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The continuity of solution set of a multivalued equation and applications in control problem

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Abstract

In this paper, we prove the existence, unbounded continuity of positive set for a multivalued equation containing a parameter of the form $x \in A \circ F(\lambda, x)$ and give applications in the control problem with multi-point boundary conditions and second order derivative operator

$$\begin{cases} u''(t) + g(\lambda, t) f(u(t)) = 0, \ t \in (0, 1), \\ g(\lambda, t) \in F(\lambda, u(t)) \text{ a.e. on } J \\ u(0) = 0, u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i) \end{cases} \tag{1}$$

Keywords: multivalued operator, multivalued equation, fixed point index, control problem.

1. Introduction

The single-valued equation of the form $x = F(\lambda, x)$ in ordered spaces has been studied for a long time by many mathematician researchers and has found many successful results (see [2, 3, 4, 11, 13, 17]). It was naturally generalized to multivalued form

$$x \in F(\lambda, x). \tag{2}$$

There are many good methods approaches available, among them are principal eigenvalue - eigenvector method (see impressive results of J. R. L. Webb and K. Q. Lan in [17], Guy Degla in [2]), monotone minorant method [7, 8], the method of using the definition of topological degree (the fixed point index) for

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single-valued/multivalued mappings [1, 5, 9, 12, 14, 15, 16] and the method of combining two latter methods [8].

The solution set of (2) is well-known in two following forms

$$S = \{(\lambda, x) : x \in F(\lambda, x)\}, \tag{3}$$

or

$$S = \{x : \exists \lambda, x \in F(\lambda, x)\}. \tag{4}$$

In this paper, we prove the existence, unbounded continuity of the solution set S of the form (4), with the equation $x \in A \circ F(\lambda, x)$, where F is a multivalued function containing a parameter λ and A is a linear mapping. We establish sufficient conditions for the set S to be unbounded continuous branch and emanating from zero, i.e., it contains the elements of S on the boundary of any set Ω, which is open, bounded, and contains zero.

Our method is to combine the method of using the definition of topological degree for multivalued mapping and the method of evaluating solutions. The others authors using only one. This is the fundamental difference between our work and the authors mentioned above.

By the abstract result obtained, we apply for the control problem with second order derivative and multi-point boundary conditions. The such problem has attracted the increasing attention of many researchers. We will solve this problem to illustrate the method.

The paper is organized as follows. In Section 2, we recall some notations and useful lemma. In Section 3, the main results are stated. In Section 4, we present the existence of solutions to the control problem.

2. Preliminaries

Let $(E, K, \|.\|)$ be a real Banach space ordering by the cone K, i.e., K is a closed convex subset of E such that $\lambda K \subset K$ for $\lambda \geq 0$, $K \cap (-K) = \{0\}$, and $x \leq y$ iff $y - x \in K$ for $x, y \in X$. For nonempty subsets A, B of E we write $A \succcurlyeq_2 B$ (or, $B \preccurlyeq_2 A$) iff for every $x \in A$, there exists $y \in B$ satisfying $x \geq y$ (or, $y \leq x$) and we also write $A \preccurlyeq_1 B$ iff for every $x \in A$, there exists $y \in B$ such that $x \leq y$. The cone K is said to be normal if there exists a constant $N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq N\|y\|$. Throughout this article we always assume that K is normal cone with $N = 1$. For $A \subset E$, the all nonempty closed convex (resp., closed) subsets of A is denoted by $cc(A)$ (resp., $c(A)$). Let Ω be an open subset of E, denote $\Omega_K = K \cap \Omega$, $\partial_K \Omega = K \cap \partial\Omega$ and $\dot{K} = K \backslash \{0\}$, where $\partial\Omega$ is the boundary of Ω in E. A mapping $T : K \cap \overline{\Omega} \to cc(K)$ is said to be compact iff $T(B)$ is relatively compact for any bounded subset B of $K \cap \overline{\Omega}$, where $T(B) = \cup_{x \in B} T(x)$. T is called upper semicontinuous (in short, u.s.c.) if $\{x \in K \cap \overline{\Omega} : T(x) \subset W\}$ is open in $K \cap \overline{\Omega}$ for every open subset W of K. Further, if $x \notin T(x)$ for all $x \in \partial_K \Omega$, the fixed point index of T in Ω with respect to K is defined which is an integer denoted by $i_K(T, \Omega)$ (see e.g. [5]). The following lemma on the computation of the index were taken in [5, proof of Theorem 3.2].

