{g}}_XZ),\\(\nabla^{\mathrm{g}}_Xg)(\varphi Y,Z)&=X(g(\varphi Y,Z))-g(\nabla^{\mathrm{g}}_X\varphi Y,Z)-g(\varphi Y,\nabla^{\mathrm{g}}_XZ)=0,\end{aligned}$$

therefore, taking into account (5.1), one concludes

$$(\nabla^{\mathrm{g}}_X\Phi_\varphi)(Y,Z)=g((\nabla^{\mathrm{g}}_X\varphi)Y,Z)=\frac{\sqrt{5}}{2\alpha}g((\nabla^{\mathrm{g}}_XJ_\varphi)Y,Z).$$

Then, identity (5.4) and the above one allow to claim that (M,φ,g) is a quasi-Kähler Norden golden manifold if and only if

$$g((\nabla^{\mathrm{g}}_XJ_\varphi)Y,Z)+g((\nabla^{\mathrm{g}}_YJ_\varphi)Z,X)+g((\nabla^{\mathrm{g}}_ZJ_\varphi)X,Y)=0,\ \ \forall X,Y,Z\in\mathfrak{X}(M), \tag{5.5}$$

which is a well-known condition that assures that the almost Norden manifold (M,J_φ,g) belongs to class $\mathcal{W}_3$ (see [11]). Thus (M,φ,g) satisfies (5.4) if and only if (M,J_φ,g) satisfies (5.5); i.e., (M,φ,g) is a quasi-Kähler Norden golden manifold if and only if (M,J_φ,g) is a quasi-Kähler Norden manifold.

In the Introduction we have seen that an α-golden metric manifold (M,φ,g) satisfies the Kähler condition $\nabla^{\mathrm{g}}\varphi=0$ if and only if its corresponding α-metric manifold (M,J_φ,g) satisfies the corresponding Kähler condition $\nabla^{\mathrm{g}}J_\varphi=0$. The same runs for the integrability condition because of the formula (1.11). All of these facts support our decision of classifying α-golden metric manifolds according to the classification of its corresponding α-metric manifold, as we have done in Definition 5.1.

By straightforward calculations using identity (1.11) and the above identities, one obtains the below classification according to Theorem 3.8, in terms of the tensors fields $\nabla^{\mathrm{g}}\varphi$ and N_φ and the 1-form $\delta\varphi$.

Theorem 5.4. *Let (M,φ,g) be a $2n$-dimensional α-golden metric manifold. Then one has the following classes of this kind of manifolds:*

i) *The class $\mathcal{W}_0$ or Kähler golden manifolds characterized by the condition*

$$\nabla^{\mathrm{g}}\varphi=0.$$

ii) *The class $\mathcal{W}_1$ characterized by the condition*

$$\begin{aligned}g((\nabla^{\mathrm{g}}_X\varphi)Y,Z)=&\frac{1}{5n}\left(g(X,Y)\delta\varphi\left(\frac{5-\alpha}{2}Z+\alpha\varphi Z\right)+g(X,Z)\delta\varphi\left(\frac{5-\alpha}{2}Y+\alpha\varphi Y\right)\right)\\&+\frac{\alpha}{5n}\left(g(X,\varphi Y)\delta\varphi(Z-2\varphi Z)+g(X,\varphi Z)\delta\varphi(Y-2\varphi Y)\right),\end{aligned}$$

for all X,Y,Z vector fields on M.

iii) *The class $\mathcal{W}_2$ characterized by the conditions*

$$\delta\varphi=0,\ \ N_\varphi=0.$$

iv) *The class $\mathcal{W}_3$ or quasi-Kähler golden manifolds characterized by the condition*

$$(\nabla^{\mathrm{g}}_X\varphi)\varphi X+(\nabla^{\mathrm{g}}_{\varphi X}\varphi)X-(\nabla^{\mathrm{g}}_X\varphi)X=0,\ \ \forall X\in\mathfrak{X}(M).$$

v) *The class $\mathcal{W}_1\oplus\mathcal{W}_2$ or integrable golden manifolds characterized by the condition*

$$N_\varphi=0.$$

vi) *The class $\mathcal{W}_2\oplus\mathcal{W}_3$ characterized by the condition*

$$\delta\varphi=0.$$

vii) *The class* $\mathcal{W}_1 \oplus \mathcal{W}_3$ *characterized by the condition*

$$g((\nabla^{\mathrm{g}}_X \varphi)\varphi X + (\nabla^{\mathrm{g}}_{\varphi X} \varphi)X - (\nabla^{\mathrm{g}}_X \varphi)X, Y) = \frac{1}{n}(g(X, \varphi X)\delta\varphi(Y) - g(X, X)\delta\varphi(\varphi Y)),$$

for all vector fields X, Y *on* M.

