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Research Article

Johns modules and quasi-Johns modules

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Abstract

A right Johns ring is a right Noetherian ring in which every right ideal is a right annihilator. It is known that in a Johns ring R the Jacobson radical $J(R)$ of R is nilpotent and $\operatorname{Soc}(R)$ is an essential right ideal of R. Moreover, every right Johns ring R is right Kasch, that is, every simple right R-module can be embedded in R. For a $M \in R$-Mod we use the concept of M-annihilator and define a *Johns module* (resp. *quasi-Johns*) as a Noetherian module M such that every submodule is an M-annihilator. A module M is called *quasi-Johns* if any essential submodule of M is an M-annihilator and the set of essential submodules of M satisfies the ascending chain condition. In this paper we extend classical results on Johns rings, as those mentioned above and we also provide new ones. We investigate when a Johns module is Artinian and we give some information about its prime submodules.

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1. Introduction

In [13] Baxter Johns studies right Noetherian rings in which every right ideal is a right annihilator. In [13, Theorem 1] it is stated that if a ring R is right Noehterian and every right ideal is a right annihilator then R is right Artinian. Fifteen years later, in [10] Faith and Menal give a counterexample to the theorem of Johns [13, Theorem 1]. Faith and Menal's example is a trivial extension of a non Artinian right Noetherian domain A and a simple A-module W. Faith and Menal define a *right Johns ring* as a right Noetherian ring in which every right ideal is a right annihilator. A Johns ring R shares nice properties with Artinian rings, namely the Jacobson radical $J(R)$ of R is nilpotent [13, Lemma 1] and $\operatorname{Soc}(R_R)$ is an essential right ideal of R [13, Lemma 4]. Moreover, every right Johns ring R is a right Kasch ring, that is, every simple right R-module can be embedded in R. Also, it can be shown that for a right Johns ring R, the factor ring $R/J(R)$ is a right V-ring [10, Theorem 2.3]. Some positive answers to the Johns statement, regarding when

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a right Johns ring is right Artinian are also presented in [10]. For example, if R is a right Johns ring with finite left uniform dimension or $R/J(R)$ is semisimple Artinian (i.e. semilocal) then R is right Artinian [10, Proposition 3.3]. In 2012, L. Shen studied those rings whose their right essential ideals are right annihilators and satisfy acc, and called them *quasi-Johns rings* [18].

The aim of this paper is to translate the concept of Johns rings to modules and extend the results given by Johns, Faith and Menal, and Shen as well as to provide new results in the module theoretic context. For, given an R-module M, we will consider *M-annihilators*, that is, submodules of the form $\bigcap\{\operatorname{Ker} f \mid f \in X\}$ with X a subset of $\operatorname{End}_R(M)$. So, we define a *Johns module* as a Noetherian module M such that every submodule is an M-annihilator. We also make use of *a product of modules*. This product is defined as follows: Let M and N be two modules and let $K \leq M$. The product of K with N is given by

$$K_M N = \sum\{f(K) \mid f \in \operatorname{Hom}_R(M, N)\}.$$

This product extends the product of an ideal of a ring R with an R-module. In particular, this product defines a product of submodules of a given module. The product $-_M-$ in general is not associative nor distributive over sums from the left. General properties of this product are listed in [8, Proposition 1.3]. In order to get a nice product, that is, associative and distributive, of submodules of a module M, we impose on M the condition of being projective in $\sigma[M]$, where $\sigma[M]$ is the full subcategory of R-Mod consisting of all M-subgenerated modules. With the above product, given a module M, it is possible to define prime, semiprime and nilpotent submodule, in a natural way. Very close to Johns modules are Kasch modules. A module M is *Kasch* if every simple module in $\sigma[M]$ can be embedded in M. These modules extend the concept of left Kasch ring and were introduced in [1]. We will provide new characterizations of Kasch modules and show that under some circumstances, Kasch modules and Johns modules coincide.

