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On the Ricci Curvature of Normal-Metric Contact Pair Manifolds

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Abstract — In this study, we work on normal-metric contact pair manifolds under certain conditions related to the Ricci curvature. We obtain some results for generalized quasi-Einstein normal-metric contact pair manifolds. We prove that such manifolds are not pseudo-Ricci symmetric. Finally, we investigate Ricci solitons on normal-metric contact pair manifolds.

Keywords — *Normal-metric contact pair manifold, generalized quasi-Einstein, Ricci curvature, Ricci symmetric*

Mathematics Subject Classification (2020) — 53D10, 53C15

1. Introduction

A real contact manifold is defined by a contact form η which is a volume form on a real $(2p+1)-$dimensional differentiable manifold M. The kernel of η defines $2p-$dimensional a non-integrable distribution of TM:

$$\mathcal{D} = \{X : \eta(X) = 0,\ X \in \Gamma(TM)\}$$

We also recall $\mathcal{D}$ contact or horizontal distribution. Let ξ be a vector field on M, which is dual vector of η. Then, for $(1,1)-$tensor field ϕ, M is called an almost-contact metric manifold if following conditions are satisfied:

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2)$$

for all $X_1, X_2 \in \Gamma(TM)$, where I is identity map on TM and g is a Riemannian metric [1]. Moreover, we call g compatible metric. Similar to the Kähler manifold, we have a second fundamental form on an almost-contact metric manifold $\Omega(X_1, X_2) = d\eta(X_1, X_2)$. Fruthermore, $d\eta(X_1, X_2) = g(X_1, \phi X_2)$ and in this case we recall g is an associated metric. An almost-contact structure is normal if $N(\phi X_1, \phi X_2) + 2d\eta(X_1, X_2)\xi = 0$, where $N(\phi X_1, \phi X_2)$ is the Nijenhuis tensor field of ϕ. A normal almost-contact metric manifold is called a Sasakian manifold.

In 1959, Kobayashi [2] defined the complex analogue of a real contact manifold. Later, in the 1980s Ishihara and Konishi [3] proved that a complex contact manifold carried an almost-contact structure. A complex almost-contact metric manifold is a complex odd $(2p+1)-$dimensional complex manifold with $(J, \phi, \phi \circ J, \xi, -J \circ \xi, \eta, \eta \circ J, g)$ structure such that

$$\phi^2 = (\phi J)^2 = -I + \eta \otimes \xi - (\eta \circ J) \otimes (J \circ \xi),$$
$$\eta(\xi) = 1 \ \ \eta(-J \circ \xi) = 0, (\eta \circ J)(-J \circ \xi) = 1, \ (\eta \circ J)(\xi) = 0,$$
$$g(\phi X_1, X_2) = -g(X_1, \phi X_2), \quad g((\phi \circ J)X_1, X_2) = -g(X_1, (\phi \circ J)X_2)$$

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where g is an Hermitian metric on M, J is a natural almost complex structure. The normality of complex almost-contact metric manifolds was given by Ishihara and Konishi [3] and Korkmaz [4]. Normal complex contact metric manifolds have been studied by several authors [4–7]. A normal complex contact manifold with a globally defined holomorphic 1-form is called complex Sasakian manifolds. This type of manifolds have been worked in [8–11]

It is wel-known that an odd-dimensional sphere S^{2p+1} carries a contact structure. Calabi and Eckmann showed that the product of two odd-dimensional spheres $M = S^{2p+1} \times S^{2q+1}$ is a complex manifold [12]. These kinds of manifolds recall Calabi-Eckman manifolds. These manifolds have some significant properties in complex geometry. Blair, Ludden, and Yano [13] studied complex manifolds whose complex structures are similar to the complex structure on M. In [13], the authors defined a new structure on Hermitian manifolds called bicontact manifolds. They proved that *"A Hermitian bicontact manifold is locally the product of two normal contact manifolds M^{2p+1} and M^{2q+1}."* Hermitian bicontact manifolds were studied by Abe [14]. Abe obtained many useful results for complex manifolds by using the notion of Hermitian bicontact manifolds.

In 2005, Bande and Hadjar [15] gave the definition of a contact pair manifold, and this definition was similar to bicontact manifolds. Then, they constructed an almost-contact structure on a contact pair manifold and defined the associated metric [16]. In 2013, the normality of almost-contact metric pair structure was studied [17]. Later, some details of the normal contact metric pair (NMCP) manifolds were studied by Bande, Hadjar and Blair in [18–20]. In 2020, one of the authors [21] defined the notion of generalized quasi-Einstein normal-metric contact pair manifold and obtained some results on curvature relations. Besides, same author worked on certain flatness conditions [22] and some semi-symmetry conditions [23]. In [24], NMCP manifolds were studied under conditions of the generalized quasi-conformal curvature tensor.

