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Research Article

Universal central extensions of braided crossed modules of Lie algebras

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Abstract

In this paper, we give a natural braiding on the universal central extension of a Lie crossed module with a given braiding in the category of Lie crossed modules. We also construct the universal central extension of a braided Lie crossed module in the category of braided Lie crossed modules, showing that, when one of these constructions exists, both of them exist and coincide.

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1. Introduction

The concept of central extension of groups or Lie algebras is highly relevant in mathematics, and it plays a fundamental role in several areas of physics as well. This notion was extended to crossed modules of groups or Lie algebras. The study of central extensions in the categories of crossed modules was initiated in [13] for groups and in [2] for Lie algebras, and it remains a current research topic, as shown by the different literature tackling this issue.

Crossed modules of groups (Lie algebras) are algebraic objects equivalent to strict 2-groups, or equivalently categorical groups (strict 2-Lie algebras or categorical Lie algebras), and also simplicial objects with associated Moore complex of length 1. Since crossed modules of groups and Lie algebras are a generalisation of groups and Lie algebras, it is natural to search, in the category of crossed modules of groups or Lie algebras, extensions of classical results in the theory of groups or Lie algebras.

Joyal and Street defined in [12] the concept of braiding for monoidal categories as a natural isomorphism $\tau_{A,B}\colon A\otimes B \to B\otimes A$, generalising the idea of the usual tensor product of vector spaces. The notion of braiding for categorical groups provides an equivalent category to the category of braided crossed modules of groups (see [4, 12]).

The concept of braiding in Lie crossed modules was introduced in [14] (see also [6]). In [6], it is proved that the category of categorical Lie algebras is equivalent to the category of braided Lie crossed modules.

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In [8], Fukushi gave a braided version of the results on universal central extensions of crossed modules of groups provided by Norrie in [13]. He found a natural braiding on the universal central extension of a crossed module of groups which behaves well with one braided crossed module. However, it is not the archetype of universal central extension in the category of braided crossed modules since, in this category, it is necessary to add additional restrictions including the braiding on the notions of centre and commutator.

In this paper, we will devise a braided version of the results given by Casas and Ladra in [2] for braided crossed modules of Lie K-algebras; more precisely, we will study universal central extensions in the category of braided Lie crossed modules $\boldsymbol{BXLie}$. For that purpose, we will need the definition of centre and commutator given by Huq in [9] in the braided context.

This text is organised as follows. In the first section, we provide some definitions such as braiding, central extensions in the category of Lie crossed modules $\boldsymbol{XLie}$, $\boldsymbol{B}$-central extensions in $\boldsymbol{BXLie}$, and the non-abelian tensor product of Lie algebras, necessary for developing the work. In Section 3, we construct the universal $\boldsymbol{B}$-central extension for a $\boldsymbol{B}$-perfect braided Lie crossed module and prove that a braided Lie crossed module admits a universal $\boldsymbol{B}$-central extension if and only if it is $\boldsymbol{B}$-perfect. In Section 4, we construct the universal $\mathfrak{U}$-central extension for a perfect braided Lie crossed module, where $\mathfrak{U}\colon \boldsymbol{BXLie} \longrightarrow \boldsymbol{XLie}$ is the forgetful functor. In Section 5, we study the relation between the universal $\boldsymbol{B}$-central extension and the universal $\mathfrak{U}$-central extension of a braided Lie crossed module. Finally, we prove that both universal extensions exist and coincide for a $\boldsymbol{B}$-perfect braided Lie crossed module.

Note that the framework of this paper is different from that given in [3], since the category $\boldsymbol{XLie}$ is not a Birkhoff subcategory of $\boldsymbol{BXLie}$.

2. Preliminaries

Throughout this paper, we will suppose that K is a field.

Definition 2.1. A *Lie crossed module* is a pair $(M \xrightarrow{\partial} N, \cdot)$ where:

- M and N are Lie algebras, together with a Lie left-action $\cdot$ of N on M, i.e. a K-bilinear map $\cdot\colon N \times M \to M$, $(n, m) \mapsto n \cdot m$, satisfying

$$[n, n'] \cdot m = n \cdot (n' \cdot m) - n' \cdot (n \cdot m),$$
$$n \cdot [m, m'] = [n \cdot m, m'] + [m, n \cdot m'], \qquad m, m' \in M,\ n, n' \in N;$$

- $\partial\colon M \to N$ is a Lie K-homomorphism that satisfies the following properties:
  - ∂ is an N-equivariant Lie K-homomorphism:

$$\partial(n \cdot m) = \mathrm{ad}(n)(\partial(m)) = [n, \partial(m)],$$

  - ∂ satisfies the Peiffer identity:

$$\partial(m) \cdot m' = \mathrm{ad}(m)(m') = [m, m'].$$

Example 2.2.

(1) The identity map $M \xrightarrow{\mathrm{Id}_M} M$ with the adjoint action, $m \cdot m' = [m, m']$, is a Lie crossed module.

(2) Any central extension of Lie algebras $M \overset{\partial}{\twoheadrightarrow} N$ is a crossed module, with the action $\partial(m) \cdot m' = [m, m']$. Conversely, a simply connected crossed module (i.e. ∂ is surjective) is a central extension.

In particular, $M \overset{\mathrm{ad}}{\twoheadrightarrow} \mathrm{IDer}(M)$, $m \mapsto \mathrm{ad}(m)$, with the action, $\mathrm{ad}(m) \cdot m' = [m, m']$, is a Lie crossed module, where $\mathrm{IDer}(M)$ are the inner derivations of a Lie algebra M.

Definition 2.3. Let $(M \xrightarrow{\partial} N, \cdot)$ and $(L \xrightarrow{\delta} H, *)$ be two Lie crossed modules. A *morphism* of Lie crossed modules is a pair of Lie K-homomorphisms (f_1, f_2), $f_1 \colon M \to L$ and $f_2 \colon N \to H$, such that:

$$f_1(n \cdot m) = f_2(n) * f_1(m), \quad \text{for all} \quad m \in M, n \in N, \tag{XLieH1}$$
$$\delta \circ f_1 = f_2 \circ \partial. \tag{XLieH2}$$

The category of Lie crossed modules is a semi-abelian category in the sense of [11], and it will be denoted by ***XLie***.

The notion of the centre of an object was defined in [9], in a category with specific properties. This construction only needs that the category has finite products and zero object.

The category ***XLie*** has centres in the sense of Huq [9], and they were constructed in [2].

Definition 2.4. The *centre* of a Lie crossed module $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot)$ is the crossed submodule $Z(\mathcal{M}) = (M^N \xrightarrow{\partial|_{M^N}} \mathrm{st}_N(M) \cap Z(N), \cdot_Z)$, where:

- $M^N = \{m \in M \mid n \cdot m = 0,\ n \in N\}$,
- $Z(N) = \{n \in N \mid [n, n'] = 0,\ n' \in N\}$ is the centre of the Lie K-algebra N,
- $\mathrm{st}_N(M) = \{n \in N \mid n \cdot m = 0,\ m \in M\}$,

and $\cdot_Z$ is the induced action, which means that it is the zero action by the definition of M^N.

The notions of commutator of a Lie crossed module and a perfect Lie crossed module were introduced in [2]. This notion of commutator coincides in the category ***XLie*** with the idea of commutator given by Huq in [9] in a category with products, zero objects, kernels and cokernels.

If L is a Lie K-algebra and $S \subset L$, we denote $\langle S \rangle_L$ the Lie subalgebra of L generated by S, that is, the intersection of all subalgebras containing S.

Definition 2.5. Let $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot)$ be a Lie crossed module. The *commutator crossed submodule* is $[\mathcal{M}, \mathcal{M}] = (D_N(M) \xrightarrow{\partial|_{D_N(M)}} [N, N], \cdot_C)$, where $\cdot_C$ is the induced action, and

- $D_N(M) = \langle \{n \cdot m \mid n \in N,\ m \in M\} \rangle_M$,
- $[N, N] = \langle \{[n, n'] \mid n, n' \in N\} \rangle_N$ is the commutator of the Lie K-algebra N.

Definition 2.6. We will say that a Lie crossed module $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot)$ is *perfect* if it coincides with its commutator crossed submodule $\mathcal{M} = [\mathcal{M}, \mathcal{M}]$, i.e. $M = D_N(M)$ and $N = [N, N]$.

Definition 2.7. An *extension* in ***XLie*** is a regular epimorphism, i.e. a surjective morphism.

Following the theory in [10], we have three kinds of extensions: trivial, normal and central.

An extension $\Phi \colon \mathcal{X} \twoheadrightarrow \mathcal{M}$ is *trivial* if the induced square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathcal{X}_{\mathrm{ab}} & \xrightarrow[\Phi_{ab}]{} & \mathcal{M}_{\mathrm{ab}} \end{array}$$

is a pullback in ***XLie***, where $\mathcal{M}_{\mathrm{ab}} = \frac{\mathcal{M}}{[\mathcal{M}, \mathcal{M}]}$.

An extension is *normal* if one of the projections of the kernel pair is trivial.

An extension $\Phi\colon \mathcal{X} \twoheadrightarrow \mathcal{M}$ is *central* if there exists another extension $\Psi\colon \mathcal{Y} \twoheadrightarrow \mathcal{M}$ such that π_2 (also denoted $\Psi^*(\Phi)$) in the pullback

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{M}} \mathcal{Y} & \xrightarrow{\pi_1} & \mathcal{X} \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \Phi} \\ \mathcal{Y} & \xrightarrow[\Psi]{} & \mathcal{M}, \end{array}$$

is trivial.