The paper is divided in five sections. The first section is this introduction and in Section 2 we present some preliminaries needed for the development of the other sections. In Section 3, we are interested in to give conditions on a module M in order to know when M is Artinian. Also, we investigate under which conditions an Artinian module is Noetherian and vice-versa. For example, for an Artinian module M projective in $\sigma[M]$, if $\operatorname{Rad}(M)$ has finite length, then M is Noetherian (Corollary 3.9). Also, the concepts of being Artinian and Noetherian coincide in a module M projective and generator of $\sigma[M]$ such that $\operatorname{Rad}(M)$ is a nilpotent submodule and $M/\operatorname{Rad}(M)$ is semisimple (Proposition 3.15). Section 4 is devoted to study those modules M whose all their submodules are M-annihilators. The section starts first with those modules whose their essential submodules are M-annihilators. We prove that the non M-singular modules such that every essential submodule is an M-annihilator are precisely the semisimple modules (Proposition 4.4). If M is a module such that every essential submodule is an M-annihilator, then $\operatorname{Rad}(M)$ and $\operatorname{Soc}(M)$ are M-annihilators (Corollary 4.8). We call a module M *annular* if every submodule of M is an M-annihilator. It is proved that an annular module M is a Kasch module (Corollary 4.22), and the concepts coincide provided that M is M-injective (Proposition 4.24). In the last section, Section 5, we introduce the quasi-Johns and Johns modules. Some examples are given and some Johns', Faith and Menal's and Shen's results are extended. We show that (essential) fully invariant submodules of (quasi-) Johns modules inherit the property (Proposition 5.3 and Proposition 5.9). For a quasi-projective Johns module M, it is proved that the lattice of fully invariant submodules of M has finite length (Proposition 5.16), as a consequence, $\operatorname{Rad}(M)$ is a nilpotent submodule of M (Corollary 5.17). Moreover, every commutative Johns ring is Artinian (Corollary 5.19). It is also proved that for a Johns module M, $M/\operatorname{Rad}(M)$ is cosemisimple provided that M is quasi-projective and generates all its submodules. As a corollary, the spectrum of M (the set of prime submodules) is finite and consists of the annihilators of the simple modules in $\sigma[M]$ (Corollary 5.27). We finish the paper with some conditions on a Johns module M implying that M is Artinian (Theorem 5.29 and Proposition 5.34). As consequences, we prove that if every maximal left ideal of a Johns ring R is two-sided then R is left Artinian (Corollary 5.31), and if R is a left fully bounded Johns ring, then R is left Artinian (Corollary 5.33).

Throughout this paper R will denote an associative ring with unit and all modules will be left R-modules. The category of all R-modules is denoted by R-Mod. The homomorphisms will act from the left and will commute with the elements of R. The notations $N \leq M$ and $N \leq^{\text{ess}} M$ will stand for N is a submodule of M and N is an essential submodule of M, respectively. $\mathrm{Rad}(M)$ and $\mathrm{Soc}(M)$ denote the radical (intersection of all maximal submodules) and socle (sum of all simple submodules) of M respectively.

2. Preliminaries

Let M and N be left R-modules. It is said that M is *N-projective* if for any epimorphism $\rho : N \to L$ and any homomorphism $\alpha : M \to L$, there exists $\overline{\alpha} : M \to N$ such that $\rho\overline{\alpha} = \alpha$. It is true that if M is N_i-projective for a finite family of modules $\{N_1, ..., N_\ell\}$, then M is $\bigoplus_{i=1}^{\ell} N_i$-projective [19, 18.2]. In general, this is not true for arbitrary families. A module M is called *quasi-projective* if, M is M-projective. In most of the results in this paper, we will assume that M *is projective in the category* $\sigma[M]$, where $\sigma[M]$ is the category consisting of all M-subgenerated modules. This hypothesis is satisfied by every finitely generated quasi-projective module for example [19, 18.2]. The condition that M is projective in $\sigma[M]$ is equivalent to say that M is $M^{(\Lambda)}$-projective for every index set Λ [19, 18.3]. The category $\sigma[M]$ is a Grothendieck category where the subobjects, factors and coproducts coincide with those in R-Mod. The *M-injective hull* $E^{[M]}(N)$ of a module $N \in \sigma[M]$ can be described as the trace of M in the injective hull of N in R-Mod. For the module M, we will write $\widehat{M}$ instead of $E^{[M]}(M)$. Finally, consider a generator U of $\sigma[M]$ and $\{N_i\}_{i\in I}$ a family of modules in $\sigma[M]$. *The product* of the family $\{N_i\}_{i\in I}$ in $\sigma[M]$ is given by the trace of U in the product of the family taken in R-Mod.