viii) *The class* $\mathcal{W}$ *or the whole class of* α*-golden metric manifolds.*

Remark 5.5. The defining conditions of all classes listed in the above theorem allow to recover the classification of almost Norden golden manifolds obtained in [6, Thm. 24] in the case of $\alpha = -1$, and the classification of almost golden Riemannian manifolds with null trace obtained in [9, Thm. 4.2] in the case of $\alpha = 1$, except the characterization of the classes $\mathcal{W}_3$ and $\mathcal{W}_1 \oplus \mathcal{W}_3$ if $\alpha = -1$. The characterizations showed in the quoted theorem are obtained rewriting identities (3.19) and (3.22) in terms of the almost golden complex structure φ. Given X, Y, Z vector fields on M, bearing in mind (1.6), (5.1) and (5.3), identity (3.22) reads as follows

$$\begin{aligned}\underset{XYZ}{\mathfrak{S}}(\nabla^{\mathrm{g}}_X \varphi)(Y, Z) = {} & \frac{1}{5n}\underset{XYZ}{\mathfrak{S}} g(X, Y)\delta\varphi((5-\alpha)Z + 2\alpha\varphi Z) \\ & + \frac{1}{5n}\underset{XYZ}{\mathfrak{S}} g(X, \varphi Y)\delta\varphi(2\alpha Z - 4\alpha\varphi Z),\end{aligned}$$

and, it is easy to prove that the second Nijenhuis tensor of J_φ vanishes if and only if

$$(\nabla^{\mathrm{g}}_X \varphi)\varphi Y + (\nabla^{\mathrm{g}}_{\varphi X} \varphi)Y + (\nabla^{\mathrm{g}}_Y \varphi)\varphi X + (\nabla^{\mathrm{g}}_{\varphi Y} \varphi)X = (\nabla^{\mathrm{g}}_X \varphi)Y + (\nabla^{\mathrm{g}}_Y \varphi)X.$$

The above conditions also characterize α-golden metric manifolds of the classes $\mathcal{W}_1 \oplus \mathcal{W}_3$ and $\mathcal{W}_3$ respectively. These defining conditions when $\alpha = -1$ are the characterizations obtained in [6, Thm. 24].

Bearing in mind the definition of the first canonical connection of an α-metric structure given in (1.12), identity (5.1) allows to introduce the first canonical connection of an α-golden metric manifold (M, φ, g) as the first canonical connection of the α-metric manifold (M, J_φ, g), which is the corresponding manifold by the bijection between these kind of metric manifolds.

Definition 5.6. Let (M, φ, g) be an α-golden metric manifold and let (J_φ, g) be the α-metric structure induced by (φ, g). The first canonical connection ∇^0 of (M, φ, g) is defined as follows

$$\begin{aligned}\nabla^0_X Y &= \nabla^{\mathrm{g}}_X Y + \frac{(-\alpha)}{2}(\nabla^{\mathrm{g}}_X J_\varphi)J_\varphi Y \\ &= \nabla^{\mathrm{g}}_X Y + \frac{\alpha}{5}(\nabla^{\mathrm{g}}_X \varphi)Y - \frac{2\alpha}{5}(\nabla^{\mathrm{g}}_X \varphi)\varphi Y, \ \forall X, Y \in \mathfrak{X}(M).\end{aligned} \tag{5.6}$$

This connection was first introduced in [10, Def. 4.3] in the golden case and later it was introduced in the complex golden case in [16, Def. 20] and [20, Eq. (23)]. In this last case, the first canonical connection of an almost complex golden manifold was introduced in the first article in the same way of identity (5.6), thus it was introduced as follows

$$\nabla^0_X Y = \nabla^{\mathrm{g}}_X Y + \frac{1}{2}(\nabla^{\mathrm{g}}_X J_\varphi)J_\varphi Y, \quad \forall X, Y \in \mathfrak{X}(M),$$

while in the second article it was introduced in a quite different manner, taking into account identity (3.1), this connection it was defined as follows

$$\begin{aligned}\nabla^0_X Y &= \nabla^{\mathrm{g}}_X Y - \frac{1}{2}J_\varphi(\nabla^{\mathrm{g}}_X J_\varphi)Y \\ &= \nabla^{\mathrm{g}}_X Y + \frac{1}{5}(\nabla^{\mathrm{g}}_X \varphi)Y - \frac{2}{5}\varphi(\nabla^{\mathrm{g}}_X \varphi)Y, \ \forall X, Y \in \mathfrak{X}(M).\end{aligned}$$

The first canonical connection ∇^0 parallelizes the α-structure J_φ and the metric g. Identity (5.1) allows to claim that ∇^0 also parallelizes the α-golden structure φ. Thus, ∇^0

is an example of adapted connection on an α-golden metric manifold, notion previously introduced on almost golden Riemannian manifolds in [10, Def. 4.1], and later, on almost Norden golden manifolds in [6, Def. 18], [16, Def. 8] and [20, Sect. 3] as follows.