In our semi-abelian context, the concepts of normal and central extension are equivalent, and a more practical characterization is the following:

An extension $\mathcal{X} = (X \xrightarrow{\delta} S, *) \xrightarrow{f=(f_1,f_2)} \mathcal{M} = (M \xrightarrow{\partial} N, \cdot)$ is central if and only if $\ker(f) = (\ker(f_1) \xrightarrow{\delta|_{\ker(f_1)}} \ker(f_2), *_{\ker})$ is a crossed submodule of the centre of $\mathcal{X}$, $Z(\mathcal{X})$.

The central extensions in ***XLie*** of an object $\mathcal{M}$ constitute another category, whose morphisms are the morphisms of Lie crossed modules $\Theta\colon \mathcal{X} \longrightarrow \mathcal{Y}$ making commutative the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Theta} & \mathcal{Y} \\ & {\scriptstyle \Phi}\searrow \quad \swarrow{\scriptstyle \Psi} & \\ & \mathcal{M}. & \end{array}$$

A central extension $\mathcal{U} \twoheadrightarrow \mathcal{M}$ is said to be *universal* (of $\mathcal{M}$) if it is the initial object in the category of central extensions of $\mathcal{M}$. From the definition, it is clear that the universal central extension is unique up to isomorphisms.

The universal central extension is entirely related to the concept of the non-abelian tensor product. In the case of groups, Brown and Loday in [1] defined the non-abelian tensor product of groups and proved that the universal central extension is the non-abelian tensor product $G \otimes G$ with the epimorphism $G \otimes G \twoheadrightarrow G$ sending $g_1 \otimes g_2$ to its commutator $[g_1, g_2]$. The same happens in Lie algebras' case with the non-abelian tensor product of Lie algebras introduced by Ellis in [5].

In the cases of crossed modules of groups [13] and Lie crossed modules [2], the notions of the non-abelian tensor products are also needed.

For the case of non-abelian tensor product of Lie K-algebras, the definition introduced by Ellis in [5] is the following one:

Definition 2.8. Let M and N be two Lie K-algebras such that M acts on N by $\cdot$ and N acts on M with $*$.

The *non-abelian tensor product*, denoted by $M \otimes N$, is defined as the Lie K-algebra generated by the symbols $m \otimes n$ with $m \in M$, $n \in N$ and the relations:

$$\lambda(m \otimes n) = \lambda m \otimes n = m \otimes \lambda n, \tag{T1}$$

$$(m + m') \otimes n = m \otimes n + m' \otimes n, \tag{T2}$$
$$m \otimes (n + n') = m \otimes n + m \otimes n',$$

$$[m, m'] \otimes n = m \otimes (m' \cdot n) - m' \otimes (m \cdot n), \tag{T3}$$
$$m \otimes [n, n'] = (n' * m) \otimes n - (n * m) \otimes n',$$

$$[m \otimes n, m' \otimes n'] = -(n * m) \otimes (m' \cdot n'), \tag{T4}$$

where $m, m' \in M$, $n, n' \in N$, $\lambda \in K$.

When we talk about $M \otimes M$ we will assume that M acts on itself by the adjoint action.

Now we will consider braidings in Lie crossed modules, whose definitions were introduced in [6, 7, 14] (the notions of [6] and [14] coincide for a field K with $\mathrm{char}(K) \neq 2$).

Definition 2.9. A *braiding* (or *Peiffer lifting*) on the Lie crossed module $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot)$ is a K-bilinear map $\{-,-\}\colon N \times N \to M$ satisfying:

$$\partial\{n, n'\} = [n, n'], \tag{BLie1}$$

$$\{\partial m, \partial m'\} = [m, m'], \tag{BLie2}$$

$$\{\partial m, n\} = -n \cdot m, \tag{BLie3}$$

$$\{n, \partial m\} = n \cdot m, \tag{BLie4}$$

$$\{n, [n', n'']\} = \{[n, n'], n''\} - \{[n, n''], n'\}, \tag{BLie5}$$

$$\{[n, n'], n''\} = \{n, [n', n'']\} - \{n', [n, n'']\}, \tag{BLie6}$$

for $m, m' \in M$, $n, n', n'' \in N$.

If $\{-,-\}$ is a braiding on $\mathcal{M}$ we will say that $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is a *braided Lie crossed module.*

Example 2.10.

(1) There is a canonical braiding on $(M \xrightarrow{\mathrm{Id}_M} M, [-,-])$, given by $\{m, m'\} = [m, m']$.

(2) There is a canonical braiding on the crossed module $(M \otimes M \xrightarrow{\partial} M, \cdot)$, with $\partial\colon m \otimes m' \mapsto [m, m']$, $m \cdot (m_1 \otimes m_2) = m \otimes [m_1, m_2]$. It is given by $\{m, m'\} = m \otimes m'$.

(3) Let $(M \overset{\partial}{\twoheadrightarrow} N, \cdot)$ be a simply connected Lie crossed module. There is a canonical braiding on $(M \overset{\partial}{\twoheadrightarrow} N, \cdot)$, given by $\{\partial(m), \partial(m')\} = [m, m']$.
In particular, $M \overset{\mathrm{ad}}{\twoheadrightarrow} \mathrm{IDer}(M)$, with the braiding $\{\mathrm{ad}(m), \mathrm{ad}(m')\} = [m, m']$, is a braided crossed module.

Definition 2.11. A *morphism of braided Lie crossed modules*

$$f = (f_1, f_2)\colon (M \xrightarrow{\partial} N, \cdot, \{-,-\}) \longrightarrow (M' \xrightarrow{\partial'} N', \cdot', \{\!\{-,-\}\!\})$$

is a homomorphism of Lie crossed modules that preserves the braiding, i.e.

$$f_1(\{n, n'\}) = \{\!\{f_2(n), f_2(n')\}\!\}, \qquad \text{for} \quad n, n' \in N. \tag{BXLieH3}$$

We will denote the category of braided Lie crossed modules by ***BXLie***. We have a faithful forgetful functor $\mathfrak{U}$: ***BXLie*** $\to$ ***XLie***.

In the case of the braiding category ***BXLie***, the idea of braiding changes a little the concepts of centre and commutator from the category of Lie crossed modules ***XLie***, appearing the following subobjects using the definition given by Huq [9] in the general case.

Definition 2.12. The ***B****-centre* of a braided Lie crossed module $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is the braided crossed submodule $Z_{\boldsymbol{B}}(\mathcal{M}) = (M^N \xrightarrow{\partial|_{M^N}} \mathrm{Z}_B(N), \cdot_Z, \{-,-\}_Z)$, where

$$\mathrm{Z}_B(N) = \{n \in N \mid \{n, n'\} = 0 = \{n', n\},\ n' \in N\},$$

$\cdot_Z$ is the induced action and $\{-,-\}_Z$ is the induced braided, i.e. the zero action and the zero braiding by the definition of M^N and $\mathrm{Z}_B(N)$.

The ***B***-centre is the centre [9] in the category ***BXLie***.

Remark 2.13. It is easy to show that the following inclusions of subalgebras are true:

$$M^N \subset Z(M), \qquad Z_B(N) \subset Z(N) \cap \mathrm{st}_N(M).$$

Besides, if we use the properties (BLie3) and (BLie4), then we have that $M^N = \{m \in M \mid \partial(m) \in Z_B(N)\}$.

Definition 2.14. Let $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a braided Lie crossed module. The ***B****-commutator braided crossed submodule* is given by

$$[\mathcal{M}, \mathcal{M}]_{\boldsymbol{B}} = (B_N(M) \xrightarrow{\partial|_{B_N(M)}} [N, N], \cdot_C, \{-,-\}_C),$$

where $\cdot_C$ and $\{-,-\}_C$ are the induced operations, and $B_N(M) = \langle\{\{n,n'\} \mid n,n' \in N\}\rangle_M$.

The $\boldsymbol{B}$-commutator is the commutator [9] in the category $\boldsymbol{BXLie}$, and therefore it is a crossed ideal (see [2]).

Remark 2.15. $B_N(M)$ is an ideal of M, and we have the following inclusion of subalgebras:

$$[M,M] \subset D_N(M) \subset B_N(M).$$

Definition 2.16. We will say that a braided Lie crossed module Lie $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is *$\boldsymbol{B}$-perfect* if it coincides with its $\boldsymbol{B}$-commutator braided crossed submodule $\mathcal{M} = [\mathcal{M},\mathcal{M}]_{\boldsymbol{B}}$, i.e. $M = B_N(M)$ and $N = [N,N]$.

Definition 2.17. An *extension of braided Lie crossed modules* is a morphism $\mathcal{X} \xrightarrow{(f_1,f_2)} \mathcal{Y}$ in $\boldsymbol{BXLie}$ such that f_1 and f_2 are surjective morphisms.

Besides, we will say that it is *$\boldsymbol{B}$-central* (central extension in the category $\boldsymbol{BXLie}$) if $\ker(f_1,f_2)$ is a braided crossed submodule of $Z_{\boldsymbol{B}}(\mathcal{X})$, i.e. the kernel is "inside" the $\boldsymbol{B}$-centre.

Definition 2.18. We will say that an extension $\mathcal{X} \xrightarrow{(f_1,f_2)} \mathcal{Y}$ in $\boldsymbol{BXLie}$ of braided Lie crossed modules is a *$\mathfrak{U}$-central extension* if $\mathfrak{U}(\mathcal{X}) \xrightarrow{\mathfrak{U}(f_1,f_2)} \mathfrak{U}(\mathcal{Y})$ is central in $\boldsymbol{XLie}$, i.e. $\ker\big(\mathfrak{U}(f_1,f_2)\big)$ is a crossed submodule of the centre of $\mathfrak{U}(\mathcal{X})$, $Z\big(\mathfrak{U}(\mathcal{X})\big)$.