Definition 2.1. Let M be a module and $L, K \leq M$. *The product of K with L in M* is defined as

$$K_M L = \sum\{f(K) \mid f \in \mathrm{Hom}_R(M, L)\}.$$

If M is projective in $\sigma[M]$, this product gives an associative operation in the lattice of submodules of M. Moreover, if N and K are fully invariant submodules of M, then $N_M K$ is fully invariant. For $N \leq M$, *the powers of N* are defined recursively as follows: $N^1 = N$ and $N^{\ell+1} = N^{\ell}{}_M N$. † It is said that a submodule $N \leq M$ is *nilpotent* if $N^\ell = 0$ for some $\ell > 0$. Some general properties of this product are listed in [8, Proposition 1.3]. Using this product, prime and semiprime submodules can be defined in a natural way, as follows:

Definition 2.2. Let N be a proper fully invariant submodule of a module M.

(1) It is said that N is a *prime submodule* (of M) if for any fully invariant submodules $K, L \leq M$ such that $K_M L \leq N$, then $K \leq N$ or $L \leq N$. It is said that M is a *prime module* if 0 is a prime submodule.
(2) It is said that N is a *semiprime submodule* (of M) if for any fully invariant submodule $K \leq M$ such that $K_M K \leq N$, then $K \leq N$. It is said that M is a *semiprime module* if 0 is a semiprime submodule.

It is clear that an intersection of prime submodules is a semiprime submodule. If M is projective in $\sigma[M]$, in [19, 22.3] it is shown that $\mathrm{Rad}(M) \neq M$, that is, M has maximal

†Do not confuse the notation with the direct product of $\ell+1$ copies of N which will be denoted by $N^{(\ell+1)}$.

submodules. Moreover, if M is projective in $\sigma[M]$, then $\operatorname{Rad}(M)$ is a semiprime submodule by [14, Propositions 3.4 and 3.6].

Since there is a product of submodules, it is natural to define an annihilator. Given a module M and a submodule $N \leq M$, the *annihilator of N in M* is defined as the submodule $\operatorname{Ann}_M(N) = \bigcap\{\operatorname{Ker} f \mid f \in \operatorname{Hom}_R(M, N)\}$. It can be seen that $\operatorname{Ann}_M(N)$ is a fully invariant submodule of M and it is the greatest submodule of M such that $\operatorname{Ann}_M(N)_M N = 0$ [3].

3. Some results of Artinian and Noetherian modules

In this section, using the product of submodules of a given module M, we are interested in some conditions which imply that the module M is Artinian and also when being Artinian and Noetherian coincide for the module M. We start with the following easy lemma.

Lemma 3.1. *Let M be an R-module. Let $N \in \sigma[M]$ such that there exists a chain of submodules*

$$0 = A_{n+1} \subseteq A_n \subseteq \cdots \subseteq A_2 \subseteq A_1 = N$$

whose quotients A_i/A_{i+1} have finite length. Then N is Artinian if and only if N is Noetherian.

Proof. Suppose N is Artinian. Since every quotient A_i/A_{i+1} has finite length then they are Noetherian. Consider the short exact sequence

$$0 \to A_n \to A_{n-1} \to A_{n-1}/A_n \to 0$$

Since A_n and A_{n-1}/A_n are Noetherian then A_{n-1} is Noetherian. Continuing with this process M is Noetherian. $\square$

Proposition 3.2. *Let M be an R-module such that any submodule has a maximal submodule. If M is Artinian then M is Noetherian.*

Proof. Consider the descending chain

$$M \geq \operatorname{Rad}(M) \geq \operatorname{Rad}(\operatorname{Rad}(M)) \geq \cdots$$

By hypothesis, this is a strict descending chain. Since M is Artinian, the chain must stop in finitely many steps,

$$M > \operatorname{Rad}(M) > \operatorname{Rad}(\operatorname{Rad}(M)) > \cdots > \operatorname{Rad}^n(M) > 0.$$

It follows from [19, 31.2] that every quotient of this chain is a semisimple module. Thus, by Lemma 3.1 M is Noetherian. $\square$

Recall that a ring R is left *Max* if every left R-module has a maximal submodule. In particular, every left perfect ring is left Max.

