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ON (p,q)-ANALOG OF STANCU OPERATORS OF ROUGH λ-STATISTICALLY ρ-CAUCHY CONVERGENCE OF TRIPLE SEQUENCE SPACES

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Abstract. In this work, using the concept of natural density, we introduce the (p,q)-analogue of the Stancu-beta operators of rough λ-statistically ρ-Cauchy convergence on triple sequence spaces. We define the set of Bernstein Stancu beta opeators of rough statistical limit points of a triple sequence spaces and obtain to λ-statistical convergence criteria associated with this set. Also, we examine the relations between the set of Bernstein-Stancu beta operators of rough λ-statistically ρ-Cauchy convergence of triple sequences.

1. Introduction

We introduce the (p,q)-analogue of the Stancu-beta operators and study their approximation properties.

The idea of statistical convergence was introduced by Steinhaus and also independently by Fast for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence.

Let K be a subset of the set of positive integers $\mathbb{N}$ and let us denote the set $K_{ij\ell} = \{(m,n,k) \in K : m \leq i, n \leq j, k \leq \ell\}$. Then the natural density of K is given by

$$\delta_3(K) = \lim_{i,j,\ell \to \infty} \frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

First applied the concept of (p,q)-calculus in approximation theory and introduced the (p,q)-analogue of Bernstein operators. Later, based on (p,q)-integers, some approximation results for Bernstein-Stancu operators, Bernstein-Kantorovich operators, (p,q)-Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein-Schurer operators etc.

Very recently, Khalid et al. have given a nice application in computer-aided geometric design and applied these Bernstein basis for construction of (p,q)-Bezier

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curves and surfaces based on (p,q)-integers which is further generalization of q-Bezier curves and surfaces.

Motivated by the above mentioned work on (p,q)-approximation and its application, in this paper we study statistical approximation properties of Bernstein-Stancu beta operators based on (p,q)-integers.

Now we recall some basic definitions about (p,q)-integers. For any $u,v,w \in \mathbb{N}$, the (p,q)-integer $[uvw]_{p,q}$ is defined by

$$[0]_{p,q} := 0 \text{ and } [uvw]_{p,q} = \frac{p^{uvw} - q^{uvw}}{p-q} \text{ if } u,v,w \geq 1,$$

where $0 < q < p \leq 1$. The (p,q)-factorial is defined by

$$[0]_{p,q}! := 1 \text{ and } [uvw]!_{p,q} = [1]!_{p,q}[6]!_{p,q}...[uvw]!_{p,q} \text{ if } u,v,w \geq 1.$$

Also the (p,q)-binomial coefficient is defined by

$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} = \frac{[uvw]!_{p,q}}{[mnk]!_{p,q}\,[(u-m)+(v-n)+(w-k)]!_{p,q}}$$

for all $u,v,w,m,n,k \in \mathbb{N}$ with $u \geq m, v \geq n, w \geq k$.

The formula for $(p,q)-$binomial expansion is as follows:

$$\begin{aligned}(ax+by)_{p,q}^{uvw} &= \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w} p^{\frac{(u-m)(u-m-1)+(v-n)(v-n-1)+(w-k)(w-k-1)}{6}} q^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}} \\ &\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} a^{(u-m)+(v-n)+(w-k)} b^{m+n+k} x^{(u-m)+(v-n)+(w-k)} y^{m+n+k},\end{aligned}$$

$$\begin{aligned}(x+y)_{p,q}^{uvw} &= (x+y)(px+qy)\left(p^6x+q^6y\right)\cdots \\ &\qquad \left(p^{(u-1)+(v-1)+(w-1)}x + q^{(u-1)+(v-1)+(w-1)}y\right),\end{aligned}$$

$$\begin{aligned}(1-x)_{p,q}^{uvw} &= (1-x)(p-qx)\left(p^6-q^6x\right)\cdots \\ &\qquad \left(p^{(u-1)+(v-1)+(w-1)} - q^{(u-1)+(v-1)+(w-1)}x\right), \text{ and}\end{aligned}$$

$$(x)_{p,q}^{mnk} = x\,(px)\left(p^6x\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x\right) = p^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}}.$$