In this study, we work on NMCP manifolds under certain conditions related to the Ricci curvature. We obtain some results for generalized quasi-EinsteinNMCP manifolds. We prove that such manifolds are not Ricci pseudo-symmetric. Finally, we work on the notion of Ricci solitons.

2. Preliminary

In this section, we give a brief survey on normal metric contact pair manifolds. For details see [15–17].

Definition 2.1. A differentiable manifold $M^{2p+2q+2}$ is called a contact pair manifold if we have

- $\alpha_1 \wedge (d\alpha_1)^p \wedge \alpha_2 \wedge (d\alpha_2)^q \neq 0$,
- $(d\alpha_1)^{p+1} = 0$ and $(d\alpha_2)^{q+1} = 0$.

for two 1-form α_1, α_2 [15]. We recall (α_1, α_2) as (p, q)-type contact pairs.

Two canonical examples of contact pair manifolds are given below.

Example 2.2. Let $x_1, ..., x_{2p+1}, y_1, ..., y_{2q+1}$ be the coordinate functions on $\mathbb{R}^{2p+2q+2}$. Then, two 1-form

$$\alpha_1 = dx_{2p+1} + \sum_{i=1}^{p} x_{2i-1} dx_{2i}, \quad \alpha_2 = dy_{2q+1} + \sum_{j=1}^{q} y_{2i-1} dy_{2i}$$

defines a (p, q)-type contact pairs. $(\mathbb{R}^{2p+2q+2}, \alpha_1, \alpha_2)$ is an example of contact pair manifolds.

Example 2.3. Let (M_1^{2p+1}, α_1) and $M_2^{2q+1}, \alpha_2)$ be two contact manifolds and M be the product of M_1^{2p+1} and M_2^{2q+1}. Then, (α_1, α_2) is a (p, q)-type contact pairs. $(M = M_1^{2p+1} \times M_2^{2q+1}, \alpha_1, \alpha_2)$ is called as product contact pairs.

As we know, the kernel of contact form defines a distribution which we recall *contact distribution*. For contact pairs, since we have two 1−forms α_1 and α_2, we have two integrable subbundle of TM as $\mathcal{D}_1 = \ker \alpha_1$, $\mathcal{D}_2 = \ker \alpha_2$. We can naturally associate it to the distribution of vectors on which α_1 and

$d\alpha_1$ vanish, and the one of vectors on which α_2 and $d\alpha_2$ vanish. (α_1, α_2) of Pfaffian forms of constant classes $2p+1$ and $2q+1$, whose characteristic foliations are transverse and complementary, such that α_1 and α_2 restrict to contact forms on the leaves of the characteristic foliations of α_1 and α_2, respectively. We determine $\mathcal{F}_1$ and $\mathcal{F}_2$ of these foliations. These distributions are involutive. Moreover, they are of codimension $2p+1$ and $2q+1$, respectively, and their leaves are contact manifolds [15]. This allow us to use the name of contact pairs. These two characteristic foliations of M are denoted by

$$\mathcal{F}_1 = \mathcal{D}_1 \cap kerd\alpha_1 \text{ and } \mathcal{F}_2 = \mathcal{D}_2 \cap kerd\alpha_2$$

The Reeb vector fields of contact pair (α_1, α_2) are determined by the following equations:

$$\alpha_1(Z_1) = \alpha_2(Z_2) = 1,\ \alpha_1(Z_2) = \alpha_2(Z_1) = 0$$
$$i_{Z_1}d\alpha_1 = i_{Z_1}d\alpha_2 = i_{Z_2}d\alpha_2 = 0$$

where i_X is the contraction with the vector field X.

Let's define two subbundle of TM by

$$T\mathcal{G}_i = kerd\alpha_i \cap ker\alpha_1 \cap ker\alpha_2,\ i = 1, 2$$

then we can write

$$T\mathcal{F}_i = T\mathcal{G}_i \oplus \mathbb{R}Z_1$$

and so

$$TM = T\mathcal{G}_1 \oplus T\mathcal{G}_2 \oplus \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$$

Thus, the horizontal and vertical subbundles are defined by $\mathcal{H} = T\mathcal{G}_1 \oplus T\mathcal{G}_2$ and $\mathcal{V} = \mathbb{R}Z_1 \oplus \mathbb{R}Z_2$, respectively. Finally, we have $TM = \mathcal{H} \oplus \mathcal{V}$ [16].