Corollary 3.3. *Let M be an R-module over a left Max ring R. If M is Artinian then M is Noetherian.*

Proposition 3.4. *Let M be a quasi-projective module and N be a fully invariant submodule of M. If $M/N \cong \bigoplus_{i\in I} S_i$ is an homogeneous semisimple module, then N is a prime submodule.*

Proof. It is not difficult to see that an homogeneous semisimple module is a prime module, in fact, it has no nontrivial fully invariant submodules. It follows from [16, Proposition 18] that N is a prime submodule. $\square$

Corollary 3.5. *Let M be a quasi-projective module. If $M/\operatorname{Rad}(M)$ is homogeneous semisimple, then $\operatorname{Rad}(M)$ is a prime submodule.*

The following example shows that Proposition 3.4 might not be true if the module is not quasi-projective.

Example 3.6. Consider the ring $R = \mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$, the trivial extension of the ring $\mathbb{Z}_2$ by the $\mathbb{Z}_2$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The ring R is a finite local ring, hence there is only one simple R-module up to isomorphism, say S. Let M denote an injective hull of S in R-Mod. Following the notation in [8, Example 1.12], $\{S, N, K, L\}$ are the proper fully invariant submodules of M. In [5, Sec. 3, Ex. 4] it is proved that those submodules are all the proper submodules of M. We have that $\operatorname{Rad}(M) = S$ and $M/\operatorname{Rad}(M) \cong S \oplus S$. On the other hand, $K_M L = S$ but $K \nsubseteq S$ neither $L \nsubseteq S$. Thus $\operatorname{Rad}(M)$ is not a prime submodule. Note that M is not quasi-projective.

Remark 3.7. Recall that a module M is said to be *retractable* if $\operatorname{Hom}_R(M, N) \neq 0$ for all $0 \neq N \leq M$. It can be seen that every semiprime module M projective in $\sigma[M]$, is retractable [5, Lemma 1.24].

Proposition 3.8. *Let M be projective in $\sigma[M]$ and Artinian. If N is a semiprime submodule of M and has finite length then M is Noetherian.*

Proof. If $N = 0$ then M is an Artinian semiprime module, then M is semisimple by [5, Theorem 1.17]. Suppose $0 \neq N$. Since N is a semiprime submodule, by [17] M/N is a semiprime module, so again by [5, Theorem 1.17] M/N is semisimple. Hence M has a chain $0 \leq N \leq M$ with quotients of finite length. By Lemma 3.1 M is Noetherian. □

Corollary 3.9. *Let M be projective in $\sigma[M]$ and Artinian. If* $\operatorname{Rad}(M)$ *has finite length then M is Noetherian.*

Definition 3.10. A module M is called *semiprimary* if $\operatorname{Rad}(M)$ is nilpotent and $M/\operatorname{Rad}(M)$ is semisimple.

Remark 3.11. Let M and N be R-modules. Since Rad is a preradical, then $f(\operatorname{Rad}(M)) \leq \operatorname{Rad}(N)$ for all $f \in \operatorname{Hom}_R(M, N)$. Thus $\operatorname{Rad}(M)_M N \leq \operatorname{Rad}(N)$.

Compare the following lemma with The Nakayama's Lemma.