The Bernstein operator of order (r,s,t) is given by

$$B_{rst}(f,x) = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

The (p,q)-Bernstein operators are defined as follows:

$$B_{rst,p,q}(f,x) = \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{6}}} \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} \binom{r}{m}\binom{s}{n}\binom{t}{k} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}} x^{m+n+k} \quad (1.1)$$
$$\cdot \prod_{u=0}^{(r-m-1)+(s-n-1)+(t-k-1)} (p^u - q^u x) f\left(\frac{[mnk]_{p,q}}{p^{(m-r)+(n-s)+(k-t)}[rst]_{p,q}}\right), x \in [0,1].$$

Also, we have

$$(1-x)_{p,q}^{rst} = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} (-1)^{m+n+k} p^{\frac{(r-m)(r-m-1)+(s-n)(s-n-1)+(t-k)(t-k-1)}{6}} q^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}} \binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}.$$

(p,q)-Bernstein-Stancu operators are defined as follows:

$$S_{rst,p,q}(f,x) = \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{6}}} \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} \binom{r}{m}\binom{s}{n}\binom{t}{k} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{6}} x^{m+n+k}. \quad (1.2)$$
$$\prod_{u=0}^{(r-m-1)+(s-n-1)+(t-k-1)} (p^u - q^u x) f\left(\frac{p^{(r-m)+(s-n)+(t-k)}[mnk]_{p,q}+\eta}{[rst]_{p,q}+\mu}\right), x \in [0,1].$$

Note that for $\eta = \mu = 0$, (p,q)-Bernstein-Stancu operators given by (1.2) reduces into (p,q)-Bernstein operators. Also for $p=1$, (p,q)-Bernstein-Stancu operators given by (1.1) turn out to be q-Bernstein-Stancu operators.

The definite integrals of a function f are defined by

$$\int_0^a \int_0^b \int_0^c f(xyz)\, d_{pq}x d_{pq}y d_{pq}z = (q-p)\, abc \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{p^{m+n+k}}{q^{(m+1)+(n+1)+(k+1)}} f\left(\frac{p^{m+n+k}}{q^{(m+1)+(n+1)+(k+1)}} abc\right),$$
$$\text{when } \left|\frac{p}{q}\right| < 1 \text{ and}$$

$$\int_0^a \int_0^b \int_0^c f(xyz)\, d_{pq}x d_{pq}y d_{pq}z = (p-q)\, abc \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{m+n+k}}{p^{(m+1)+(n+1)+(k+1)}} f\left(\frac{q^{m+n+k}}{p^{(m+1)+(n+1)+(k+1)}} abc\right),$$
$$\text{when } \left|\frac{q}{p}\right| < 1.$$

There are two (p,q)-analogues of the classical exponential function defined as follows:

$$e_{pq}(x)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{p^{\frac{u(u-1)+v(v-1)+w(w-1)}{2}}}{[uvw]_{pq}!}x^{u+v+w} \text{ and}$$
$$E_{pq}(x)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{\frac{u(u-1)+v(v-1)+w(w-1)}{2}}}{[uvw]_{pq}!}x^{u+v+w}.$$