It is immediate that every $\boldsymbol{B}$-central extension in the category $\boldsymbol{BXLie}$ is a $\mathfrak{U}$-central extension. The next example shows that not every $\mathfrak{U}$-central extension is a $\boldsymbol{B}$-central extension. Furthermore, it manifests that the concepts of $\boldsymbol{B}$-centre and $\boldsymbol{B}$-commutator of a braided crossed module are different from the notions of centre and commutator.

Example 2.19. Let $M \neq 0$ be an abelian K-Lie algebra of finite dimension n, i.e. M is isomorphic to K^n.

Using Example 2.10 (2), we have that $(M \otimes M \xrightarrow{\partial} M, \cdot)$, with $\partial = 0$, $m \cdot (m_1 \otimes m_2) = m \otimes [m_1,m_2] = m \otimes 0 = 0$ and $\{m,m'\} = m \otimes m'$ is a braided Lie crossed module.

Note that, since M is abelian, we have that $M \otimes M$ is isomorphic to the usual tensor product as vector spaces.

(i) Let $\mathcal{X} = (M \otimes M \xrightarrow{0} M, 0, - \otimes -)$ be the braided Lie crossed module, where the tensor product is the usual one. It is easy to show that the correspondent subalgebras are the following ones:

- $(M \otimes M)^M = M \otimes M$, $\mathrm{st}_M(M \otimes M) = M$, $Z(M) = M$, and $Z_B(M) = 0$.
- $D_M(M \otimes M) = 0$, $B_M(M \otimes M) = M \otimes M$, and $[M,M] = 0$.

So, the centre $Z\big(\mathfrak{U}(\mathcal{X})\big) = (M \otimes M \xrightarrow{0} M)$ and the $\boldsymbol{B}$-centre $Z_{\boldsymbol{B}}(\mathcal{X}) = (M \otimes M \xrightarrow{0} 0, -\otimes-)$ are different in $\boldsymbol{XLie}$, i.e. $Z\big(\mathfrak{U}(\mathcal{X})\big) \neq \mathfrak{U}\big(Z_{\boldsymbol{B}}(\mathcal{X})\big)$. On the other hand, the commutator $[\mathfrak{U}(\mathcal{X}),\mathfrak{U}(\mathcal{X})] = (0 \xrightarrow{0} 0)$ and the $\boldsymbol{B}$-commutator $[\mathcal{X},\mathcal{X}]_{\boldsymbol{B}} = (M \otimes M \xrightarrow{0} 0, - \otimes -)$ are also different in $\boldsymbol{XLie}$, i.e. $[\mathfrak{U}(\mathcal{X}),\mathfrak{U}(\mathcal{X})] \neq \mathfrak{U}([\mathcal{X},\mathcal{X}]_{\boldsymbol{B}})$.

(ii) Now, we will show a $\mathfrak{U}$-central extension that is not a $\boldsymbol{B}$-central extension.

In particular, we have for $M = K^3$ and $M = K^2$, the braided Lie crossed modules $\mathcal{Y} = (K^3 \otimes K^3 \xrightarrow{0} K^3, 0, - \otimes -)$ and $\mathcal{Z} = (K^2 \otimes K^2 \xrightarrow{0} K^2, 0, - \otimes -)$.

By taking the projection $K^3 \xrightarrow{\pi} K^2$, $(x,y,z) \mapsto (x,y)$, we have that $\pi \otimes \pi \colon K^3 \otimes K^3 \to K^2 \otimes K^2$ is surjective and $\mathcal{Y} \xrightarrow{(\pi\otimes\pi,\pi)} \mathcal{Z}$ is an extension of braided Lie crossed modules.

It is immediate that $\ker(\pi \otimes \pi) \subset (K^3 \otimes K^3)^{K^3} = K^3 \otimes K^3$ and $\ker(\pi) \subset Z(K^3) \cap \mathrm{st}_{K^3}(K^3 \otimes K^3) = K^3$, i.e. $\ker(\pi \otimes \pi, \pi) \subset Z\big(\mathfrak{U}(\mathcal{Y})\big)$, and so the extension is a $\mathfrak{U}$-central extension.

However $0 \neq \ker(\pi) = \{(x,y,z) \in K^3 \mid x = y = 0\} \not\subseteq Z_B(K^3) = 0$, and therefore the extension $\mathcal{Y} \xrightarrow{(\pi\otimes\pi,\pi)} \mathcal{Z}$ is not $\boldsymbol{B}$-central.

3. The universal *B*-central extension for *B*-perfect braided Lie crossed modules

Similar to Lie crossed modules we have the following definition of the universal central extension of braided Lie crossed modules.

A ***B***-central extension $\mathcal{U} \xrightarrow{u} \mathcal{M}$ of $\mathcal{M}$ in ***BXLie*** is *universal* if it is the initial object in the category of ***B***-central extensions of $\mathcal{M}$, i.e. if for any other ***B***-central extension $\mathcal{Z} \xrightarrow{f} \mathcal{M}$ in ***BXLie***, there is a unique morphism $h\colon \mathcal{U} \to \mathcal{Z}$ such that $u = f \circ h$.

In this section, we will find the expression of this universal initial object when it exists, and we will try to characterize this fact.

Lemma 3.1. *If* $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a braided Lie crossed module, then* $N \otimes N \xrightarrow{\Phi_1} M$ *defined by* $n \otimes n' \mapsto \{n, n'\}$*, and* $N \otimes N \xrightarrow{\Phi_2} N$ *defined by* $n \otimes n' \mapsto [n, n']$*, are Lie* K*-homomorphisms.*

Besides, Φ_1 *and* Φ_2 *are simultaneously surjective if and only if the braided Lie crossed module* $\mathcal{M}$ *is* ***B****-perfect.*

Proof. Since Φ_1 and Φ_2 are determined by generators, we only need to prove that they are well defined to prove that they are Lie K-homomorphisms.

First, we will prove that the two morphisms preserve the relations (T1)–(T4).

(T1) and (T2) are preserved because $[-,-]$ and $\{-,-\}$ are K-bilinear.

Since the two actions on $N \otimes N$ are the Lie bracket of N, $[-,-]$, we can rewrite the relations (T3) and (T4) to obtain the following ones:

$$[n_1, n_2] \otimes n_3 = n_1 \otimes [n_2, n_3] - n_2 \otimes [n_1, n_3], \tag{T3}$$
$$n_1 \otimes [n_2, n_3] = [n_3, n_1] \otimes n_2 - [n_2, n_1] \otimes n_3,$$
$$[(n_1 \otimes n_2), (n_3 \otimes n_4)] = [n_1, n_2] \otimes [n_3, n_4]. \tag{T4}$$

Starting with (T3) we have:

$$\begin{aligned}\Phi_1([n_1, n_2] \otimes n_3) &= \{[n_1, n_2], n_3\} = \{n_1, [n_2, n_3]\} - \{n_2, [n_1, n_3]\} \\ &= \Phi_1(n_1 \otimes [n_2, n_3]) - \Phi_1(n_2 \otimes [n_1, n_3]) = \Phi_1(n_1 \otimes [n_2, n_3] - n_2 \otimes [n_1, n_3]),\end{aligned}$$

where we have used (BLie6).

We will see now the second relation in (T3):

$$\begin{aligned}\Phi_1(n_1 \otimes [n_2, n_3]) &= \{n_1, [n_2, n_3]\} = \{[n_1, n_2], n_3\} - \{[n_1, n_3], n_2\} \\ &= \Phi_1([n_1, n_2] \otimes n_3 - [n_1, n_3] \otimes n_2) = \Phi_1(-[n_2, n_1] \otimes n_3 + [n_3, n_1] \otimes n_2) \\ &= \Phi_1([n_3, n_1] \otimes n_2 - [n_2, n_1] \otimes n_3),\end{aligned}$$

where we have used (BLie5).

For Φ_2 is true using a similar argument together with the Jacobi identity in both equalities.

The proof of (T4) for Φ_2 follows since both equalities are $[[n_1, n_2], [n_3, n_4]]$ after applying Φ_2.

For Φ_1 we have the following equalities:

$$\begin{aligned}\Phi_1([n_1 \otimes n_2, n_3 \otimes n_4]) &= \Phi_1([n_1, n_2] \otimes [n_3, n_4]) = \{[n_1, n_2], [n_3, n_4]\} = \{\partial\{n_1, n_2\}, \partial\{n_3, n_4\}\} \\ &= [\{n_1, n_2\}, \{n_3, n_4\}] = [\Phi_1(n_1 \otimes n_2), \Phi_1(n_3 \otimes n_4)]\end{aligned}$$

where we have used (T4), (BLie1) and (BLie2).

So, Φ_1 and Φ_2 are well defined and are Lie K-homomorphisms.

For the second part, we have that $\operatorname{Im}\Phi_1 = B_N(M)$ and $\operatorname{Im}\Phi_2 = [N, N]$. Therefore, Φ_1 and Φ_2 are simultaneously surjective if and only if the braided Lie crossed module is ***B***-perfect. □

Lemma 3.2. *Let* $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *be a braided Lie crossed module, and the braided Lie crossed module* $(N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N, [-,-], [-,-])$ *(see Example 2.10 (1)). Then* $(\Phi_1, \Phi_2)\colon (N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N, [-,-], [-,-]) \longrightarrow (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a morphism in* ***BXLie****, with* Φ_1 *and* Φ_2 *defined in Lemma 3.1,.*

Besides, $\ker(\Phi_1) \subset (N \otimes N)^{(N\otimes N)}$ *and* $\ker(\Phi_2) \subset \mathrm{Z}_B(N \otimes N)$.