Lemma 3.12. *Let N be an R-module such that* $\operatorname{Rad}(N) \neq N$. *Then $A_M N \neq N$ for all $A \leq \operatorname{Rad}(M)$.*

Proof. Let $A \leq \operatorname{Rad}(M)$. By [8, Proposition 1.3] and Remark 3.11

$$A_M N \leq \operatorname{Rad}(M)_M N \leq \operatorname{Rad}(N) \neq N.$$

□

Proposition 3.13. *Let M be projective in $\sigma[M]$ and Artinian. Then M is semiprimary.*

Proof. By [19, 31.2] $M/\operatorname{Rad}(M)$ is semisimple. Consider the chain of powers of $\operatorname{Rad}(M)$

$$\operatorname{Rad}(M) \geq \operatorname{Rad}(M)^2 \geq \cdots$$

Since M is Artinian, there exists $n > 0$ such that $\operatorname{Rad}(M)^n = \operatorname{Rad}(M)^{n+1}$. Suppose $\operatorname{Rad}(M)^n \neq 0$. Set $\Gamma = \{K \leq M \mid \operatorname{Rad}(M)^n_M K \neq 0\}$. Since $\operatorname{Rad}(M) \in \Gamma$, $\Gamma \neq \emptyset$. Then Γ has minimal elements. Let $K \in \Gamma$ be a minimal one, then $0 \neq \operatorname{Rad}(M)^n_M K$. We have that $K = \sum\{Rk \mid k \in K\}$, hence by [6, Lemma 2.1]

$$0 \neq \operatorname{Rad}(M)^n{}_M \left(\sum Rk\right) = \sum \left(\operatorname{Rad}(M)^n_M Rk\right).$$

Thus there exists $k \in K$ such that $\operatorname{Rad}(M)^n_M Rk \neq 0$. It follows that $K = Rk$. Therefore

$$\operatorname{Rad}(M)^n_M(\operatorname{Rad}(M)_M Rk) = (\operatorname{Rad}(M)^n_M \operatorname{Rad}(M))_M Rk = \operatorname{Rad}(M)^n_M Rk \neq 0.$$

Since Rk is a minimal element in Γ, $\operatorname{Rad}(M)_M Rk = Rk$ but this is a contradiction to Lemma 3.12. □

In the following propositions we will suppose M is a *progenerator of* $\sigma[M]$, that is, M is projective in $\sigma[M]$ and a generator of $\sigma[M]$, in order to get stronger results. Since it is not assumed that M is finitely generated, even in this case the category $\sigma[M]$ might not be equivalent to a category R-Mod for some ring R, hence $\sigma[M]$ still gives us a more general framework.

Remark 3.14. Let $N \in \sigma[M]$. Suppose M is a progenerator of $\sigma[M]$ and $M/\operatorname{Rad}(M)$ is semisimple. Since M is projective in $\sigma[M]$, it follows that

$$\operatorname{Rad}(M)_M\left(\frac{N}{\operatorname{Rad}(M)_M N}\right) = 0.$$

By [7, Proposition 1.5], $\frac{N}{\operatorname{Rad}(M)_M N} \in \sigma[M/\operatorname{Rad}(M)]$ so $\frac{N}{\operatorname{Rad}(M)_M N}$ is semisimple, hence $\operatorname{Rad}\left(\frac{N}{\operatorname{Rad}(M)_M N}\right) = 0$. By Remark 3.11, $\operatorname{Rad}(N) = \operatorname{Rad}(M)_M N$. Thus $N/\operatorname{Rad}(N)$ is semisimple.

Proposition 3.15. *Suppose M is a progenerator of $\sigma[M]$ and semiprimary. Let $N \in \sigma[M]$. Then N is Artinian if and only if N is Noetherian.*

Proof. Let $N \in \sigma[M]$. Since M is semiprimary, there exists $n > 1$ such that $\operatorname{Rad}(M)^n = 0$. We have the following chain

$$0 \le \operatorname{Rad}(N)^{n-1} \le \operatorname{Rad}(N)^{n-2} \le \cdots \le \operatorname{Rad}(N) \le N$$

By Remark 3.14, every quotient of this chain is semisimple. By Lemma 3.1 we have the result. $\square$