It is easily seen that $e_{pq}(x)\ E_{pq}(-x)=1$. For $m,n,k\in\mathbb{N}$, the (p,q)-beta and the (p,q)-Gamma functions are defined by

$$B_{pq}(m,n)=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{x^{m-1}+x^{n-1}+x^{k-1}}{(1+x)^{m+n}}d_{pq}x$$

and

$$\Gamma_{pq}(u)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}p^{\frac{u(u-1)+v(v-1)+w(w-1)}{2}}E_{pq}(-q(xyz))\,d_{pq}x,$$
$$\Gamma_{pq}(u+1,v+1,w+1)=[uvw]_{pq}!,$$

respectively. The functions are connected through

$$B_{pq}(m,n,k)=q^{\frac{6-[m(m-1)+n(n-1)+k(k-1)]}{2}}p^{\frac{-[m(m-1)+n(n-1)+k(k-1)]}{2}}\frac{\Gamma_{pq}(m)\,\Gamma_{pq}(n)\,\Gamma_{pq}(k)}{\Gamma_{pq}(m+n+k)}. \quad (1.3)$$

If $p=1$ then the above notions of (p,q)-calculus reduce to the corresponding notations of q-calculus.

Let $0<q<p<1$ and $x\in[0,\infty)$. We introduce the (p,q)-Stancu-beta operators as follows:

$$S_{uvw,pq}(f,x)=\frac{1}{B_{pq}\left([uvw]_{pq}x,[uvw]_{pq}+3\right)}.$$
$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{(rst)^{[uvw]_{pq}(x-1)}}{(1+(rst))^{[uvw]_{pq}x+[uvw]_{pq}+3}}.$$
$$f\left(p^{[uvw]_{pq}x},q^{[uvw]_{pq}x}rst\right)d_{pq}r\,d_{pq}s\,d_{pq}t.$$

Throughout the paper, $\mathbb{R}^3$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence of Bernstein-Stancu beta operators $(S_{uvw,p,q}(f,x))$ such that $(S_{uvw,p,q}(f,x))\in\mathbb{R}$, $m,n,k\in\mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu-beta operators $(S_{uvw,p,q}(f,x))$ is said to be statistically convergent to $0\in\mathbb{R}$, written as $st_3-\lim S_{uvw,p,q}(f,x)=f(x)$, provided that the set

$$K_\epsilon:=\left\{(m,n,k)\in\mathbb{N}^3:|S_{uvw,p,q}(f,x)-(f,x)|\geq\epsilon\right\}$$

has natural density zero for any $\epsilon>0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein-Stancu-beta operators. i.e., $\delta_3(K_\epsilon)=0$. That is,

$$\lim_{u,v,w\to\infty}\frac{1}{uvw}\left|\{m\leq u,n\leq v,k\leq w:|S_{uvw,p,q}(f,x)-(f,x)|\geq\epsilon\}\right|=0.$$

In this case, we write $\delta_3 - \lim S_{uvw,p,q}(f,x) = (f,x)$ or $S_{uvw,p,q}(f,x) \xrightarrow{st_3} (f,x)$.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers , χ_A-the characteristic function of $A \subset \mathbb{N}$. A subset A of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d_3(A) = \lim_{i,j,\ell\to\infty} \frac{1}{ij\ell} \sum_{m=1}^{i} \sum_{n=1}^{j} \sum_{k=1}^{\ell} \chi_A(K).$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu [11], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [10] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

In this paper, we introduce the notion of Bernstein-Stancu beta operators of rough λ-statistically ρ-Cauchy sequences convergence. Defining the set of Bernstein-Stancu beta operators of rough λ-statistical limit points of a sequence, we obtain to λ-statistical convergence criteria associated with this set. Later, we prove that this set of rough λ-statistically ρ-Cauchy convergence of a triple sequence spaces.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N}^3 \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Esi et al.* [2, 3, 4, 5], *Dutta et al.* [6], *Esi et al.* [7, 8], *Sahiner et al.* [12, 13], *Subramanian et al.* [14], *Debnath et al.* [9] and many others.

Throughout the paper let β be a nonnegative real number.