Proof. For the proof, we will denote the action $[-,-]$ of $N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N$ as $*$, and its braiding as $[\![-,-]\!]$.

First, we will show (XLieH1). Let $n \otimes n', n'' \otimes n''' \in N \otimes N$.

$$\begin{aligned}\Phi_1((n \otimes n') * (n'' \otimes n''')) &= \Phi_1([n \otimes n', n'' \otimes n''']) = \Phi_1([n, n'] \otimes [n'', n''']) \\ &= \{[n, n'], [n'', n''']\} = \{[n, n'], \partial\{n'', n'''\}\} = [n, n'] \cdot \{n'', n'''\} \\ &= \Phi_2(n \otimes n') \cdot \Phi_1(n'' \otimes n'''),\end{aligned}$$

where we have used (BLie1) and (BLie4).

Now, we will show (XLieH2).

$$\partial \circ \Phi_1(n \otimes n') = \partial\{n, n'\} = [n, n'] = \Phi_2(\mathrm{Id}_{N\otimes N}(n \otimes n')),$$

where we have used (BLie2).

Now, we will prove (BXLieH3).

$$\begin{aligned}\Phi_1([\![n \otimes n', n'' \otimes n''']\!]) &= \Phi_1([n \otimes n', n'' \otimes n''']) = \Phi_1([n, n'] \otimes [n'', n''']) \\ &= \{[n, n'], [n'', n''']\} = \{\Phi_2(n \otimes n'), \Phi_2(n'' \otimes n''')\}.\end{aligned}$$

So, (Φ_1, Φ_2) is a morphism in ***BXLie***. We will now prove that the inclusions hold.

If $n \otimes n' \in \ker(\Phi_1)$ then $\{n, n'\} = 0$. Using (BLie1) we have that $0 = \partial\{n, n'\} = [n, n']$.

Since $(N \otimes N)^{(N\otimes N)} = \{x \in N \otimes N \mid (n'' \otimes n''') * x = 0,\ n'' \otimes n''' \in N \otimes N\}$ (it is enough to work on generators), we have for generators $x = n \otimes n'$

$$(n'' \otimes n''') * (n \otimes n') = [n'' \otimes n''', n \otimes n'] = [n'', n'''] \otimes [n, n'] = [n'', n'''] \otimes 0 = 0.$$

Therefore, we have that $n \otimes n' \in (N \otimes N)^{(N\otimes N)}$ and $\ker(\Phi_1) \subset (N \otimes N)^{(N\otimes N)}$.

For the second inclusion, we take $n \otimes n' \in \ker(\Phi_2)$, i.e. $[n, n'] = 0$.

Since it is enough to work on generators, we have that

$$\mathrm{Z}_B(N \otimes N) = \{x \in N \otimes N \mid [\![x, n'' \otimes n''']\!] = 0 = [\![n'' \otimes n''', x]\!],\ n'' \otimes n''' \in N \otimes N\}.$$

Taking into account that for generators $x = n \otimes n'$

$$\begin{aligned}[\![n'' \otimes n''', n \otimes n']\!] &= [n'' \otimes n''', n \otimes n'] = [n'', n'''] \otimes [n, n'] = [n'', n'''] \otimes 0 = 0, \\ [\![n \otimes n', n'' \otimes n''']\!] &= [n \otimes n', n'' \otimes n'''] = [n, n'] \otimes [n'', n'''] = 0 \otimes [n'', n'''] = 0,\end{aligned}$$

we deduce $n \otimes n' \in Z_B(N \otimes N)$, which proves that $\ker(\Phi_2) \subset Z_B(N \otimes N)$. □

Corollary 3.3. *The morphism given in Lemma 3.2 is a* ***B****-central extension if and only if* $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a* ***B****-perfect braided Lie crossed module.*

Proof. It will be a ***B***-central extension if and only if (Φ_1, Φ_2) is an extension, since Lemma 3.2 establishes the two inclusions and they have the restricted operations as a braided Lie crossed module.

Moreover, (Φ_1, Φ_2) is an extension if and only if Φ_1 and Φ_2 are simultaneously surjective, and by Lemma 3.1 that it happens if and only if the braided Lie crossed module $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is ***B***-perfect. □

Proposition 3.4. *If* $(X_1 \xrightarrow{\delta} X_2, *, (\!(-,-)\!)) \xrightarrow{f=(f_1,f_2)} (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a* ***B****-central extension, then we have a morphism in* ***BXLie****,* $h = (h_1, h_2)\colon (N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N, [-,-], [-,-]) \longrightarrow (X_1 \xrightarrow{\delta} X_2, *, (\!(-,-)\!))$*, defined by:*

- $h_1\colon N\otimes N \to X_1$, $n\otimes\eta \mapsto (\!(\overline{n},\overline{\eta})\!)$, *where* $\overline{n},\overline{\eta}\in X_2$ *are elements such that* $f_2(\overline{n})=n$ *and* $f_2(\overline{\eta})=\eta$*;*
- $h_2\colon N\otimes N \to X_2$, $n\otimes\eta \mapsto [\overline{n},\overline{\eta}]$, *where* $\overline{n},\overline{\eta}\in X_2$ *are elements such that* $f_2(\overline{n})=n$ *and* $f_2(\overline{\eta})=\eta$*.*

Besides, $f\circ h=\Phi=(\Phi_1,\Phi_2)$*, i.e.* h *is a morphism between the extensions* $\Phi=(\Phi_1,\Phi_2)$ *and* $f=(f_1,f_2)$ *if* $(M\xrightarrow{\partial}N,\cdot,\{-,-\})$ *is* $\boldsymbol{B}$*-perfect (see Lemmas 3.1 and 3.2).*

Proof. We need to prove that h_1 and h_2 are well defined.

We will start with h_1. We will take $\overline{n},\widetilde{n},\overline{\eta},\widetilde{\eta}\in X_2$ such that $f_2(\overline{n})=f_2(\widetilde{n})=n$ and $f_2(\overline{\eta})=f_2(\widetilde{\eta})=\eta$ and prove that $(\!(\overline{n},\overline{\eta})\!)=(\!(\widetilde{n},\widetilde{\eta})\!)$.

Since $f_2(\overline{n})=f_2(\widetilde{n})$ and $f=(f_1,f_2)$ is a $\boldsymbol{B}$-central extension, we have that $\overline{n}-\widetilde{n}\in \ker(f_2)\subset \mathrm{Z}_B(X_2)$. By the definition of $\mathrm{Z}_B(X_2)$ we get that $(\!(\overline{n}-\widetilde{n},\overline{\eta})\!)=0$ and so $(\!(\overline{n},\overline{\eta})\!)=(\!(\widetilde{n},\overline{\eta})\!)$.

Using an analogue reasoning, we have that $\overline{\eta}-\widetilde{\eta}\in\ker(f_2)\subset \mathrm{Z}_B(X_2)$, and, $(\!(\widetilde{n},\overline{\eta}-\widetilde{\eta})\!)=0$. So $(\!(\widetilde{n},\overline{\eta})\!)=(\!(\widetilde{n},\widetilde{\eta})\!)$.

With both equalities, we have that $(\!(\overline{n},\overline{\eta})\!)=(\!(\widetilde{n},\overline{\eta})\!)=(\!(\widetilde{n},\widetilde{\eta})\!)$, and h_1 is independent of the choice.

Since $\mathrm{Z}_B(X_2)\subset \mathrm{Z}(X_2)$ we can change the proof for h_1 taking the equalities for $[-,-]$ instead of $(\!(-,-)\!)$ which proves that h_2 is independent of the choice.

We can use an analogue argument as in Lemma 3.1 to prove that h_1 and h_2 are well defined, i.e. they preserve the relations. So, they are Lie K-homomorphisms since they are determined on generators.

To prove that $h=(h_1,h_2)$ is a morphism of braided Lie crossed modules, we also use similar reasoning as the one done in Lemma 3.2, since we can make the changes in the choice inside the braidings and brackets.

To finish, if $n\otimes\eta\in N\otimes N$, then

$$f_1\circ h_1(n\otimes\eta)=f_1((\!(\overline{n},\overline{\eta})\!))=\{f_2(\overline{n}),f_2(\overline{\eta})\}=\{n,\eta\}=\Phi_1(n\otimes\eta),$$
$$f_2\circ h_2(n\otimes\eta)=f_2([\overline{n},\overline{\eta}])=[f_2(\overline{n}),f_2(\overline{\eta})]=[n,\eta]=\Phi_2(n\otimes\eta).$$

Therefore, $f\circ h=\Phi$. □

Lemma 3.5. *If* N *is a perfect Lie* K*-algebra, i.e.* $N=[N,N]$*, then* $(N\otimes N\xrightarrow{\mathrm{Id}_{N\otimes N}}N\otimes N,[-,-],[-,-])$ *is a* $\boldsymbol{B}$*-perfect braided Lie crossed module.*

In particular, if $(M\xrightarrow{\partial}N,\cdot,\{-,-\})$ *is a* $\boldsymbol{B}$*-perfect braided Lie crossed module, then* $(N\otimes N\xrightarrow{\mathrm{Id}_{N\otimes N}}N\otimes N,[-,-],[-,-])$ *is a* $\boldsymbol{B}$*-perfect braided Lie crossed module.*

Proof. Since the braiding in $(N\otimes N\xrightarrow{\mathrm{Id}_{N\otimes N}}N\otimes N,[-,-],[-,-])$ is the bracket, we have that $[N\otimes N,N\otimes N]=B_{N\otimes N}(N\otimes N)$, and so it is enough to prove that $[N\otimes N,N\otimes N]=N\otimes N$.