Definition 5.7. Let (M, φ, g) be an α-golden metric manifold and let ∇ be a connection on M. One says that ∇ is adapted to the α-golden metric structure (φ, g) if $\nabla\varphi = 0$ and $\nabla g = 0$.

Identity (5.1) allows to claim that the set of adapted connections is invariant under the bijection between α-metric and α-golden metric manifolds.

Proposition 5.8. *Let (M, φ, g) be an α-golden metric manifold and let ∇ be a connection on M. The connection ∇ is adapted to the α-golden metric structure (φ, g) if and only if ∇ is adapted to its induced α-metric structure (J_φ, g).*

Once introduced the first canonical connection of an α-golden metric manifold, the previous identities allow to obtain defining conditions of all classes of this kind of metric manifolds rewriting the characterizations of the corresponding classes of α-metric manifolds showed in Theorem 4.7 using the α-golden structure φ instead of its corresponding α-structure J_φ.

Theorem 5.9. *Let (M, φ, g) be a $2n$-dimensional α-golden metric manifold. The classes given in Theorem 5.4 can be characterized by means of the torsion tensor and the torsion form of the first canonical connection as follows:*

i) *The class $\mathcal{W}_0$ or Kähler golden manifolds characterized by the condition*
$$\mathrm{T}^0(X,Y) = 0, \ \forall X, Y \in \mathfrak{X}(M).$$

ii) *The class $\mathcal{W}_1$ characterized by the condition*
$$\begin{aligned}\mathrm{T}^0(X,Y) = {}& \frac{1}{5n}\left(t^0\left(\frac{5-\alpha}{2}X + \alpha\varphi X\right)Y - t^0\left(\frac{5-\alpha}{2}Y + \alpha\varphi Y\right)X\right)\\ &+ \frac{\alpha}{5n}(t^0(X - 2\varphi X)\varphi Y - t^0(Y - 2\varphi Y)\varphi X), \ \forall X, Y \in \mathfrak{X}(M).\end{aligned}$$

iii) *The class $\mathcal{W}_2$ characterized by the conditions*
$$t^0(X) = 0, \ \frac{1+5\alpha}{2}\mathrm{T}^0(X,Y) + 2\mathrm{T}^0(\varphi X, \varphi Y) = \mathrm{T}^0(\varphi X, Y) + \mathrm{T}^0(X, \varphi Y), \ \forall X, Y \in \mathfrak{X}(M),$$
for all X, Y vector fields on M.

iv) *The class $\mathcal{W}_3$ or quasi-Kähler golden manifolds characterized by the condition*
$$\frac{1+5\alpha}{2}g(\mathrm{T}^0(X,Y),X) = -2g(\mathrm{T}^0(\varphi X,Y),\varphi X) + g(\mathrm{T}^0(\varphi X,Y),X) + g(\mathrm{T}^0(X,Y),\varphi X),$$
for all X, Y vector fields on M.

v) *The class $\mathcal{W}_1 \oplus \mathcal{W}_2$ or integrable golden manifolds characterized by the condition*
$$\frac{1+5\alpha}{2}\mathrm{T}^0(X,Y) + 2\mathrm{T}^0(\varphi X, \varphi Y) = \mathrm{T}^0(\varphi X, Y) + \mathrm{T}^0(X, \varphi Y), \ \forall X, Y \in \mathfrak{X}(M).$$

vi) *The class $\mathcal{W}_2 \oplus \mathcal{W}_3$ characterized by the condition*
$$t^0(X) = 0, \ \forall X \in \mathfrak{X}(M).$$

vii) *The class $\mathcal{W}_1 \oplus \mathcal{W}_3$ characterized by the condition*
$$\begin{aligned}\frac{1+5\alpha}{2}g(\mathrm{T}^0(X,Y),X) = {}& -2g(\mathrm{T}^0(\varphi X,Y),\varphi X) + g(\mathrm{T}^0(\varphi X,Y),X) + g(\mathrm{T}^0(X,Y),\varphi X)\\ &- \frac{1}{n}\left(g(X,X)t^0\left(\frac{5\alpha-1}{2}Y + \varphi Y\right) + g(X,\varphi X)t^0(Y - 2\varphi Y)\right),\end{aligned}$$
for all X, Y vector fields on M.

viii) *The class $\mathcal{W}$ or the whole class of α-golden metric manifolds.*

Now we will introduce a second distinguished adapted connection on α-golden metric manifolds using the well-adapted connection of α-metric manifolds (see (1.13)) as follows.