Any $X \in \Gamma(TM)$ could be written as $X = X^{\mathcal{H}} + X^{\mathcal{V}}$, where $X^{\mathcal{H}} \in \mathcal{H},\ X^{\mathcal{V}} \in \mathcal{V}$. In another way, we can write $X = X^1 + X^2$ for $X^1 \in T\mathcal{F}_1$ and $X^2 \in T\mathcal{F}_2$. Furthermore, we can state $X^1 = X^{1^h} + \alpha_2(X^1)Z_2$ and $X^2 = X^{2^h} + \alpha_1(X^2)Z_1$, where X^{1^h} and X^{2^h} are horizontal parts of X^1 and X^2, respectively. From all these decomposition of X finally we get

$$X = X^{1^h} + X^{2^h} + \alpha_1(X^2)Z_1 + \alpha_2(X^1)Z_2$$
$$\alpha_1(X^{1^h}) = \alpha_1(X^{2^h}) = 0, \quad \alpha_2(X^{1^h}) = \alpha_2(X^{2^h}) = 0$$

Let's define $(1,1)-$tensor field ϕ such as

$$\phi^2 = -I + \alpha_1 \otimes Z_1 + \alpha_2 \otimes Z_2, \quad \phi Z_1 = \phi Z_2 = 0,\ \alpha_1(\phi) = \alpha_2(\phi) = 0$$

If $\phi T\mathcal{F}_i = T\mathcal{F}_i$, then ϕ is said to be decomposable, i.e $\phi = \phi_1 + \phi_2$. With the decomposability of ϕ, we have that (α_1, Z_1, ϕ_1) (resp. (α_2, Z_2, ϕ_2)) induces an almost-contact structure on the leaves of $\mathcal{F}_2$ (resp.$\mathcal{F}_1$) [16]. Throughout this study, it is assume that ϕ is decomposable. We recall $(\phi_1, \phi_2, g, \alpha_2, Z_2, \phi_2)$ the contact pair structure

A Riemannian metric g on $(M, \phi, Z_1, Z_2, \alpha_1, \alpha_2)$ is called compatible if $g(\phi X_1, \phi X_2) = g(X_1, X_2) - \alpha_1(X_1)\alpha_1(X_2) - \alpha_2(X_1)\alpha_2(X_2)$ for all $X_1, X_2 \in TM$, and associated if $g(X_1, \phi X_2) = (d\alpha_1 + d\alpha_2)(X_1, X_2)$ and $g(X_1, Z_i) = \alpha_i(X_1)$, for $i = 1, 2$. 4-tuple $(\alpha_1, \alpha_2, \phi, g)$ is called metric contact pair structure on M.

Normality of almost-contact structure is an important notion in contact geometry. As we know a normal contact metric manifold is called as Sasakian manifold. A Sasakian manifold can be seen as odd-dimensional Kähler manifolds. Similarly, we have many subclasses of complex contact manifolds which are normal. A complex Sasakian manifold is also a normal complex contact metric manifold [11]. The normality of a MCP manifold was studied in [17]. We have two almost complex structures:

$$\mathcal{J} = \phi - \alpha_2 \otimes Z_1 + \alpha_1 \otimes Z_2, \quad \mathcal{T} = \phi + \alpha_2 \otimes Z_1 - \alpha_1 \otimes Z_2$$

$\mathcal{J}$ and $\mathcal{T}$ are called almost complex structure associated $(\alpha_1, \alpha_2, \phi)$. If $\mathcal{J}$ and $\mathcal{T}$ are integrable, then M is normal. On the other hand, the integrability of $\mathcal{J}$ and $\mathcal{T}$ is determined by the following condition

$$[\phi, \phi](X_1, X_2) + 2d\alpha_1(X_1, X_2)Z_1 + 2d\alpha_2(X_1, X_2)Z_2 = 0,$$

for all $X_1, X_2 \in \Gamma(TM)$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ [17]. For the sake of brevity, we use the abbreviation of NMCP instead of the term of *normal metric contact pair*.