Proposition 3.16. *Let M be progenerator of $\sigma[M]$ and suppose $M/\operatorname{Rad}(M)$ is semisimple. Let $N \in \sigma[M]$. Then $\operatorname{Soc}(N)$ is the largest submodule of N such that $\operatorname{Rad}(M)_M \operatorname{Soc}(N) = 0$. This implies that $\operatorname{Soc}(N) = \sum\{L \le N \mid \operatorname{Rad}(M)_M L = 0\}$.*

Proof. It is clear that $\operatorname{Rad}(M)_M \operatorname{Soc}(N) = 0$. Now let $L \le N$ such that $\operatorname{Rad}(M)_M L = 0$. Then $\operatorname{Rad}(M) \le \operatorname{Ann}_M(L)$, so by [7, Proposition 1.5] $L \in \sigma[M/\operatorname{Rad}(M)]$. Hence L is semisimple which implies that $L \le \operatorname{Soc}(N)$. $\square$

4. Modules whose their submodules are annihilators

Definition 4.1. Let M be an R-module. An M*-annihilator* is a submodule of M of the form

$$\mathbf{r}_M(X) = \bigcap\{\operatorname{Ker} f \mid f \in X\}$$

for some $X \subseteq \operatorname{End}_R(M)$.

These kind of submodules have appeared many times in the literature, we are taking the name from [9, pp. 2]. Other authors call these submodules *right annihilators in M* or just *annihilators in M*. We will focus first on those modules M whose their essential submodules are M-annihilators.

Definition 4.2. Let $N \le M$ and $S = \operatorname{End}_R(M)$. For every subset $Y \subseteq N$ we define

$$\mathbf{l}^N_M(Y) = \{f : N \to M | f(Y) = 0\}.$$

In case $N = M$, we write $\mathbf{l}^M_M = \mathbf{l}_S$.

Remark 4.3. For every submodule $N \le M$ and every subset $Y \subseteq N$, $\mathbf{l}^N_M(Y)$ is a left S-module. In the case Y is a fully invariant submodule of M, $\mathbf{l}_S(Y)$ is an ideal of S.

Notice that

$$\mathbf{l}_S\left(\mathbf{r}_M(\mathbf{l}_S(Y))\right) = \mathbf{l}_S(Y)$$

for all $Y \subseteq M$ and

$$\mathbf{r}_M(X) = \mathbf{r}_M(\mathbf{l}_S(\mathbf{r}_M(X)))$$

for all $X \subseteq S$

Let M and N be two modules with N in $\sigma[M]$. It is said that N is *M-singular* if there exists an exact sequence $0 \to K \to L \to N \to 0$ in $\sigma[M]$ such that $K \leq^{\text{ess}} L$. The class of all M-singular modules is a pretorsion class and hence every module U in $\sigma[M]$ has a maximal M-singular submodule denoted $\mathcal{Z}(U)$. It is said that the module N is *non M-singular* if $\mathcal{Z}(N) = 0$. Non M-singular modules are known also as *polyform modules* [20, Sec. 10 and 11].

Proposition 4.4. *The following conditions are equivalent for a module M:*

(a) *M is non M-singular (polyform) and every essential submodule is an M-annihilator.*
(b) *M is semisimple.*

Proof. Let $L \leq^{\text{ess}} M$. By hypothesis, $L = \bigcap_{f\in X} \operatorname{Ker} f$ for some $X \subseteq \operatorname{End}_R(M)$. Since L is essential, $\operatorname{Ker} f \leq^{\text{ess}} M$ for all $f \in X$. It follows that $X = \{0\}$ because M is non M-singular. Hence $L = M$. Thus, M is semisimple. The converse is clear. $\square$

Proposition 4.5. *The following conditions are equivalent for a module M:*

(a) *Every essential submodule of M is an M-annihilator.*
(b) *Every submodule of M is a direct summand of an M-annihilator.*
(c) *$\mathbf{r}_M(\mathbf{l}_S(N)) = N$ for every essential submodule N of M.*

Proof. (a)⇒(b) Let N be any submodule of M. If $N \leq^{\text{ess}} M$, then N is an M-annihilator by hypothesis. If N is not essential in M, there exists a submodule L such that $N \oplus L \leq^{\text{ess}} M$.