Moreover, it is enough to prove that the generators $[n_1,n_2]\otimes[n_3,n_4]$ are inside $[N\otimes N,N\otimes N]$ since $N=[N,N]$. Using (T4) we have that $[n_1,n_2]\otimes[n_3,n_4]=[n_1\otimes n_2,n_3\otimes n_4]\in[N\otimes N,N\otimes N]$.

For the second part, if $(M\xrightarrow{\partial}N,\cdot,\{-,-\})$ is $\boldsymbol{B}$-perfect, then $N=[N,N]$, and we conclude using the first part. □

Proposition 3.6. *Let* $(Y_1\xrightarrow{\varrho}Y_2,\star,[\![-,-]\!])\xrightarrow{\Psi=(\Psi_1,\Psi_2)}(M\xrightarrow{\partial}N,\cdot,\{-,-\})$ *be a morphism of braided Lie crossed modules such that* $(Y_1\xrightarrow{\varrho}Y_2,\star,[\![-,-]\!])$ *is* $\boldsymbol{B}$*-perfect.*

If $(X_1\xrightarrow{\rho}X_2,*,(\!(-,-)\!))\xrightarrow{f=(f_1,f_2)}\!\!\!\!\!\rightarrow(M\xrightarrow{\partial}N,\cdot,\{-,-\})$ *is a* $\boldsymbol{B}$*-central extension and exists* $(Y_1\xrightarrow{\varrho}Y_2,\star,[\![-,-]\!])\xrightarrow{h=(h_1,h_2)}(X_1\xrightarrow{\rho}X_2,*,(\!(-,-)\!))$ *such that* $\Psi=f\circ h$*, then* h *is the unique that satisfies the equality.*

Proof. Suppose that there are morphisms $g = (g_1, g_2), h = (h_1, h_2)\colon (Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!]) \longrightarrow (X_1 \xrightarrow{\rho} X_2, *, (\!(-,-)\!))$ such that $\Psi = f \circ h = f \circ g$, i.e. $\Psi_1 = f_1 \circ h_1 = f_1 \circ g_1$ and $\Psi_2 = f_2 \circ h_2 = f_2 \circ g_2$.

If $y \in Y_2$ then $f_2 \circ h_2(y) = f_2 \circ g_2(y)$, i.e. $h_2(y) - g_2(y) \in \ker(f_2)$. Then there is $k_y \in \ker(f_2)$ such that $h_2(y) = g_2(y) + k_y$. Since f is a $\boldsymbol{B}$-central extension we have that $\ker(f_2) \subset \mathrm{Z}_B(X_2) \subset \mathrm{Z}(X_2)$. If we take $y, z \in Y_2$, and since $k_y, k_z \in \mathrm{Z}(X_2)$, we have

$$[k_y, g_2(z)] = [k_y, k_z] = [g_2(y), k_z] = 0.$$

Using this fact, we have:

$$\begin{aligned} h_2([y,z]) &= [h_2(y), h_2(z)] = [g_2(y) + k_y, g_2(z) + k_z] \\ &= [g_2(y), g_2(z)] + [k_y, g_2(z)] + [k_y, k_z] + [g_2(y), k_z] = [g_2(y), g_2(z)] = g_2([y,z]). \end{aligned}$$

So, $g_2 = h_2$ since $(Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!])$ is $\boldsymbol{B}$-perfect.

Besides, since $\ker(f_2) \subset \mathrm{Z}_B(X_2)$, for $y, z \in Y_2$, we have that:

$$\begin{aligned} &h_1([\![y,z]\!]) = (\!(h_2(y), h_2(z))\!) = (\!(g_2(y) + k_y, g_2(z) + k_z)\!) \\ &= (\!(g_2(y), g_2(z))\!) + (\!(k_y, g_2(z))\!) + (\!(k_y, k_z)\!) + (\!(g_2(y), k_z)\!) = (\!(g_2(y), g_2(z))\!) = g_1([\![y,z]\!]), \end{aligned}$$

where we have used that $k_y, k_z \in \mathrm{Z}_B(X_2)$.

Therefore, $g_1 = h_1$ because $(Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!])$ is $\boldsymbol{B}$-perfect, i.e. $Y_1 = B_{Y_2}(Y_1)$ is generated by the images of the braiding. □

Corollary 3.7. *If $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is a $\boldsymbol{B}$-perfect Lie braided crossed module, then*

$$\mathcal{U} = (N \otimes N \xrightarrow{\mathrm{Id}_{N\otimes N}} N \otimes N, [-,-], [-,-]) \xrightarrow{\Phi=(\Phi_1,\Phi_2)} \mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\}) \quad \text{(UBCE)}$$

is the universal $\boldsymbol{B}$-central extension of $\mathcal{M}$, where Φ_1, Φ_2 were defined in Lemma 3.1.

Proof. Since $\mathcal{M}$ is $\boldsymbol{B}$-perfect, Corollary 3.3 states that the morphism $\mathcal{U} \xrightarrow{\Phi} \mathcal{M}$ is a $\boldsymbol{B}$-central extension.

We need to prove that it is universal.

If we have another $\boldsymbol{B}$-central extension $\mathcal{X} \xrightarrow{f} \mathcal{M}$ then by Proposition 3.4 there is h such that $\Phi = f \circ h$.

The uniqueness of this morphism is given by Proposition 3.6. We can use the previous proposition since $\mathcal{U}$ is $\boldsymbol{B}$-perfect by Lemma 3.5 and the fact that $\mathcal{M}$ is $\boldsymbol{B}$-perfect. □

Let us see the converse of Corollary 3.7.

Proposition 3.8. *Let $(Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!]) \xrightarrow{\Psi=(\Psi_1,\Psi_2)} (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be an extension in* ***BXLie*** *such that $(Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!])$ is $\boldsymbol{B}$-perfect. Then $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is $\boldsymbol{B}$-perfect.*

Proof. Ψ_1 and Ψ_2 are surjective maps since Ψ is an extension, and $Y_1 = B_{Y_2}(Y_1)$ and $Y_2 = [Y_2, Y_2]$ because $(Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!])$ is $\boldsymbol{B}$-perfect.

Since the elements $[\![y,z]\!]$, with $y, z \in Y_2$ are the generators of Y_1, we have that $\Psi_1([\![y,z]\!])$ are the generators of $\operatorname{Im}\Psi_1 = M$. Since $\Psi_1([\![y,z]\!]) = \{\Phi_2(y), \Phi_2(z)\}$, we get that the generators of M are braided elements and $M = B_N(M)$.

We know that the elements $[y,z]$, with $y, z \in Y_2$, are the generators of Y_2. Therefore, $\Phi_2([y,z]) = [\Phi_2(y), \Phi_2(z)]$ are the generators of $\operatorname{Im}\Phi_2 = N$, and then $N = [N,N]$.

So, $(M \xrightarrow{\partial} N \cdot, \{-,-\})$ is $\boldsymbol{B}$-perfect. □

Lemma 3.9. *Let $\mathcal{Y} = (Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!]) \xrightarrow{\Psi} \mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a $\boldsymbol{B}$-central extension in* ***BXLie*** *such that $\mathcal{Y}$ is not $\boldsymbol{B}$-perfect. Then exists another $\boldsymbol{B}$-central extension $\mathcal{X} \xrightarrow{f} \mathcal{M}$ and two different morphisms $h, g\colon \mathcal{Y} \to \mathcal{X}$ such that $\Psi = f \circ h = f \circ g$.*

Proof. Let $[\mathcal{Y},\mathcal{Y}]_{\boldsymbol{B}} = (B_{Y_2}(Y_1) \xrightarrow{\varrho|_{B_{Y_2}(Y_1)}} [Y_2,Y_2], \star_C, [\![-,-]\!]_C) \xrightarrow{i=(i_1,i_2)} (Y_1 \xrightarrow{\varrho} Y_2, \star, [\![-,-]\!])$ be the inclusion morphism of the $\boldsymbol{B}$-commutator braided crossed submodule.

Since $[\mathcal{Y},\mathcal{Y}]_{\boldsymbol{B}}$ is a crossed ideal, it is the commutator [9] in the category ***BXLie***, by taking the cokernel of i we have the Lie crossed module $\overline{\mathcal{Y}} = (\frac{Y_1}{B_{Y_2}(Y_1)} \xrightarrow{\overline{\varrho}} \frac{Y_2}{[Y_2,Y_2]}, \overline{\star}, [\![-,-]\!])$. We will denote in the same way, by abuse of notation, the braidings in $\mathcal{Y}$ and in its quotient $\overline{\mathcal{Y}}$. We will represent the elements in $\frac{Y_1}{B_{Y_2}(Y_1)}$ as $\overline{x}$, $x \in Y_1$, and the ones in $\frac{Y_2}{[Y_2,Y_2]}$ as $\overline{y}$, $y \in Y_2$.

We take now the product in the category ***BXLie*** and we construct $\mathcal{M} \times \overline{\mathcal{Y}}$. We denote as $\pi^1 = (\pi_1^1, \pi_2^1)$ the first projection morphism. Since π_1^1 and π_1^2 are surjective maps, we have that $\mathcal{M} \times \overline{\mathcal{Y}} \stackrel{\pi^1}{\twoheadrightarrow} \mathcal{M}$ is an extension. We will denote the braiding in the product as $\{\!\{-,-\}\!\}$.

We will prove that it is a $\boldsymbol{B}$-central extension, i.e. we need to prove the inclusions $\ker(\pi_1^1) \subset (M \times \frac{Y_1}{B_{Y_2}(Y_2)})^{(N \times \frac{Y_2}{[Y_2,Y_2]})}$ and $\ker(\pi_2^1) \subset \mathrm{Z}_B\,(N \times \frac{Y_2}{[Y_2,Y_2]})$.