2. Definitions and Preliminaries

Definition 2.1. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,pq,}(f,x))$ be a triple sequence of Bernstein-Stancu beta operators of real numbers is said to be β-convergent to (f,x) denoted by $S_{uvw,p,q}(f,x) \to^{\beta} (f,x)$, provided that*

$$\forall \epsilon > 0\ \exists (u_\epsilon, v_\epsilon, w_\epsilon) \in \mathbb{N}^3 : u \geq u_\epsilon, v \geq v_\epsilon, w \geq w_\epsilon \Rightarrow |S_{uvw,p,q}(f,x) - (f,x)| < \beta + \epsilon.$$

The set

$$\mathrm{LIM}^{\beta} x = \left\{(f,x) \in \mathbb{R}^3 : S_{uvw,p,q}(f,x) \to^{\beta} (f,x)\right\}$$

is called the β-limit set of the triple sequences.

Definition 2.2. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu-beta operators of real numbers is said to be β-convergent if $\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x) \neq \phi$. In this case, β is called the Bernstein-Stancu-beta operators of rough convergence degree of the triple sequence spaces. For $\beta = 0$, we get the ordinary convergence.*

Definition 2.3. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu beta operators of real numbers is said to be β-statistically convergent to (f,x), denoted by $S_{uvw,p,q}(f,x) \rightarrow^{uvw} (f,x)$, provided that the set*

$$\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \epsilon\right\}$$

has natural density zero for every $\epsilon > 0$, or equivalently, if the condition

$$st - \limsup |S_{uvw,p,q}(f,x) - (f,x)| \leq \beta$$

is satisfied.

In addition, we can write $S_{uvw,p,q}(f,x) \rightarrow^{uvw} (f,x)$ if and only if the inequality

$$|S_{uvw,p,q}(f,x) - (f,x)| < \beta + \epsilon$$

holds for every $\epsilon > 0$ and almost all (u,v,w). Here β is called the Bernstein-Stancu beta operators of roughness of degree. If we take $\beta = 0$, then we obtain the ordinary statistical convergence.

In a similar fashion to the idea of classic Bernstein-Stancu beta operators of rough convergence, the idea of Bernstein-Stancu beta operators of rough statistical convergence of a triple sequence spaces can be interpreted as follows:

Assume that a Bernstein-Stancu beta operators of triple sequence space $(S_{uvw,p,q}(g,x))$ is statistically convergent and cannot be measured or calculated exactly; one has to do with an approximated (or statistically approximated) triple sequence spaces $(S_{uvw,p,q}(f,x))$ satisfying $|S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \beta$ for all u,v,w (or for almost all u,v,w, i.e.,

$$\delta\left(\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| > \beta\right\}\right) = 0.$$

Then the triple sequence spaces x is not statistically convergent any more, but as the inclusion

$$\begin{aligned} &\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(g,x) - (f,x)| \geq \epsilon\right\} \\ &\supseteq \left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \epsilon\right\} \end{aligned} \tag{2.1}$$

holds and we have

$$\delta\left(\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(g,x) - (f,x)| \geq \epsilon\right\}\right) = 0,$$

i.e., we get

$$\delta\left(\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \epsilon\right\}\right) = 0,$$

i.e., the triple sequence spaces x is β-statistically convergent in the sense of definition 2.3

In general, the Bernstein-Stancu beta operators of rough statistical limit may not unique for the Bernstein-Stancu beta operators of roughness degree $r > 0$. So we have to consider the so called Bernstein-Stancu beta operators of rough ness of β-statistical limit set is defined by

$$st - \mathrm{LIM}^{\beta}\, s_{uvw,p,q}(f,x) = \left\{(f,x) \in \mathbb{R}^3 : S_{uvw,p,q}(f,x) \rightarrow^{uvw} (f,x)\right\}.$$

The Bernstein-Stancu-beta operators of triple sequence space $S_{uvw,p,q}(f,x)$ is said to be Bernstein-Stancu beta operators of rough β-statistically convergent provided that $st{-}\mathrm{LIM}^{\beta}\, S_{uvw,p,q}(f,x) \neq \phi$. It is clear that if $st{-}\mathrm{LIM}^{\beta}\, S_{uvw,p,q}(f,x) \neq$

