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Research Article

A new approach to a theorem of Eng

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Abstract

The main aim of this work is to give a case-free algebraic proof for a theorem of Eng on the Poincaré polynomial of parabolic quotients of finite Coxeter groups evaluated at -1.

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1. Introduction

Let (W, S) be a finite Coxeter system. For any subset J of S, if the simple reflections in J generate W_J, then W_J is called *a standard parabolic subgroup* of W. Denote by X_J the set of distinguished left coset representatives W_J in W and let w_0 be the longest element of W. In [3, Theorem 1], Eng proved the following theorem about -1 phenomenon.

Theorem 1.1. *For any $J \subseteq S$, we have*

$$\sum_{w \in X_J} (-1)^{l(w)} = |\{w \in X_J : w_0 w W_J = w W_J\}|, \tag{1.1}$$

where l denotes the length function on W.

Eng proved this theorem by using case by case techniques. In [5], Reiner gave a case-free geometric proof of Theorem 1.1, which applies only to Weyl groups not all finite Coxeter groups. Then in [6], Reiner, Stanton, White presented the first case-free algebraic proof of the theorem given above. To give another proof of Theorem 1.1 algebraically, we use a different method than that of Reiner, Stanton, White. Our approach depends more on the structure of the descent algebra of a finite Coxeter group introduced by Solomon in [7]. Our proof is new and avoids case by case considerations. Set $x_J = \sum_{w \in X_J} w$ for any subset J of S. Then $\{x_J \mid J \subseteq S\}$ forms a basis for a subalgebra of the group algebra $\mathbb{Q}W$ called *the descent algebra* of W. We denote by $\sum(W)$ the descent algebra corresponding to W. The ascent set of w is defined by

$$\mathcal{R}(w) = \{s \in S : l(ws) > l(w)\}.$$

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For any subset I of S, put $Y_I = \{w \in W : \mathcal{R}(w) = I\}$. We consider the element $y_I = \sum_{w \in Y_I} w$ in $\mathbb{Q}W$. Then

$$x_I = \sum_{I \subseteq J} y_J$$

and by Möbius inversion formula

$$y_I = \sum_{I \subseteq J} (-1)^{|J-I|} x_J.$$

Thus the set $\{y_I : I \subseteq S\}$ is a basis of $\sum(W)$, see [7]. In [7], Solomon also defined an algebra map from $\sum(W)$ to $\mathbb{Q}\mathrm{Irr}W$ as follows:

$$\Phi : \sum(W) \to \mathbb{Q}\mathrm{Irr}(W), x_J \mapsto \mathrm{Ind}_{W_J}^W 1_{W_J},$$

where $\mathbb{Q}\mathrm{Irr}W$, 1_{W_J} and $\mathrm{Ind}_{W_J}^W 1_{W_J}$ denote the algebra generated by irreducible characters of W, the trivial character of W_J and the permutation character of W_J in W, respectively. Taking into account the sign character of W, which is defined as $\varepsilon : W \to \mathbb{N}$, $\varepsilon(w) = (-1)^{l(w)}$ (see [4]), it is well-known from [7] that $y_\emptyset = w_0 \in \sum(W)$ and

$$\Phi(w_0) = \varepsilon. \tag{1.2}$$

The sign character ε of W is actually irreducible and equals to the Steinberg character of W which is given by the formula $St_W = \sum_{J \subseteq S} (-1)^{|J|} \mathrm{Ind}_{W_J}^W 1_{W_J}$ due to [2]. In [1, Main Theorem], Blessenohl, Hohlweg, Schocker showed the symmetry property for the descent algebra $\sum(W)$, that is,

$$\Phi(x)(y) = \Phi(y)(x) \tag{1.3}$$

for all $x, y \in \sum(W)$.

2. Proof of Theorem 1.1

Proof. When we extend linearly the sign character ε of W to the group algebra $\mathbb{Q}W$ and use the equations (1.2), (1.3), then we conclude that

$$\begin{aligned}\sum_{w \in X_J} (-1)^{l(w)} &= \sum_{w \in X_J} \varepsilon(w) = \varepsilon(\sum_{w \in X_J} w) \\ &= \varepsilon(x_J) = \Phi(w_0)(x_J) = \Phi(x_J)(w_0) \\ &= Ind_{W_J}^W 1_{W_J}(w_0) = |\{w \in X_J : w_0 w W_J = w W_J\}|.\end{aligned}$$

Therefore, we obtain the equation (1.1) and so we complete a case-free algebraic proof of the Theorem 1.1. □

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