Lemma 2.1. [8, 5, proof of Theorem 3.2] *Let $T : K \cap \overline{\Omega} \to cc(K)$ be an u.s.c. compact multivalued operator. Then*

1. $i_K(T, \Omega) = 0$ *if there exists* $u \in \dot{K}$ *such that* $x \notin T(x) + ku$ *for all* $x \in \partial_K \Omega$ *and* $k \geq 0$.
2. $i_K(T, \Omega) = 1$ *if* $kx \notin T(x)$ *for all* $k \geq 1$.

We review the results using to prove our abstract results.

Lemma 2.2. [6, Proposition 2.22] *Assume that $T : D \subset E \to c(E)$ is an u.s.c. multivalued operator and a net $(x_\epsilon, y_\epsilon) \to (x, y)$ with $y_\epsilon \in T(x_\epsilon)$. Then $y \in T(x)$.*

Lemma 2.3. [5, Theorem 2.1] *Assume that multivalued operator $H : [0, 1] \times K \cap \overline{\Omega} \to cc(K)$ is u.s.c. compact satisfying $x \notin H(t, x)$ for all $(t, x) \in [0, 1] \times \partial_K \Omega$. Then, $i_K(H(0, .), \Omega) = i_K(H(1, .), \Omega)$.*

3. Abstract results

Lemma 3.1. *Let $T : [0, \infty) \times K \to cc(K)$ be an u.s.c compact operator and $\Omega \ni 0$ be an open bounded subset of E. Assume that the following conditions are satisfied*

1. *$tx \in T(0, x)$ for some $x \in \dot{K}$ implies $t < 1$,*
2. *there exists $\lambda_0 > 0$ such that $i_k(T(\lambda, .), \Omega) = 0$ for all $\lambda \geq \lambda_0$.*

Then the set $\{x \in \partial_K\Omega : \exists\lambda > 0, x \in T(\lambda, x)\}$ is nonempty.

Proof. We define

$$\alpha = \sup\{\lambda > 0 : i_K(T(\lambda, .), \Omega) \neq 0\}$$

It is clear that $\alpha > 0$. Indeed, assume that the following assertion holds

$$\forall\epsilon > 0, \exists(t_\epsilon, x_\epsilon) \in [0, 1] \times \partial_K\Omega : x_\epsilon \in (1 - t_\epsilon)T(\epsilon, x_\epsilon) + t_\epsilon T(0, x_\epsilon). \tag{5}$$

Since T is compact, without loss of generality we may assume that $t_\epsilon \to t, x_\epsilon \to x$ when $\epsilon \to 0$. From (5) by Lemma 2.2 it follows that

$$x \in (1 - t)T(0, x) + tT(0, x) \subset T(0, x).$$

This contradicts the first condition. Thus, there exists $\epsilon > 0$ such that $(t, x) \notin H(t, x)$ for all $(t, x) \in [0, 1] \times \partial_K\Omega$, where

$$H(t, x) = (1 - t)T(\epsilon, x) + tT(0, x).$$

Using Lemma 2.3 we have

$$i_K(T(0, .), \Omega) = i_K(T(\epsilon, .), \Omega).$$

By Lemma 2.1 from the first condition it follows $i_K(T(0, .), \Omega) = 1$. Thus $i_K(T(\epsilon, .), \Omega) = 1$, we deduce $\lambda_0 > \alpha \geq \epsilon > 0$.
Next, for any $\epsilon \in (0, \alpha)$, there exists $\lambda_\epsilon \in (\alpha - \epsilon, \alpha]$ with $i_K(T(\lambda_\epsilon, .), \Omega) \neq 0$. Consider multivalued operator H_ϵ defined by

$$H_\epsilon(t, x) = (1 - t)T(\lambda_\epsilon, x) + tT(\alpha + \epsilon, x).$$

Now, we prove

$$\{x \in \partial_K\Omega : \exists\lambda > 0, x \in T(\lambda, x)\} \neq \emptyset.$$

Assume on the contrary, that

$$\{x \in \partial_K\Omega : \exists\lambda > 0, x \in T(\lambda, x)\} = \emptyset. \tag{6}$$