ϕ. We have

$$\begin{aligned} & st-\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x) \\ = & \left[st-\limsup S_{uvw,p,q}(f,x)-\beta, st-\liminf S_{uvw,p,q}(f,x)+\beta\right] \end{aligned} \tag{2.2}$$

We know that $\mathrm{LIM}^{\beta}=\phi$ for an unbounded triple sequence spaces might be rough statistically convergent. For instance, define

$$S_{uvw,p,q}(f,x)=\begin{cases} (-1)^{uvw} & , \text{ if} u \neq i^3, v \neq j^3, w \neq \ell^3 \ (i,j,\ell \in \mathbb{N}) \\ uvw & , \text{ otherwise} \end{cases}$$

in $\mathbb{R}$. Because the set $\{1,64,739,\ldots\}$ has natural density zero, we have

$$st-\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x)=\begin{cases} \phi & , \text{ if } \beta<1 \\ [1-\beta,\beta-1] & , \text{ otherwise} \end{cases}$$

and $\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x)=\phi$ for all $\beta \geq 0$.

As can be seen by the example above, the fact that $st-\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x) \neq \phi$ does not imply $\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x) \neq \phi$. Because a finite set of natural numbers has natural density zero, $\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x) \neq \phi$ implies $st-\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x) \neq \phi$. Therefore, we get $\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x) \subseteq st-\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x)$. This obvious fact means $\left\{\beta \geq 0 : \mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x) \neq \phi\right\} \subseteq \left\{\beta \geq 0 : st-\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x) \neq \phi\right\}$ in this language of sets and yields immediately

$$\inf\left\{\beta \geq 0 : \mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x) \neq \phi\right\} \geq \inf\left\{\beta \geq 0 : st-\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x) \neq \phi\right\}.$$

Moreover, it also yields directly

$$\dim\left(\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x)\right) \leq \dim\left(st-\mathrm{LIM}^{\beta} s_{uvw,p,q}(f,x)\right).$$

Note. *The Bernstein-Stancu beta operators of rough statistical limit of a triple sequence spaces is unique for the roughness degree $\beta>0$.*

Definition 2.4. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu-beta operators of real numbers is β-convergent, i.e., $\mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x) \neq \phi$. Take an arbitrary $L \in \mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x)$, for all $\epsilon>0 \; \exists$ an $u_\epsilon, v_\epsilon, w_\epsilon \in \mathbb{N}^3$ such that $u \geq u_\epsilon, v \geq v_\epsilon, w \geq w_\epsilon$ implies*

$$\left|S_{uvw,p,q}(f,x)-(f,x)\right| \leq \beta+\frac{\epsilon}{2} \text{ and } \left|S_{uvw,p,q}(g,x)-(g,x)\right| \leq \beta+\frac{\epsilon}{2}$$
$$\begin{aligned} \Rightarrow \left|S_{uvw,p,q}(f,x)-S_{uvw,p,q}(g,x)\right| & \leq \left|S_{uvw,p,q}(f,x)-(f,x)\right|+\left|S_{uvw,p,q}(g,x)-(g,x)\right| \\ & \leq \beta+\frac{\epsilon}{2}+\frac{\epsilon}{2} \leq 2\beta+\epsilon. \end{aligned}$$

Hence the Bernstein-Stancu beta operators of triple sequence spaces is a ρ-Cauchy sequence with $\rho=2\beta$. This Cauchy degree cannot be generally decreased. Indeed, let $z \in \mathbb{R}^3$ with $|z|=\beta$ and $S_{uvw,p,q}(f,x)=(-1)^{u+v+w} z$ then Bernstein-Stancu-beta operators of roughness is β-convergent with $0 \in \mathrm{LIM}^{\beta} S_{uvw,p,q}(f,x)$, and $\rho=2\beta$ is its minimal Cauchy degree.