The curvature properties of a NMCP manifold are given by

$$\begin{aligned} R(X_1, Z)X_2 &= -g(\phi X_1, \phi X_2)Z, \\ R(X_1, X_2, Z, X_3) &= d\alpha_1(\phi X_3, X_1)\alpha_1(X_2) + d\alpha_2(\phi X_3 X_1)\alpha_2(X_2) \\ &\quad - d\alpha_1(\phi X_3, X_2)\alpha_1(X_1) - d\alpha_2(\phi X_3, X_2)\alpha_2(X_1) \\ R(X_1, Z)Z &= -\phi^2 X_1 \end{aligned}$$

for $X_1, X_2, X_3 \in \Gamma(TM)$ and $Z = Z_1 + Z_2$ for the Reeb vector fields Z_1, Z_2, R is the Riemannian curvature tensor [18]. Moreover, the Ricci curvature of M has the following properties [18];

$$Ric(X_1, Z) = 0, \quad \text{for } X_1 \in \Gamma(\mathcal{H}) \tag{1}$$

$$Ric(Z, Z) = 2p + 2q. \tag{2}$$

$$Ric(Z_1, Z_1) = 2p, \ Ric(Z_2, Z_2) = 2q, \ Ric(Z_1, Z_2) = 0 \tag{3}$$

Definition 2.4. An NMCP manifold is called a generalized quasi-Einstein (GQE) manifold if the Ricci curvature of M has the following form:

$$Ric(X_1, X_2) = \lambda g(X_1, X_2) + \beta\alpha_1(X_1)\alpha_1(X_2) + \gamma\alpha_2(X_1)\alpha_2(X_2)$$

where λ, β, and γ are scalar fields on M and $X_1, X_2 \in \Gamma(TM)$ [21].

Thus, from (2) and (3) we have

$$Ric(X_1, X_2) = \lambda g(X_1, X_2) + (2p - \lambda)\alpha_1(X_1)\alpha_1(X_2) + (2q - \lambda)\alpha_2(X_1)\alpha_2(X_2)$$

for all $X_1, X_2 \in \Gamma(TM)$.

3. Certain Conditions on the Ricci Curvature of Nomral-metric Contact Pair Manifolds

Ricci curvature Ric, which is defined as the trace of Riemannian curvature tensor, has a major role in the Riemannian geometry. In this section, we work on NMCP manifolds with certain conditions related to the Ricci curvature.

We recall a Riemannian manifold as flat if it has zero curvature. Furthermore, a Riemannian manifold is said to be Ricci-flat if $Ric = 0$.

Theorem 3.1. An NMCP manifold could not be Ricci-flat.

Proof. Let M be an NMCP manifold. Suppose that it is Ricci-flat, i.e for every X_1, X_2 vector fields we have $Ric(X_1, X_2) = 0$. Then, from (2) we get $2p + 2q = 0$, which is impossible. Thus, there is a contradiction. The manifold could not be Ricci-flat. □

An normal-metric contact pair manifoldmanifold is Ricci symmetric if $\nabla Ric = 0$. Let M be a GQE NMCP manifold with constant λ. From the Riemannian geometry we have following wel-known relation;

$$(\nabla_X Ric)(X_1, X_2) = \nabla_X Ric(X_1, X_2) - Ric(\nabla_X X_1, X_2) - Ric(X_1, \nabla_X X_2)$$

for all $X, X_1, X_2 \in \Gamma(TM)$. Then, using (2) we obtain

$$(\nabla_X Ric)(X_1, X_2) = (2p - \lambda)g(\phi_1 X, X_1)\alpha_1(X_2) + (2q - \lambda)g(\phi_2 X, X_2)\alpha_1(X_1)$$

If X_1 and X_2 are horizontal vector fields, we get $(\nabla_X Ric)(X_1, X_2) = 0$.

Corollary 3.2. On the horizontal bundle of a GQE NMCP manifold with constant λ, $\nabla Ric = 0$.

If we take $X_1 = a_1 Z_1 + a_2 Z_2, X_2 = b_1 Z_1 + b_2 Z_2$ for coefficients $a_i, b_i,\ i = 1,2$ since $g(\phi_1 X, X_1) = -g(X, \phi_1 X_1)$ and $g(\phi_2 X, X_2) = -g(X, \phi_2 X_2)$, we get $(\nabla_X Ric)(X_1, X_2) = 0$.

Corollary 3.3. On the vertical bundle of a GQE NMCP manifold with constant λ, $\nabla Ric = 0$.