Then, the fixed point index of $T(\alpha + \epsilon, .)$ is well defined and it is equal 0 from the definition of α. If

$$x \notin H_\epsilon(t, x) \text{ for all } (t, x) \in [0, 1] \times \partial_K\Omega, \tag{7}$$

by Lemma 2.3 we obtain

$$i_K(T(\lambda_\epsilon, .), \Omega) = i_K(T(\alpha + \epsilon, .), \Omega). \tag{8}$$

This is a contradiction. Therefore (7) is impossible, i.e., there is $(t_\epsilon, x_\epsilon) \in [0, 1] \times \partial_K\Omega$ satisfying

$$x_\epsilon \in (1 - t_\epsilon)T(\lambda_\epsilon, x_\epsilon) + t_\epsilon T(\alpha + \epsilon, x_\epsilon). \tag{9}$$

By an argument analogous to the previous one we can find $x \in \partial_K\Omega$ with $x \in T(\alpha, x)$. This contradicts (6). The proof is complete. □

Let $(Y, K_Y, \|.\|_Y)$ be a Banach space, ordered by normal cone K_Y. Suppose that $E \subset Y, K \subset K_Y \cap E$, embedding $(E, \|.\|) \hookrightarrow (Y, \|.\|_Y)$ is continuous, and $F : [0, \infty) \times K \to cc(K_Y)$ is u.s.c. compact multivalued operator. Let $A : Y \to X$ be a compact linear operator satisfying $A(K_Y) \subset K$.

Theorem 3.1. *Assume that the following conditions are satisfied*

1. *$kx \in A \circ F(0, x)$ for some $x \in \dot{K}$ implies $k < 1$;*
2. *there are positive numbers a, b, c and a linear function $L : Y \to \mathbb{R}_+$ with $L(y) \neq 0$ for some $y \in K$ such that*

(a) $LAx \succcurlyeq_2 \{aLx\}$ and $LAx \succcurlyeq_2 \{a.\|Ax\|_Y\}$ for all $x \in K_Y$,

(b) $L(F(\lambda, x)) \succcurlyeq_2 \{b\lambda Lx - c\}$ for all $x \in K$, and

(c) there exists a function $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ increasing on the second variable with

$$\lim_{\lambda\to\infty} h(\lambda, \frac{c}{ab\lambda - 1}) = 0 \tag{10}$$

such that $(k, \lambda, x) \in [0, 1] \times [0, \infty) \times K$ with

$$x \in kA \circ F(\lambda, x) + (1 - k)b\lambda Ax \tag{11}$$

implies

$$\|x\| \leq h(\lambda, \|x\|_Y). \tag{12}$$

Then, $S = \{x \in \dot{K} : \exists\lambda > 0, x \in A \circ F(\lambda, x)\}$ is unbounded continuous branch emanating from 0.

Proof. Let $\Omega \ni 0$ be an open bounded subset of E. We will apply Lemma 3.1 with $T(\lambda, x) = A \circ F(\lambda, x)$ to prove $S \cap \partial_K\Omega \neq \emptyset$. Clearly, the condition 1. of the lemma holds. Assume that $(k, \lambda, x) \in [0, 1] \times [0, \infty) \times K$ satisfied (11), it is obvious that $x \in A[kF(\lambda, x) + (1 - k)b\lambda x]$, hence $x = A[ky_\lambda + (1 - k)b\lambda x]$ for some $y_\lambda \in F(\lambda, x)$. From 2(a) and 2(b) applying the operator L we have

$$Lx \geq aL(ky_\lambda + (1 - k)b\lambda x) \geq a(b\lambda Lx - c), \tag{13}$$

$$Lx \geq a\|A[ky_\lambda + (1 - k)b\lambda x]\|_Y = a\|x\|_Y. \tag{14}$$

We always assume λ is sufficiently large, from (13) and (14) if follows that

$$\|x\|_Y \leq \frac{c}{ab\lambda - 1}$$

which together with (12) gives

$$\|x\| \leq h(\lambda, \frac{c}{ab\lambda - 1}). \tag{15}$$

If $x \in \partial_K\Omega$, $b\|x\| > \epsilon > 0$ for some ϵ. From (15), (11) and (12) it follows that

$$x \notin H(k, x) \text{ for all } (k, x) \in [0, 1] \times \partial_K\Omega, \tag{16}$$

where $H(k, x) = kA \circ F(\lambda, x) + (1 - k)b\lambda Ax$, partially, $x \neq b\lambda Ax$.