If $a \in \ker(\pi_1^1)$ then $a = (0,\overline{x})$ with $x \in Y_1$.

If we take $(n,\overline{y}) \in N \times \frac{Y_2}{[Y_2,Y_2]}$ then:

$$(n,\overline{y})(\cdot \times \overline{\star})(0,\overline{x}) = (n\cdot 0, \overline{y}\,\overline{\star}\,\overline{x}) = (0, \overline{y \star x}).$$

But $\overline{y \star x} = \overline{0}$ since $y \star x \in D_{Y_2}(Y_1) \subset B_{Y_2}(Y_1)$.

So $\ker(\pi_1^1) \subset (M \times \frac{Y_1}{B_{Y_2}(Y_2)})^{(N \times \frac{Y_2}{[Y_2,Y_2]})}$.

If $a \in \ker(\pi_2^1)$ then $a = (0,\overline{y})$ with $y \in Y_2$. If we take $(n,\overline{y_1}) \in N \times \frac{Y_2}{[Y_2,Y_2]}$ then:

$$\{\!\{(0,\overline{y}),(n,\overline{y_1})\}\!\} = (\{0,n\}, [\![\overline{y},\overline{y_1}]\!]) = (0, \overline{[\![y,y_1]\!]}),$$
$$\{\!\{(n,\overline{y_1}),(0,\overline{y})\}\!\} = (\{n,0\}, [\![\overline{y_1},\overline{y}]\!]) = (0, \overline{[\![y_1,y]\!]}).$$

Moreover, $\overline{[\![y,y_1]\!]} = \overline{[\![y_1,y]\!]} = \overline{0}$ since $[\![y_1,y]\!], [\![y,y_1]\!] \in B_{Y_2}(Y_1)$. Therefore $\ker(\pi_2^1) \subset \mathrm{Z}_B\,(N \times \frac{Y_2}{[Y_2,Y_2]})$, and so π^1 is a $\boldsymbol{B}$-central extension.

If $i^c\colon \mathcal{Y} \twoheadrightarrow \overline{\mathcal{Y}}$ is the cokernel of i, then we have two morphisms, induced by the product, with domain $\mathcal{Y}$ and $\mathcal{M} \times \overline{\mathcal{Y}}$ as codomain. They are $h = (\Psi, 0)$ and $g = (\Psi, i^c)$. Since they are induced by the universal property of the product, we have that $\Psi = \pi^1 \circ h = \pi^1 \circ g$.

To finish the proof, we only must prove that they are different. Since the braided Lie crossed module $\mathcal{Y}$ is not $\boldsymbol{B}$-perfect and i_1^c and i_2^c are surjective we know that $i_1^c \neq 0$ or $i_2^c \neq 0$ (if both were the zero morphisms, then $\mathcal{Y}$ would be $\boldsymbol{B}$-perfect), and so $h \neq g$. □

Corollary 3.10. *If $\mathcal{M}$ is a braided Lie crossed module, then its universal $\boldsymbol{B}$-central extension, if it exists, is $\boldsymbol{B}$-perfect.*

Proof. If the universal extension is not $\boldsymbol{B}$-perfect, then using Lemma 3.9 we have another $\boldsymbol{B}$-central extension $\mathcal{X} \twoheadrightarrow \mathcal{M}$ for which there exist two different morphisms from the universal $\boldsymbol{B}$-central extension to $\mathcal{X} \twoheadrightarrow \mathcal{M}$, which contradicts the universality. □

Theorem 3.11. *A braided Lie crossed module admits a universal $\boldsymbol{B}$-central extension if and only if it is $\boldsymbol{B}$-perfect.*

Proof. It is a consequence of Corollary 3.7, Corollary 3.10 and Proposition 3.8. □

4. Braiding on a universal extension of Lie crossed modules

Universal central extensions of braided crossed modules of groups are not studied in [8]. However, the author constructed a canonical braiding on the universal central extension of a crossed module of groups [13], when the given crossed module is braided as well, and showed that it was universal in a sense that we will explain in this section.

In this part of the paper, we will consider braided Lie crossed modules extensions, but unlike the previous section, we will construct a braiding on the universal central extension of a braided Lie crossed module though as Lie crossed module and with the centre in ***XLie***, which we have called $\mathfrak{U}$-central extension. In this sense, we will obtain similar results given by Fukushi in [8] for crossed modules of groups in the category ***BXLie***.

Casas and Ladra in [2] proved that the universal central extension of a perfect Lie crossed module $(M \xrightarrow{\partial} N, \cdot)$ in ***XLie*** is given by:

$$(N \otimes M \xrightarrow{\mathrm{Id}_N \otimes \partial} N \otimes N, *) \xrightarrow{c=(c_1,c_2)} (M \xrightarrow{\partial} N, \cdot), \tag{UCE}$$

where $N \otimes M$ is given by the actions $\cdot$ of N on M and $m \star n = [\partial(m), n]$ of M on N; the action of $N \otimes N$ on $N \otimes M$ is given by $(n \otimes n') * (n'' \otimes m) = [[n, n'], n''] \otimes m + n'' \otimes [n, n'] \cdot m$ for $n, n', n'' \in N, m \in M$; and the morphisms are $c_1(n \otimes m) = n \cdot m$ and $c_2(n \otimes n') = [n, n']$.

Proposition 4.1. *If $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is a braided Lie crossed module then $\{\!\{-,-\}\!\} : (N \otimes N) \times (N \otimes N) \to N \otimes M$, defined on generators by $\{\!\{n \otimes n', n'' \otimes n'''\}\!\} = [n, n'] \otimes \{n'', n'''\}$, is a braiding for the Lie crossed module $(N \otimes M \xrightarrow{\mathrm{Id}_N \otimes \partial} N \otimes N, *)$.*

Proof. The braiding $\{\!\{-,-\}\!\}$ is well defined since it preserves the relations (T1) and (T2) using the K-bilinearity of $[-,-]$ and $\{-,-\}$, and (T3) and (T4) are fulfilled too since $\{-,-\}$ and $[-,-]$ satisfy it.

It is sufficient to prove the axioms of braidings. Let $n, n, n', n'' \in N$, $m, m' \in M$. Then

$$\begin{aligned}&(\mathrm{Id}_N \otimes \partial)(\{\!\{n \otimes n', n'' \otimes n'''\}\!\}) = (\mathrm{Id}_N \otimes \partial)([n, n'] \otimes \{n'', n'''\}) = [n, n'] \otimes \partial\{n'', n'''\}\\ &= [n, n'] \otimes [n'', n'''] = [n \otimes n', n'' \otimes n'''] \text{ (BLie1)},\end{aligned}$$

$$\begin{aligned}&\{\!\{(\mathrm{Id}_N \otimes \partial)(n \otimes m), (\mathrm{Id}_N \otimes \partial)(n' \otimes m')\}\!\} = \{\!\{n \otimes \partial(m), n' \otimes \partial(m')\}\!\} = [n, \partial(m)] \otimes \{n', \partial(m')\}\\ &= -(m \star n) \otimes (n' \cdot m') = [n \otimes m, n' \otimes m'] \text{ (BLie2)},\end{aligned}$$

$$\begin{aligned}&\{\!\{(\mathrm{Id}_N \otimes \partial)(n \otimes m), n' \otimes n''\}\!\} = \{\!\{n \otimes \partial(m), n' \otimes n''\}\!\} = [n, \partial(m)] \otimes \{n', n''\} = -(m \star n) \otimes \{n', n''\}\\ &= n \otimes [m, \{n', n''\}] - \{n', n''\} \star n \otimes m = n \otimes \{\partial(m), \partial(\{n', n''\})\} - [\partial(\{n', n''\}), n] \otimes m\\ &= -n \otimes [n', n''] \cdot m - [[n', n''], n] \otimes m = -(n' \otimes n'') * (n \otimes m) \text{ (BLie3)},\end{aligned}$$

where we have used the second relation of (T3) in the third equality.

$$\begin{aligned}&\{\!\{n' \otimes n'', (\mathrm{Id}_N \otimes \partial)(n \otimes m)\}\!\} = \{\!\{n' \otimes n'', n \otimes \partial(m)\}\!\} = [n', n''] \otimes \{n, \partial(m)\}\\ &= [n', n''] \otimes (n \cdot m) = n \otimes [n, n'] \cdot m + [[n', n''], n] \otimes m = (n' \otimes n'') * (n \otimes m) \text{ (BLie4)},\end{aligned}$$

where we have used the first relation of (T3) in the third equality.

$$\begin{aligned}&\{\!\{n_1 \otimes n_1', [n_2 \otimes n_2', n_3 \otimes n_3']\}\!\} = \{\!\{n_1 \otimes n_1', [n_2 \otimes n_2'] \otimes [n_3 \otimes n_3']\}\!\} = [n_1, n_1'] \otimes \{[n_2, n_2'], [n_3, n_3']\}\\ &= [n_1, n_1'] \otimes [\{n_2, n_2'\}, \{n_3, n_3'\}] = (\{n_3, n_3'\} \star [n_1, n_1']) \otimes \{n_2, n_2'\} - (\{n_2, n_2'\} \star [n_1, n_1']) \otimes \{n_3, n_3'\}\\ &= [\partial(\{n_3, n_3'\}), [n_1, n_1']] \otimes \{n_2, n_2'\} - [\partial(\{n_2, n_2'\}), [n_1, n_1']] \otimes \{n_3, n_3'\}\\ &= [[n_3, n_3'], [n_1, n_1']] \otimes \{n_2, n_2'\} - [[n_2, n_2'], [n_1, n_1']] \otimes \{n_3, n_3'\}\\ &= -[[n_1, n_1'], [n_3, n_3']] \otimes \{n_2, n_2'\} + [[n_1, n_1'], [n_2, n_2']] \otimes \{n_3, n_3'\}\\ &= -\{\!\{[n_1, n_1'] \otimes [n_3, n_3'], n_2 \otimes n_2'\}\!\} + \{\!\{[n_1, n_1'] \otimes [n_2, n_2'], n_3 \otimes n_3'\}\!\}\\ &= \{\!\{[n_1 \otimes n_1', n_2 \otimes n_2'], n_3 \otimes n_3'\}\!\} - \{\!\{[n_1 \otimes n_1', n_3 \otimes n_3'], n_2 \otimes n_2'\}\!\} \text{ (BLie5)},\end{aligned}$$