Definition 5.10. Let (M, φ, g) be an α-golden metric manifold and let (J_φ, g) be the α-metric structure induced by (φ, g). The well-adapted connection ∇^{w} of (M, φ, g) is the unique connection satisfying $\nabla^{\mathrm{w}} J_\varphi = 0$, $\nabla^{\mathrm{w}} g = 0$ and the equality

$$g(\mathrm{T}^{\mathrm{w}}(X,Y),Z) - g(\mathrm{T}^{\mathrm{w}}(Z,Y),X) = \alpha(g(\mathrm{T}^{\mathrm{w}}(J_\varphi Z,Y),J_\varphi X) - g(\mathrm{T}^{\mathrm{w}}(J_\varphi X,Y),J_\varphi Z)), \quad (5.7)$$

for all vector fields X, Y, Z on M.

The above connection was previously introduced on almost golden Riemannian and almost Norden golden manifolds in [10, Thm. 4.5] and [6, Def. 21].

Given X, Y, Z vector fields on M, identity (1.6) carries to the following equalities

$$g(\mathrm{T}^{\mathrm{w}}(J_\varphi X,Y),J_\varphi Z) = \frac{1}{5}(g(\mathrm{T}^{\mathrm{w}}(X,Y),Z-2\varphi Z) - 2g(\mathrm{T}^{\mathrm{w}}(\varphi X,Y),Z-2\varphi Z)),$$
$$g(\mathrm{T}^{\mathrm{w}}(J_\varphi Z,Y),J_\varphi X) = \frac{1}{5}(g(\mathrm{T}^{\mathrm{w}}(Z,Y),X-2\varphi X) - 2g(\mathrm{T}^{\mathrm{w}}(\varphi Z,Y),X-2\varphi X)),$$

then, according to (5.7), one obtains the next expression of the well-adapted connection of (M, φ, g)

$$\begin{aligned} g(\mathrm{T}^{\mathrm{w}}(X,Y),Z) - g(\mathrm{T}^{\mathrm{w}}(Z,Y),X) = {} & \frac{2\alpha}{5+\alpha}(g(\mathrm{T}^{\mathrm{w}}(X,Y),\varphi Z) - g(\mathrm{T}^{\mathrm{w}}(Z,Y),\varphi X)) \\ & + \frac{2\alpha}{5+\alpha} g(\mathrm{T}^{\mathrm{w}}(\varphi X,Y),Z-2\varphi Z) \\ & - \frac{2\alpha}{5+\alpha} g(\mathrm{T}^{\mathrm{w}}(\varphi Z,Y),X-2\varphi X). \end{aligned} \quad (5.8)$$

Remark 5.11. Among all adapted connections on an α-golden metric manifold, identities (5.7) and (5.8) characterize the well-adapted connection. Nevertheless, none of them provide an explicit expression of this adapted connection similar to (5.6) which defines the first canonical connection of one of these manifolds. The expressions of the well-adapted connection of an α-metric manifold in the integrable and quasi-Kähler cases showed in (4.23) and (4.25) respectively, jointly with identities (1.6) and (5.1), lead to an explicit expression of well-adapted connection of an α-golden metric in both cases as below. If (M, φ, g) belongs to the classes $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_1 \oplus \mathcal{W}_2$ then the well-adapted connection reads as follows

$$\nabla^{\mathrm{w}}_X Y = \nabla^{\mathrm{g}}_X Y + \frac{\alpha}{5}(\nabla^{\mathrm{g}}_X \varphi)Y - \frac{2\alpha}{5}(\nabla^{\mathrm{g}}_X \varphi)\varphi Y, \;\; \forall X, Y, \in \mathfrak{X}(M),$$

while if (M, φ, g) is of class $\mathcal{W}_3$ the expression of the well-adapted connection is the next one

$$\nabla^{\mathrm{w}}_X Y = \nabla^{\mathrm{g}}_X Y - \frac{2\alpha}{5}(\nabla^{\mathrm{g}}_X \varphi)\varphi Y + \frac{\alpha}{5}((\nabla^{\mathrm{g}}_X \varphi)Y + (\nabla^{\mathrm{g}}_{\varphi Y} \varphi)X - (\nabla^{\mathrm{g}}_Y \varphi)\varphi X), \;\; \forall X, Y \in \mathfrak{X}(M).$$

Moreover, bearing in mind (1.11) and Remark 4.10 one can claim that (M, φ, g) is of class $\mathcal{W}_3$ if and only if

$$\mathrm{T}^{\mathrm{w}}(X,Y) = \frac{(-\alpha)}{5} N_\varphi(X,Y), \;\; \forall X, Y \in \mathfrak{X}(M).$$

Analogously to Theorem 5.9, we finish characterizing all classes of of α-golden metric manifolds by means of the well-adapted connection previously introduced.