(b)⇒(a) Let N be an essential submodule of M. By hypothesis there exists $L \leq M$ such that $N \oplus L$ is an M-annihilator. Since N is essential, $L = 0$. Thus, N is an M-annihilator.

(a)⇔(c) follows from Remark 4.3. $\square$

Proposition 4.6. *Let M be a module such that every essential submodule is an M-annihilator and let N be a fully invariant essential submodule of M. Then, every essential submodule of N is an N-annihilator.*

Proof. Let L be an essential submodule of N. Hence, $L \leq^{\text{ess}} M$. By hypothesis there exists a subset $X \subseteq \operatorname{End}_R(M)$ such that $L = \bigcap_{f\in X} \operatorname{Ker} f$. Since N is fully invariant in M, $f|_N \in \operatorname{End}_R(N)$. Consider $Y = \{f|_N \mid f \in X\} \subseteq \operatorname{End}_R(N)$. Then,

$$\bigcap_{f|_N\in Y} \operatorname{Ker} f|_N = \bigcap_{f|_N\in Y} (\operatorname{Ker} f \cap N) = \left(\bigcap_{f\in X} \operatorname{Ker} f\right) \cap N = L \cap N = L.$$

$\square$

Proposition 4.7. *If every essential submodule of M is an M-annihilator, then every maximal submodule of M is an M-annihilator.*

Proof. Let N be a maximal submodule of M. If $N \leq^{\text{ess}} M$, then N is an M-annihilator by hypothesis. If N is not essential, there exists $L \leq M$ such that $L \cap N = 0$. Since N is maximal, $M = N \oplus L$. Let $i : L \to M$ and $\pi : M \to L$ be the canonical inclusion and the canonical projection respectively. Hence $N = \operatorname{Ker} i\pi$, that is, N is an M-annihilator. $\square$

Corollary 4.8. *Let M be a module such that every essential submodule is an M-annihilator. Then* $\operatorname{Rad}(M)$ *and* $\operatorname{Soc}(M)$ *are M-annihilators.*

Proof. Since $\operatorname{Soc}(M) = \bigcap\{N \mid N \leq^{\text{ess}} M\}$ and each essential submodule is an M-annihilator, $\operatorname{Soc}(M)$ is an M-annihilator. On the other hand, if $\operatorname{Rad}(M) = M$, then $\operatorname{Rad}(M)$ is an M-annihilator. Suppose $\operatorname{Rad}(M) \neq M$. By Proposition 4.7, every maximal submodule of M is an M-annihilator. It follows that $\operatorname{Rad}(M)$ is an M-annihilator since $\operatorname{Rad}(M)$ is the intersection of all maximal submodules. $\square$

Corollary 4.9. *Let M be a module such that every essential submodule is an M-annihilator. Then,*

(1) $\mathbf{r}_M(\mathbf{l}_S(N)) = N$ *for every maximal submodule N of M.*
(2) $\mathbf{r}_M(\mathbf{l}_S(\mathrm{Soc}(M))) = \mathrm{Soc}(M)$.
(3) $\mathbf{r}_M(\mathbf{l}_S(\mathrm{Rad}(M))) = \mathrm{Rad}(M)$.

Proof. The proofs of the statements follow directly using Remark 4.3. □

Proposition 4.10. *Let M be projective in $\sigma[M]$. Suppose that every essential submodule of M is an M-annihilator. If* $\mathrm{Hom}_R(M, \mathcal{Z}(M)) \neq 0$, *then* $\mathrm{Soc}(M) = \mathrm{Ann}_M(\mathcal{Z}(M))$.