$\{\![n_1 \otimes n_1', n_2 \otimes n_2'], n_3 \otimes n_3'\}\!\} = \{\![n_1, n_1'] \otimes [n_2, n_2'], n_3 \otimes n_3'\}\!\} = [[n_1, n_1'], [n_2, n_2']] \otimes \{n_3, n_3'\}$
$= [n_1, n_1'] \otimes [n_2, n_2'] \cdot \{n_3, n_3'\} - [n_2, n_2'] \otimes [n_1, n_1'] \cdot \{n_3, n_3'\}$
$= [n_1, n_1'] \otimes \{[n_2, n_2'], \partial(\{n_3, n_3'\})\} - [n_2, n_2'] \otimes \{[n_1, n_1'], \partial(\{n_3, n_3'\})\}$
$= [n_1, n_1'] \otimes \{[n_2, n_2'], [n_3, n_3']\} - [n_2, n_2'] \otimes \{[n_1, n_1'], [n_3, n_3']\}$
$= \{\!\{n_1 \otimes n_1', [n_2, n_2'] \otimes [n_3, n_3']\}\!\} - \{\!\{n_2 \otimes n_2', [n_1, n_1'] \otimes [n_3, n_3']\}\!\}$
$= \{\!\{n_1 \otimes n_1', [n_2 \otimes n_2', n_3 \otimes n_3']\}\!\} - \{\!\{n_2 \otimes n_2', [n_1 \otimes n_1', n_3 \otimes n_3']\}\!\}$ (BLie6).

In all equalities, we have used the properties of $\{-,-\}$ and relations of the tensor product. □

Proposition 4.2. *If* $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a braided Lie crossed module such that* $\mathfrak{U}(\mathcal{M})$ *is perfect, then* $(N \otimes M \xrightarrow{\mathrm{Id}_N \otimes \partial} N \otimes N, *, \{\!\{-,-\}\!\}) \xrightarrow{c=(c_1,c_2)} (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a* $\mathfrak{U}$*-central extension, where* $c_1(n \otimes m) = n \cdot m$ *and* $c_2(n \otimes n') = [n, n']$*, and* $\{\!\{-,-\}\!\}$ *is defined in Proposition 4.1.*

Proof. Since $\mathfrak{U}(\mathcal{M}) = (M \to N, \cdot)$ is a perfect Lie crossed module we have the central extension $(N \otimes M \xrightarrow{\mathrm{Id}_N \otimes \partial} N \otimes N, *) \xrightarrow{c=(c_1,c_2)} (M \xrightarrow{\partial} N, \cdot)$ in ***XLie*** (see [2]).

Now, we will prove that c respects the braiding.

$$c_1(\{\!\{n_1 \otimes n_1', n_2 \otimes n_2'\}\!\}) = c_1([n_1, n_1'] \otimes \{n_2, n_2'\}) = [n_1, n_1'] \cdot \{n_2, n_2'\}$$
$$= \{[n_1, n_1'], \partial(\{n_2, n_2'\})\} = \{[n_1, n_1'], [n_2, n_2']\} = \{c_2(n_1 \otimes n_1'), c_2(n_2 \otimes n_2')\}.$$

So, c is a $\mathfrak{U}$-central extension. □

Now, we will provide a result similar to what was given by Fukushi in [8] for the case of central extensions of braided crossed modules of groups. A $\mathfrak{U}$-central extension $\mathcal{V} \xrightarrow{v} \mathcal{M}$ of $\mathcal{M}$ in ***BXLie*** is *universal* if it is the initial object in the category of $\mathfrak{U}$-central extensions of $\mathcal{M}$.

Proposition 4.3. *If* $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ *is a braided Lie crossed module such that* $\mathfrak{U}(\mathcal{M})$ *is perfect, then*

$$\mathcal{V} = (N \otimes M \xrightarrow{\mathrm{Id}_N \otimes \partial} N \otimes N, *, \{\!\{-,-\}\!\}) \xrightarrow{c=(c_1,c_2)} \mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\}), \qquad \text{(UUCE)}$$

is the universal $\mathfrak{U}$*-central extension of* $\mathcal{M}$*.*

Moreover, the universal initial morphism is the same as in the universality of the non-braiding case.

Proof. Let $(X_1 \xrightarrow{\delta} X_2, \diamond, (\!(-,-)\!)) \xrightarrow{f=(f_1,f_2)} (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a $\mathfrak{U}$-central extension of braided Lie crossed modules.

Since $\mathfrak{U}(\mathcal{M})$ is perfect, we have that $\mathfrak{U}(\mathcal{V}) \xrightarrow{c=(c_1,c_2)} \mathfrak{U}(\mathcal{M})$ is the universal central extension in ***XLie*** (UCE). Therefore, there exists a unique morphism in ***XLie***

$$(N \otimes M \xrightarrow{\mathrm{Id}_N \otimes \partial} N \otimes N, *) \xrightarrow{h=(h_1,h_2)} (X_1 \xrightarrow{\delta} X_2, \diamond),$$

defined as $h_1(n \otimes m) = \overline{n} \diamond \overline{m}$ and $h_2(n \otimes n') = [\overline{n}, \overline{n'}]$ where $f_1(\overline{m}) = m$, $f_2(\overline{n}) = n$ and $f_2(\overline{n'}) = n'$, which satisfy $c = f \circ h$.

We check that h is a morphism in ***BXLie*** showing that preserves the braidings $\{\!\{-,-\}\!\}$ and $(\!(-,-)\!)$.

Let $n_1, n_2, \eta_1, \eta_2 \in N$. Then

$$h_1(\{\!\{n_1 \otimes \eta_1, n_2 \otimes \eta_2\}\!\}) = h_1([n_1, \eta_1] \otimes \{n_2, \eta_2\}) = \overline{[n_1, \eta_1]} \diamond \overline{\{n_2, \eta_2\}}$$
$$= [\overline{n_1}, \overline{\eta_1}] \diamond (\!(\overline{n_2}, \overline{\eta_2})\!) = (\!([\overline{n_1}, \overline{\eta_1}], \delta((\!(\overline{n_2}, \overline{\eta_2})\!)))\!)$$
$$= (\!([\overline{n_1}, \overline{\eta_1}], [\overline{n_2}, \overline{\eta_2}])\!) = (\!(h_2(n_1 \otimes \eta_1), h_2(n_2 \otimes \eta_2))\!),$$

since $f_2(\overline{[n_1,\eta_1]}) = [n_1,\eta_1] = [f_2(\overline{n_1}), f_2(\overline{\eta_1})] = f_2([\overline{n_1},\overline{\eta_1}])$ being f_2 a Lie K-homomorphism, and $f_1(\overline{\{n_2,\eta_2\}}) = \{n_2,\eta_2\} = \{f_2(\overline{n_2}), f_2(\overline{\eta_2})\} = f_1(\langle\!\langle \overline{n_2},\overline{\eta_2}\rangle\!\rangle)$ being f a morphism of braided Lie crossed modules.

So, the unique morphism $h = (h_1, h_2)$ in ***XLie***, in the non-braiding case, that satisfies $c = f \circ h$ also satisfies it in the braiding case, i.e in ***BXLie***. Therefore, the uniqueness of h in ***BXLie*** is a consequence of that the forgetful functor $\mathfrak{U}\colon$ ***BXLie*** $\to$ ***XLie*** is faithful. □

Let $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a braided Lie crossed module. In the next, we will prove that the universal $\mathfrak{U}$-central extension of $\mathcal{M}$ exists if and only if $\mathfrak{U}(\mathcal{M})$ is perfect in ***XLie***.