Theorem 5.12. *Let (M, φ, g) be a $2n$-dimensional α-golden metric manifold. The classes given in Theorem 5.4 can be characterized by means of the torsion tensor and the torsion form of the well-adapted connection as follows:*

i) *The class* $\mathcal{W}_0$ *or Kähler golden manifolds characterized by the condition*

$$\mathrm{T}^{\mathrm{w}}(X,Y)=0,\ \ \forall X,Y\in\mathfrak{X}(M).$$

ii) *The class* $\mathcal{W}_1$ *characterized by the condition*

$$\mathrm{T}^{\mathrm{w}}(X,Y)=\frac{1}{5n}\left(t^{\mathrm{w}}\left(\frac{5-\alpha}{2}X+\alpha\varphi X\right)Y-t^{\mathrm{w}}\left(\frac{5-\alpha}{2}Y+\alpha\varphi Y\right)X\right)$$
$$+\frac{\alpha}{5n}(t^{\mathrm{w}}(X-2\varphi X)\varphi Y-t^{\mathrm{w}}(Y-2\varphi Y)\varphi X),\ \ \forall X,Y\in\mathfrak{X}(M).$$

iii) *The class* $\mathcal{W}_2$ *characterized by the conditions*

$$t^{\mathrm{w}}(X)=0,\ \ \frac{1+5\alpha}{2}\mathrm{T}^{\mathrm{w}}(X,Y)+2\mathrm{T}^{\mathrm{w}}(\varphi X,\varphi Y)=\mathrm{T}^{\mathrm{w}}(\varphi X,Y)+\mathrm{T}^{\mathrm{w}}(X,\varphi Y),\ \ \forall X,Y\in\mathfrak{X}(M).$$

iv) *The class* $\mathcal{W}_3$ *or quasi-Kähler golden manifolds characterized by one of the equivalent conditions*

$$\mathrm{T}^{\mathrm{w}}(\varphi X,Y)+\varphi\mathrm{T}^{\mathrm{w}}(X,Y)=\mathrm{T}^{\mathrm{w}}(X,Y),$$
$$\frac{1-5\alpha}{2}\mathrm{T}^{\mathrm{w}}(X,Y)+2\mathrm{T}^{\mathrm{w}}(\varphi X,\varphi Y)=\mathrm{T}^{\mathrm{w}}(\varphi X,Y)+\mathrm{T}^{\mathrm{w}}(X,\varphi Y),\ \ \forall X,Y\in\mathfrak{X}(M).$$

v) *The class* $\mathcal{W}_1\oplus\mathcal{W}_2$ *or integrable golden manifolds characterized by the condition*

$$\frac{1+5\alpha}{2}\mathrm{T}^{\mathrm{w}}(X,Y)+2\mathrm{T}^{\mathrm{w}}(\varphi X,\varphi Y)=\mathrm{T}^{\mathrm{w}}(\varphi X,Y)+\mathrm{T}^{\mathrm{w}}(X,\varphi Y),\ \ \forall X,Y\in\mathfrak{X}(M).$$

vi) *The class* $\mathcal{W}_2\oplus\mathcal{W}_3$ *characterized by the condition*

$$t^{\mathrm{w}}(X)=0,\ \ \forall X\in\mathfrak{X}(M).$$

vii) *The class* $\mathcal{W}_1\oplus\mathcal{W}_3$ *characterized by the condition*

$$\mathrm{T}^{\mathrm{w}}(\varphi X,Y)+\varphi\mathrm{T}^{\mathrm{w}}(X,Y)-\mathrm{T}^{\mathrm{w}}(X,Y)=\frac{1}{n}(t^{\mathrm{w}}(\varphi Y)X-t^{\mathrm{w}}(Y)\varphi X),\ \ \forall X,Y\in\mathfrak{X}(M).$$

viii) *The class* $\mathcal{W}$ *or the whole class of* α*-golden metric manifold.*

Theorems 5.9 and 5.12 allow to recover the classifications of almost golden Riemannian with null trace and almost Norden golden manifolds obtained in [9, Thm. 4.3] and [6, Thm. 25] respectively.

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