Applying Lemma 2.3 we obtain $i_K(T(\lambda, .\Omega)) = i_K(\lambda bA, \Omega)$. Choose $u \in \dot{K}$ with $Lu > 0$. We now prove that

$$x \neq b\lambda Ax + su \text{ for all } (s, x) \in [0, \infty) \times \partial_K\Omega. \tag{17}$$

Assume on the contrary, that exists $(s, x) \in [0, \infty) \times \partial_K\Omega$ satisfying

$$x = b\lambda Ax + su. \tag{18}$$

This implies $s > 0$. Acting the operator L to both sides of (18) from the condition 2(b) we obtain $(1 - ab\lambda)Lx \geq su$, this is impossible. Therefore, from Lemma 2.1 it follows $i_K(b\lambda A, \Omega) = 0$. We deduce $i_K(T(\lambda, .), \Omega) = 0$. The proof is complete. □

4. Applications

Let $F : [0,\infty) \times \mathbb{R}_+ \to cc(\mathbb{R}_+)$ be an u.s.c. compact multivalued operator and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function. Denote $J = [0,1]$. We consider control problem which contains a parameter of the form

$$\begin{cases} u''(t) + g(\lambda,t)f(u(t)) = 0,\ t \in (0,1), \\ g(\lambda,t) \in F(\lambda,u(t)) \text{ a.e. on } J \\ u(0) = 0, u(1) = \sum_{i=1}^m \alpha_i u(\eta_i) \end{cases} \tag{19}$$

where, $0 < \eta_i < 1$, $\alpha_i \geq 0$, $\sum_{i=1}^m \alpha_i\eta_i < 1$.
Denote $\Lambda = \sum_{i=1}^m \alpha_i\eta_i$. for every $(t,s) \in [0,1] \times [0,1]$, we define

$$h(t,s) = \begin{cases} s(1-t), s \leq t, \\ t(1-s), s > t. \end{cases}$$

$$G(t,s) = \frac{t}{1-\Lambda}\sum_{i=1}^m \alpha_i h(\eta_i,s) + h(t,s);$$

Let $C(J)$ and $C^1(J)$, resp., be the Banach spaces of all continuous and continuous differentiable function on J. Denote $E = \left\{x \in C^1(J) : x(0) = 0\right\}$, and $Y = \{x \in C(J) : x(0) = 0\}$. Let $A : Y \to E$ be a compact linear operator defined by

$$A(u)(t) = \int_0^1 G(t,s)u(s)ds, t \in J \tag{20}$$

Instead of solving problem (20) we shall consider its equivalent form

$$x \in A \circ T(\lambda,x), \tag{21}$$

where the multivalued operator T is defined by

$$T(\lambda,x)(t) = F[\lambda,x(t)]f[x(t)],\ t \in J.$$

Theorem 4.1. *Let $\rho = \left\{\sup_{t\in J}\int_0^1 G(t,s)ds\right\}^{-1}$. Assume that there exist numbers $\alpha > 0, \beta > 0, \gamma \in (0,\rho)$ and $r \in (0,2)$ such that*

1. $F(0,x)f(x) \preccurlyeq_1 \gamma x\ \forall x > 0,$
2. $\alpha\lambda x - \beta \preccurlyeq_2 F(\lambda,x)f(x),$
3. $F(\lambda,x) \preccurlyeq_1 1 + \lambda^{\frac{r}{2}}|x|^r$ *for all* $(\lambda,x) \in (0,+\infty) \times \mathbb{R}_+$.

Then, the set S of positive solutions for (21) is unbounded continuous in $C^1(J)$, emanating from 0.