Conversely, let $\rho \geq 0$ be a Cauchy degree of some given Bernstein-Stancu-beta operators of triple sequence $(S_{uvw,p,q}(f,x))$ its convergence degree to equal $\frac{\rho}{2}$, i.e., $\mathrm{LIM}^{\frac{\rho}{2}} S_{uvw,p,q}(f,x) \neq \phi$. This condition always not true.

Definition 2.5. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu beta operators of real numbers is said to be $\beta\lambda$-statistically convergent or $\beta\lambda$ st-convergent to (f,x), denoted by $S_{uvw,p,q}(f,x) \rightarrow^{\beta\lambda st} (f,x)$, provided that the set*

$$\lim_{u,v,w} \frac{1}{\lambda_{uvw}} \left| \left\{ (u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \epsilon \right\} \right| = 0$$

Definition 2.6. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(s_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu-beta operators of real numbers and $\lambda = (\lambda_{uvw})$ be non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{(uvw)+1} \leq \lambda_{uvw}+1$, $\lambda_{111} = 1$. Hence the Bernstein-Stancu-beta operators of triple sequence is a $\rho-$ Cauchy sequence with $\rho = 2\beta$, i.e.,*

$$\lim_{uvw} \frac{1}{\lambda_{uvw}} \left| \left\{ (u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \rho + \epsilon \right\} \right| = 0$$

If $\lambda = 1$ then it is called ordinary ρ-Cauchy sequences.

3. Main Results

Theorem 3.1. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu beta operators of real numbers $\beta > 0$, a triple sequence $(S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st} (f,x) \Leftrightarrow (S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st}$ ρ-Cauchy sequence.*

Proof. Assume that $(S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st} (f,x)$. Let $\epsilon > 0$. then we can write

$$\delta\left(\left\{ (u,v,w) \in \mathbb{N}^3 : \frac{1}{\lambda_{uvw}} |S_{uvw,p,q}(f,x) - (f,x)| \geq \epsilon \right\} \right) = 0,$$

we have $\delta(K_1) = 0$ and $\delta(K_2) = 0$, where

$$K_1 = \left\{ (u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \frac{\epsilon}{2} \right\}$$

and

$$K_2 = \left\{ (u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(g,x) - (g,x)| \geq \beta + \frac{\epsilon}{2} \right\}.$$

Using the properties of natural density, we get

$$\frac{\delta\left(K_1^c \bigcap K_2^c\right)}{\lambda_{uvw}} = 1 \text{ as } u,v,w \to \infty.$$

Since the triple sequence is β convergent, $\mathrm{LIM}^\beta\, S_{uvw,p,q}(f,x) \neq \phi$, take an arbitrary $(f,x) \in \mathrm{LIM}^\beta\, S_{uvw,p,q}(f,x)$ for all $\epsilon > 0$ there exists an $u_\epsilon, v_\epsilon, w_\epsilon \in \mathbb{N}$ such that $u \geq u_\epsilon, v \geq v_\epsilon, w \geq w_\epsilon$ and $u \geq u_\epsilon, v \geq v_\epsilon, w \geq w_\epsilon$ implies $|S_{uvw,p,q}(f,x) - (f,x)| < \beta + \frac{\epsilon}{2}$ and $|S_{uvw,p,q}(g,x) - (g,x)| < \beta + \frac{\epsilon}{2}$,

$$\begin{aligned} \Rightarrow \frac{1}{\lambda_{uvw}} & \left| \left\{ (u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \geq \rho + \epsilon \right\} \right| \\ & \leq \frac{1}{\lambda_{uvw}} \left| \left\{ (u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \frac{\epsilon}{2} \right\} \right| + \\ & \qquad \frac{1}{\lambda_{uvw}} \left| \left\{ (u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(g,x) - (g,x)| \geq \beta + \frac{\epsilon}{2} \right\} \right| = 0. \end{aligned}$$

Hence, $(S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st}$ ρ-Cauchy sequence.