Let $X_1 = X_1^{1^h} + X_1^{2^h} + \alpha_1(X_1^2)Z_1 + \alpha_2(X_1^1)Z_2$ and $X_2 = X_2^{1^h} + X_2^{2^h} + \alpha_1(X_2^2)Z_1 + \alpha_2(X_2^1)Z_2$. Then, we obtain

$$(2p - \lambda)g(\phi_1 X, X_1^{1^h} + X_1^{2^h})\alpha_1(X_1^2) - (2q - \lambda)g(\phi_2 X, X_2^{1^h} + X_2^{2^h})\alpha_1(X_2^1) = 0$$

Thus, we state the following theorem.

Proposition 3.4. On a GQE NMCP manifold with constant λ, $(\nabla_X Ric)(X_1, X_2) = 0$ if and only if $(2p - \lambda)g(\phi_1 X, X_1^{1^h} + X_1^{2^h})\alpha_1(X_1) - (2q - \lambda)g(\phi_2 X, X_2^{1^h} + X_2^{2^h})\alpha_1(X_2) = 0$ for all $X_1 = X_1^{1^h} + X_1^{2^h} + \alpha_1(X_1^2)Z_1 + \alpha_2(X_1^1)Z_2, X_2 = X_2^{1^h} + X_2^{2^h} + \alpha_1(X_2^2)Z_1 + \alpha_2(X_2^1)Z_2$ and $X \in \Gamma(TM)$.

An NMCP manifold M satisfies cyclic parallel Ricci tensor if we have

$$(\nabla_{X_1} Ric)(X_2, X_3) + (\nabla_{X_2} Ric)(X_3, X_1) + (\nabla_{X_3} Ric)(X_1, X_2) = 0$$

for all $X_1, X_2, X_3 \in \Gamma(TM)$. M is also satisfies Codazzi type of Ricci tensor if we have

$$(\nabla_{X_1} Ric)(X_2, X_3) - (\nabla_{X_2} Ric)(X_1, X_3) = 0$$

for all X_1 and X_2 vector fields on M. In [21], one of the presented authors proved the following results.

Theorem 3.5. A GQE NMCP manifold with constant λ satisfies cyclic parallel Ricci tensor [21].

Theorem 3.6. A GQE NMCP manifold with constant λ does not satisfy Codazzi type of Ricci tensor [21].

In [25], the authors proved that *if the generators of GQE manifolds are Killing then the manifold satisfies cyclic parallel Ricci tensor.* Since Z_1 and Z_2 are Killing, Theorem 3.5 is compatible with this result. The same authors proved that *if a GQE manifold is Codazzi type of Ricci tensor then the integral curves of the generator vector fields are geodesic.* It is known that Z_1 and Z_2 are geodesics. But the manifold is not the Codazzi type of Ricci tensor. Theorem 3.6 is guaranteed that the converse of the second result in [25] is not satisfied.

In [26], the authors proved that *in a GQE manifold, if the associated scalars are constant and the Ricci tensor is of Codazzi type, then the associated* 1*-form are closed.* As we know, α_1 and α_2 are not closed. Thus, Theorem 3.6 is compatible with this result.

A generalization of Ricci symmetry was pointed out by the name of Ricci semi-symmetry. If $R \cdot Ric = 0$ we recall the manifold as Ricci semi-symmetric manifold. In [23], we proved following theorem

Theorem 3.7. A Ricci semi-symmetric NMCP manifold is a GQE manifold [23].

A non-flat NMCP manifold M is called a Chaki pseudo-Ricci symmetric manifold if the Ricci tensor Ric of type $(0,2)$ is non-zero and satisfies the condition

$$(\nabla_X Ric)(X_1, X_2) = 2A(X)Ric(X_1, X_2) + A(X_2)Ric(X, X_3) + A(X_3)Ric(X_2, X)$$

where A is non-zero $1-$form such that $g(X, \rho) = A(X)$ for all vector fields X; ρ being the vector field corresponding to the associated 1-form [27]. If $A = 0$, then the manifold is called Ricci symmetric.

The another study on GQE manifolds was presented in [28], the authors proved that *a pseudo-Ricci symmetric manifold cannot cyclic parallel Ricci tensor; otherwise, this manifold reduces to a Ricci symmetric manifolds.* Thus, with the considered Theorem 3.5, we can state,

Theorem 3.8. A GQE NMCP manifold with constant λ cannot be pseudo-Ricci symmetric.