Proposition 4.4. *Let $\mathcal{Y} \overset{\Psi}{\twoheadrightarrow} \mathcal{M}$ be an extension of braided Lie crossed modules such that $\mathfrak{U}(\mathcal{Y})$ is perfect in* ***XLie***. *Then $\mathfrak{U}(\mathcal{M})$ is perfect.*

Proof. Since $\mathfrak{U}(\mathcal{Y}) \overset{\mathfrak{U}(\Psi)}{\twoheadrightarrow} \mathfrak{U}(\mathcal{M})$ is an extension in ***XLie***, by [2, Proposition 2] $\mathfrak{U}(\mathcal{M})$ is perfect. □

Lemma 4.5. *Let $\mathcal{Y} = (Y_1 \xrightarrow{\rho} Y_2, \star, [\![-,-]\!]) \overset{\Psi}{\twoheadrightarrow} \mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a $\mathfrak{U}$-central extension of braided Lie crossed modules such that $\mathfrak{U}(\mathcal{Y})$ is not perfect. Then exists another $\mathfrak{U}$-central extension $\mathcal{X} \overset{f}{\twoheadrightarrow} \mathcal{M}$ in* ***BXLie*** *and two different morphisms $h, g\colon \mathcal{Y} \longrightarrow \mathcal{X}$ such that $\Psi = f \circ h = f \circ g$.*

Proof. If $(Y_1 \xrightarrow{\rho} Y_2, \star, [\![-,-]\!])$ is a braided Lie crossed module, then we know that the Lie crossed module $[\mathcal{Y},\mathcal{Y}] = (D_{Y_2}(Y_1) \xrightarrow{\rho|_{D_{Y_2}(Y_1)}} [Y_2,Y_2], \star_C)$ is a crossed ideal of $(Y_1 \xrightarrow{\rho} Y_2, \star)$. But it is itself a braided crossed submodule of $(Y_1 \xrightarrow{\rho} Y_2, \star, [\![-,-]\!])$ since, if we have $[y,y'], [z,z'] \in [Y_2,Y_2]$, then:

$$[\![[y,y'],[z,z']]\!] = [\![[y,y'], \rho([\![z,z']\!])]\!] = [y,y'] \star [\![z,z']\!] \in D_{Y_2}(Y_1).$$

Let us denote $i\colon (D_{Y_2}(Y_1) \xrightarrow{\rho|_{D_{Y_2}(Y_1)}} [Y_2,Y_2], \star_C, [\![-,-]\!]_C) \longrightarrow (Y_1 \xrightarrow{\rho} Y_2, \star, [\![-,-]\!])$.

Since $[\mathcal{Y},\mathcal{Y}]$ is a crossed ideal, we can consider $\mathcal{M} \times \operatorname{coker}(i) \overset{\pi^1}{\twoheadrightarrow} \mathcal{M}$ the extension given by the first projection. This extension is a $\mathfrak{U}$-central extension and since $\mathfrak{U}(\mathcal{Y})$ is not perfect, there are two morphisms in ***XLie***, $h, g\colon \mathcal{Y} \longrightarrow \mathcal{M} \times \operatorname{coker}(i)$ such that $\Psi = f \circ h = f \circ g$ (see [2, Lemma 4]). The product in ***BXLie*** is the same as in ***XLie*** with induced braiding, so we have that the morphisms are in ***BXLie***. □

Corollary 4.6. *If the universal $\mathfrak{U}$-central extension $\mathcal{V}$ of a braided Lie crossed module $\mathcal{M}$ exists, then $\mathfrak{U}(\mathcal{V})$ is perfect in* ***XLie*** .

Proof. If the universal extension is not perfect, then using Lemma 4.5, we have another $\mathfrak{U}$-central extension and two different morphisms from the universal $\mathfrak{U}$-central extension, which contradicts the universality. □

Corollary 4.7. *A braided Lie crossed module admits a universal $\mathfrak{U}$-central extension if and only if it is perfect as Lie crossed module.*

Proof. If the braided Lie crossed module is perfect, then using Proposition 4.3, we have its universal $\mathfrak{U}$-central extension.

If the braided Lie crossed module has a universal $\mathfrak{U}$-central extension, then using Corollary 4.6, we have that the universal $\mathfrak{U}$-central extension is perfect as Lie crossed module. Since it is an extension, we can use Lemma 4.4 and conclude that our braided Lie crossed module is perfect as a Lie crossed module. □

5. Relationship between the universal $\boldsymbol{B}$-central extension and the universal $\mathfrak{U}$-central extension in the braided case

This section will show the relation between the notions of universal $\boldsymbol{B}$-central extension and universal $\mathfrak{U}$-central extension in the case of braided Lie crossed modules.

Lemma 5.1. *Let $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a braided Lie crossed module. Then, $\mathcal{M}$ is $\boldsymbol{B}$-perfect if and only if $\mathfrak{U}(\mathcal{M})$ is perfect.*
In fact, we have that if $N = [N, N]$ then $B_N(M) = D_N(M)$.

Proof. We have $N = [N, N]$, and we need to check $B_N(M) = D_N(M)$.

If $\mathfrak{U}(\mathcal{M}) = (M \xrightarrow{\partial} N, \cdot)$ is perfect, then $D_N(M) = M$. Since $D_N(M) \subset B_N(M) \subset M$, we have that $B_N(M) = M$ and $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is $\boldsymbol{B}$-perfect.

On the other hand, since $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ is $\boldsymbol{B}$-perfect, we have that $B_N(M) = M$. So, we only need to prove that $B_N(M) \subset D_N(M)$.

If $\{n, n'\}$ is a generator of $B_N(M)$, and since $N = [N, N]$ by being $\boldsymbol{B}$-perfect, we have that on generators $\{n, n'\} = \{[n_1, n_2], [n'_1, n'_2]\}$,

$$\{n, n'\} = \{[n_1, n_2], [n'_1, n'_2]\} = \{[n_1, n_2], \partial\{n'_1, n'_2\}\} = [n_1, n_2] \cdot \{n'_1, n'_2\} \in D_N(M). \quad \square$$

Lemma 5.2. *Let $(M \xrightarrow{\partial} N, \cdot, \{-,-\}) \xrightarrow{f=(f_1,f_2)} (X_1 \xrightarrow{\delta} X_2, \diamond, (\!(-,-)\!))$ be an extension of braided Lie crossed modules with $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ $\boldsymbol{B}$-perfect. Then, f is a $\boldsymbol{B}$-central extension if and only if f is a $\mathfrak{U}$-central extension.*
In fact, if $N = [N, N]$ then $\mathrm{Z}_B(N) = Z(N) \cap \mathrm{st}_N(M)$.

Proof. If f is a $\mathfrak{U}$-central extension, then $\ker(f_1) \subset M^N$ and $\ker(f_2) \subset \mathrm{st}_N(M) \cap \mathrm{Z}(N)$. We need to prove that $\ker(f_2) \subset \mathrm{Z}_B(N)$.

Let $n \in \mathrm{st}_N(M) \cap \mathrm{Z}(N)$ and $x = [n_1, n_2] \in N = [N, N]$. We have

$$\{n, x\} = \{n, [n_1, n_2]\} = \{n, \partial\{n_1, n_2\}\} = n \cdot \{n_1, n_2\} = 0,$$
$$\{x, n\} = \{[n_1, n_2], n\} = \{\partial\{n_1, n_2\}, n\} = -n \cdot \{n_1, n_2\} = 0.$$

So, $\ker(f_2) \subset \mathrm{Z}_B(N)$ and f is a $\boldsymbol{B}$-central extension. $\square$

Theorem 5.3. *Let $\mathcal{M} = (M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a $\boldsymbol{B}$-perfect braided Lie crossed module. Then its universal $\boldsymbol{B}$-central extension $\mathcal{U} \xrightarrow{\Phi} \mathcal{M}$ and its universal $\mathfrak{U}$-central extension $\mathcal{V} \xrightarrow{c} \mathcal{M}$ are isomorphic.*

Proof. Since $\mathcal{V} \xrightarrow{c} \mathcal{M}$ is a $\mathfrak{U}$-central extension, we know using Lemmas 5.1 and 5.2 (by hypothesis $\mathcal{M}$ is $\boldsymbol{B}$-perfect) that it is a $\boldsymbol{B}$-central extension, and using the universality of $\mathcal{U}$, there is a unique morphism $\mathcal{U} \xrightarrow{h} \mathcal{V}$ such that $\Phi = c \circ h$.

Since $\mathcal{U} \xrightarrow{\Phi} \mathcal{M}$ is a $\boldsymbol{B}$-central extension is also a $\mathfrak{U}$-central extension, and so by the universality of $\mathcal{V}$, there exists a unique morphism $\mathcal{V} \xrightarrow{h'} \mathcal{U}$ such that $c = \Phi \circ h'$.

Using the universality of $\mathcal{U}$, since $\Phi \circ (h' \circ h) = c \circ h = \Phi$, we get that $h' \circ h = \mathrm{Id}_{\mathcal{U}}$.

By the same arguments using the universality of $\mathcal{V}$, we have that $h \circ h' = \mathrm{Id}_{\mathcal{V}}$. $\square$

Corollary 5.4.

(i) *Let $(M \xrightarrow{\partial} N, \cdot, \{-,-\})$ be a $\boldsymbol{B}$-perfect braided Lie crossed module. Then $N \otimes M \simeq N \otimes N$.*

(ii) *If M is a perfect Lie K-algebra, then $M \otimes (M \otimes M) \simeq M \otimes M$.*

Proof. (i) By Theorem 5.3, (UBCE) and (UUCE) are isomorphic. Therefore $N \otimes M \simeq N \otimes N$.

The isomorphism can be described explicitly using Proposition 3.4 and Proposition 4.3, and it is given by: $h_1 \colon N \otimes N \to N \otimes M$, $n \otimes n' \mapsto n \otimes \{n'_1, n'_2\}$, with $n' = [n'_1, n'_2]$, and $h_1^{-1} \colon N \otimes M \to N \otimes N$, $n \otimes m \mapsto n \otimes \partial(m)$.

(ii) If M is perfect, then the braiding Lie crossed module $(M \otimes M \xrightarrow{\partial} M, \cdot, \{-,-\})$ is $\boldsymbol{B}$-perfect (see Example 2.10 (2)), since $M \otimes M$ is generated by $m_1 \otimes m_2 = \{m_1, m_2\}$. By (i), we have $M \otimes (M \otimes M) \simeq M \otimes M$.

In this case, the isomorphism is described by: $h_1 \colon M \otimes M \to M \otimes (M \otimes M)$, $m \otimes m' \mapsto m \otimes (m'_1 \otimes m'_2)$, with $m' = [m'_1, m'_2]$, and $h_1^{-1} \colon M \otimes (M \otimes M) \to M \otimes M$, $m \otimes (m' \otimes m'') \mapsto m \otimes [m', m'']$. ☐

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