Proof. We shall apply Theorem 3.1 with the cone

$$K = \{x \in E : x(t) \geq 0\ \forall t \in J\},$$

the cone

$$K_Y = \{x \in Y : x(t) \geq 0\ \forall t \in J\}.$$

Then, Y and E, resp., are Banach spaces with the norms

$$\|x\|_Y = \sup_{t\in J}|x(t)|$$

and

$$\|x\| = \|x'\|_Y.$$

Suppose $x \in \dot{K}$ and k satisfies $kx \in A \circ T(0,x)$, we can find $u(s) \in F(0,x(s))$ such that

$$\begin{aligned}|kx(t)| &= \left|\int_0^1 G(t,s)u(s)f(x(s))ds\right|,\\ &\leq \gamma\|x\|_Y \left|\int_0^1 G(t,s)ds\right|\\ &\leq \|x\|_Y \ \forall t \in J.\end{aligned}$$

This implies $k < 1$. From the well-known results in [17], the compact linear operator A has an eigen-value $\mu_0 > 0$ with respect to a positive eigen-function u_0. We define the linear operator L on Y, by $Lx = \int_0^1 x(s)u_0(s)ds$. From the condition 2, we have

$$\begin{aligned}L(T(\lambda,x)) &\succcurlyeq_2 \int_0^1 (\alpha\lambda x(s) - \beta)u_0(s)ds\\ &\geq \alpha\lambda Lx - c,\end{aligned}$$

where $c = \beta\int_0^1 u_0(s)ds$. If y is non-negative continuous concave function on J satisfying $y(0) = 0$ and $y(1) \geq 0$, there exists number $\xi > 0$ such that $y(t) \geq \xi\|y\|_Y u_0(t)$ on J. For $x \in K_Y$, Ax is concave function with $Ax(0) = 0$ and $Ax(1) \geq 0$, we have $Ax(t) \geq \xi\|Ax\|_Y u_0(t)$. From Fubini's Theorem it follows that

$$\begin{aligned}L(Ax) &= \int_0^1 \left(\int_0^1 G(t,s)x(s)ds\right)u_0(t)dt\\ &= \int\int_{J\times J} G(t,s)x(s)u_0(t)dsdt\\ &= \int_0^1 \left(\int_0^1 G(t,s)u_0(t)dt\right)x(s)ds\\ &= \int_0^1 Au_0(s)x(s)ds\\ &= \mu_0\int_0^1 u_0(s)x(s)ds\\ &= \mu_0 Lx.\end{aligned}$$

Consequently, there is constant $a > 0$ satisfying

$$L(Ax) \geq aLx \text{ and } L(Ax) \geq a\|Ax\|_Y. \tag{22}$$

Now, assume $(k,\lambda,x) \in [0,1]\times[0,\infty)\times K$ with

$$x \in kA \circ T(\lambda,x) + (1-k)\alpha\lambda Ax. \tag{23}$$

This implies

$$-x'' \in kT(\lambda,x) + (1-k)\alpha\lambda x. \tag{24}$$

In the following the numbers $m_j, j = 0,1,2,..,6$ and m are constant numbers, not depending on λ, x and $t \in J$. By a similar argument as the proof of Theorem 3.1 we obtain

$$\|x\|_Y \leq \frac{c}{a\alpha\lambda - 1}. \tag{25}$$

Therefore we can choose m_1 such that

$$\lambda\|x\|_Y \leq m_1 \tag{26}$$

From (26), the well-known inequality

$$\|x'\|_Y^2 \leq m_2\|x\|_Y.\|x''\|_Y. \tag{27}$$

and (24) we obtain

$$\|x''\|_Y \leq m_3(1+\lambda^{\frac{r}{2}}\|x\|_Y^r)+\alpha m_1 \tag{28}$$
$$\leq m_4(1+\lambda^{\frac{r}{2}}\|x\|_Y^r). \tag{29}$$

Further, for $x \in K$, we have $\|x\|_Y \leq m_0\|x'\|_Y$. Combining this inequality, (26), (27), (29) and (29) we have

$$\|x''\|_Y \leq m_5(1+\|x''\|_Y^{\frac{r}{2}}) \leq m_6. \tag{30}$$

From (27) we can choose m such that $\|x'\|_Y \leq m\|x\|_Y^{\frac{1}{2}}$. Since $\|x\| = \|x'\|_Y$, the condition (2c) of Theorem 3.1 are satisfied with function $h(\lambda, t) = mt^{\frac{1}{2}}$. $\square$

5. Conclusion

In this paper, the unbounded continuity of positive solution set for a multivalued equation containing a parameter has established and given the application in the control problem with multi-point boundary conditions.

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