Conversely, suppose that the triple sequence spaces of Bernstein-Stancu beta operators of $(S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st}$ ρ-Cauchy sequence. For every $\epsilon > 0$, we have

$$\Rightarrow \frac{1}{\lambda_{uvw}} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \epsilon\right\}\right|$$
$$\leq \frac{1}{\lambda_{uvw}} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \frac{\epsilon}{2}\right\}\right| +$$
$$\frac{1}{\lambda_{uvw}} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(g,x) - (g,x)| \geq \beta + \frac{\epsilon}{2}\right\}\right|,$$

$$\Rightarrow \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \epsilon\right\}\right| \leq$$
$$\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - (f,x)| \geq \beta + \frac{\epsilon}{2}\right\} +$$
$$\left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(g,x) - (g,x)| \geq \beta + \frac{\epsilon}{2}\right\}\right|.$$
$$\Rightarrow (s_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st} (f,x).$$

∎

Theorem 3.2. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu beta operators of real numbers $\beta > 0$,*

$$(S_{uvw,p,q}(f,x)) \rightarrow^{mnk} (f,x) \Longrightarrow st - \liminf \left(\frac{\lambda_{uvw}}{uvw}\right) \rightarrow^{\beta\lambda st} (f,x).$$

Proof. Omitted. ∎

Theorem 3.3. *Let f be a continuous function defined on the closed interval $[0,1]$ and let $(S_{uvw,p,q}(f,x))$ be a triple sequence of Bernstein-Stancu beta operators of real numbers $\beta > 0$, if $(S_{uvw,p,q}(f,x)) \rightarrow^{mnk}$ ρ-Cauchy sequence and $\mathrm{LIM}^r\, S_{uvw,p,q}(f,x) - \liminf\left(\frac{\lambda_{uvw}}{uvw}\right) > 0$, then $(S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st}$ ρ-Cauchy sequence.*

Proof. A Bernstein-Stancu beta operators of triple sequence $(S_{uvw,p,q}(f,x))$ be β-convergent, i.e., $\mathrm{LIM}^\beta\, S_{uvw,p,q}(f,x) \neq \phi$. Take an arbitrary $(f,x) \in \mathrm{LIM}^\beta\, S_{uvw,p,q}(f,x)$ for all $\epsilon > 0$ there exists an $u_\epsilon, v_\epsilon, w_\epsilon \in \mathbb{N}$ such that $u \geq u_\epsilon, v \geq v_\epsilon, w \geq w_\epsilon \in \mathbb{N}$ and $\beta \geq u_\epsilon, v_\epsilon, w_\epsilon \in \mathbb{N}$ implies

$$\{u_\epsilon \leq u, v_\epsilon \leq v, w_\epsilon \leq w : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \rho + \epsilon\}$$
$$\supset \left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \rho + \epsilon\right\}.$$

Therefore,

$$\frac{1}{uvw} |\{u_\epsilon \leq u, v_\epsilon \leq v, w_\epsilon \leq w : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \rho + \epsilon\}|$$
$$\supset \frac{1}{uvw} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \rho + \epsilon\right\}\right|$$
$$\geq \frac{\lambda_{uvw}}{uvw} \frac{1}{\lambda_{uvw}} \left|\left\{(u,v,w) \in \mathbb{N}^3 : |S_{uvw,p,q}(f,x) - S_{uvw,p,q}(g,x)| \leq \rho + \epsilon\right\}\right|.$$

Taking limit as $u,v,w \to \infty$ and using $\mathrm{LIM}^\beta\, S_{uvw,p,q}(f,x) - \liminf\left(\frac{\lambda_{uvw}}{uvw}\right) > 0$, we get $(S_{uvw,p,q}(f,x)) \rightarrow^{\beta\lambda st}$ ρ-Cauchy sequence. ∎

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