A Riemannian manifold (M, g) is called a *Ricci soliton* if there is a smooth vector field V and a scalar $\nu \in \mathbb{R}$ such that

$$\mathcal{L}_V g + 2Ric = 2\nu g \tag{4}$$

on M, where Ric is the Ricci tensor and $\mathcal{L}_V g$ is the Lie derivative of the metric g. The Ricci soliton is called *shriking, steady, or expanding* according to $\nu < 0$, $\nu = 0$, $\nu > 0$, respectively [29]. Contact manifolds have been studied as the solution of Ricci soliton equations. For different structures, see [30–33]

Suppose that a NMCP manifold satisfies (4) with the potential vector fields Z_1 and Z_2. Since Z_1 and Z_2 are the Killing vector fields, we get

$$Ric(X_1, X_2) = \nu g(X_1, X_2)$$

for all $X_1, X_2 \in \Gamma(TM)$. Thus, M is Einstein manifold. Moreover, from (2), we get $\nu = 2p + 2q$. Thus, we state the following result:

Theorem 3.9. Let a NMCP manifold satisfy the Ricci soliton equation with the potential vector fields Z_1 (and Z_2). Then, the Ricci soliton is expanding.

Let M be a GQE NMCP manifold which satisfies the Ricci soliton equation. Thus, from (2), we obtain

$$(\mathcal{L}_V g)(X_1, X_2) = -(2\nu + 2\lambda)g(X_1, X_2) - 2(2p - \lambda)\alpha_1(X_1)\alpha_1(X_2) - 2(2q - \lambda)\alpha_2(X_1)\alpha_2(X_2) \tag{5}$$

Corollary 3.10. Let M be a GQE NMCP manifold which satisfies the Ricci soliton equation. The potential vector field is Killing if and only if $g(X_1, X_2) = \frac{2(2p-\lambda)}{(2\nu+2\lambda)}\alpha_1(X_1)\alpha_1(X_2) + \frac{2(2q-\lambda)}{(2\nu+2\lambda)}\alpha_2(X_1)\alpha_2(X_2)$ for all $X_1, X_2 \in \Gamma(TM)$.

Let $V = Z_1$ in (5), then we have

$$0 = -(2\nu + 2\lambda)g(X_1, X_2) - 2(2p - \lambda)\alpha_1(X_1)\alpha_1(X_2) - 2(2q - \lambda)\alpha_2(X_1)\alpha_2(X_2)$$

Choose $X_1 = X_2 = Z$, then we get $\nu = p + q$. Thus, we state,

Theorem 3.11. Let M be a GQE NMCP manifold, which satisfies the Ricci soliton equation. If $V = Z_1(or Z_2)$, then the Ricci soliton is expanding with $\nu = p + q$.

There are many interesting vector fields (sometimes called collineations), considered infinitesimal symmetries of geometric structure or physical quantities such as metric, curvature, energy-momentum tensors, geodesics, and light cones. These vector fields have many applications in Riemannian geometry and general relativity. One of them is Killing vectors, which are named after a Norwegian mathematician Killing, who first described these notions in 1892. The Killing vectors preserve the metric and all the derived structures. Another type is the conformal Killing vector field. A vector field X recall conformal Killing if its Lie derivative is proportional to itself $\mathcal{L}_X g = 2\mu g$, for some scalar field μ. If μ is zero, X is the Killing vector fields and if μ is constant, but not zero, the vector field is said to be homothetic, and the metric is changed by a (constant) scale factor as it moves along.

Suppose that V is conformal Killing in (5). Then, we obtain

$$-2(\nu + \lambda + \mu)g(X_1, X_2) = 2(2p - \lambda)\alpha_1(X_1)\alpha_1(X_2) + 2(2q - \lambda)\alpha_2(X_1)\alpha_2(X_2)$$

By taking $X_1 = X_2 = Z$, we get $-4(\nu + \lambda + \mu) = 4(p + q) - 4\lambda$ and so $\mu + \nu = -(p + q)$. Since ν is constant, we state the following result.

Theorem 3.12. Let M be a GQE NMCP manifold which satisfies the Ricci soliton equation. If the potential vector field is conformal Killing, then it reduces to the homothetic vector field.

4. Conclusion

Normal-metric contact pair manifolds are an important class of contact manifolds. These manifolds have many significant properties that differ from the classical contact structures. Moreover, a normal contact metric pair manifold could be a special solution to Einstein's field equations. Furthermore, we have the applications of GQE manifolds in the contact geometry thanks to normal-metric contact pair manifolds. In this paper, we study normal-metric contact pair manifolds from the Riemannian geometric perspective. We obtain some results on the Ricci curvature and the Ricci solitons. The results of the paper will be a reference for future works on contact manifolds and general relativity.

Conflicts of Interest

The authors declare no conflict of interest.

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