Proof. Given a nonzero $f : M \to \mathcal{Z}(M)$, by [7, Proposition 1.2], $\mathrm{Ker} f \leq^{\mathrm{ess}} M$. It follows that $\mathrm{Soc}(M) \subseteq \mathrm{Ann}_M(\mathcal{Z}(M))$. Now, let $N \leq^{\mathrm{ess}} M$. By hypothesis, $N = \mathbf{r}_M(X)$ for some $X \subseteq \mathrm{End}_R(M)$. This implies that $\mathrm{Ker} f \leq^{\mathrm{ess}} M$ for all $f \in X$. Therefore $f(M) \subseteq \mathcal{Z}(M)$ for all $f \in X$. Hence, we can consider $X \subseteq \mathrm{Hom}_R(M, \mathcal{Z}(M))$. Thus, $\mathrm{Ann}_M(\mathcal{Z}(M)) \subseteq \mathbf{r}_M(X) = N$. Since $\mathrm{Soc}(M)$ is the intersection of all essential submodules on M, $\mathrm{Ann}_M(\mathcal{Z}(M)) \subseteq \mathrm{Soc}(M)$. □

Definition 4.11. An R-module M is a Kasch module if $\widehat{M}$ is an injective cogenerator of $\sigma[M]$.

Proposition 4.12 ([1, Proposition 2.6])**.** *The following conditions are equivalent for an R-module M:*

(a) *M is a Kasch module.*
(b) *Every simple module in $\sigma[M]$ can be embedded in M.*
(c) *Every simple module in $\sigma[M]$ is cogenerated by M.*
(d) $\mathrm{Hom}_R(C, M) \neq 0$ *for all (cyclic) module C in $\sigma[M]$.*

Proposition 4.13. *The following conditions are equivalent for an R-module M:*

(a) *M is a Kasch module.*
(b) *Let $X < N \leq M$ with X maximal in N. Then*
$$\mathbf{l}_M^N(X) = \{f : N \to M | f(X) = 0\} \neq 0$$
(c) *Let $X < N \leq M$ with X maximal in N. Then*
$$X = \mathbf{r}_M(\mathbf{l}_M^N(X)).$$

Proof. (a)⇒(b) Since X is maximal in N, N/X is a simple module. By Proposition 4.12(b), there exists a monomorphism $\alpha : N/X \to M$. If $\pi : N \to N/X$ is the canonical projection then $0 \neq f = \alpha \circ \pi$ and $f(X) = 0$. Thus $\mathbf{l}_M^N(X) \neq 0$.

(b)⇒(c) We have that $X \subseteq \mathbf{r}_M(\mathbf{l}_M^N(X)) \subseteq N$. If $\mathbf{r}_M(\mathbf{l}_M^N(X)) = N$ then $\mathbf{l}_M^N(X) = 0$, which is a contradiction. Then $X = \mathbf{r}_M(\mathbf{l}_M^N(X))$.

(c)⇒(a) Let T be a simple module in $\sigma[M]$. Then T is isomorphic to a subfactor of M i.e. there exists $X \leq N \leq M$ such that $T \cong N/X$. By hypothesis $X = \mathbf{r}_M(\mathbf{l}_M^N(X))$, so there exists a monomorphism $T \cong N/X \to M^{\mathcal{I}}$ where $\mathcal{I} = \mathbf{l}_M^N(X)$. Hence every simple in $\sigma[M]$ is cogenerated by M. Therefore M is a Kasch module by Proposition 4.12. □

Corollary 4.14. *Consider the following conditions for a module M:*

(1) M is a Kasch module.
(2) Each maximal submodule of M is an M-annihilator.

Then (1) ⇒ (2). In addition, if M generates every simple module in $\sigma[M]$, then the two conditions are equivalent.

Proof. *(1) ⇒ (2)* follows from Proposition 4.13. For the converse, assume M generates every simple module in $\sigma[M]$. Let S be a simple module in $\sigma[M]$. Hence, there exists an epimorphism $f : M \to S$. By hypotesis $K = \mathrm{Ker} f = \bigcap_{g \in X} \mathrm{Ker}\, g$ for some subset

$X \subseteq \mathrm{End}_R(M)$. Since $N \neq M$, we can assume that $g \neq 0$ for all $g \in X$. Since N is maximal, then $N = \mathrm{Ker}\, g$ for all $g \in X$. This implies that $S \cong M/\mathrm{Ker}\, g$ can be embedded in M. Thus, M is a Kasch module by Proposition 4.12. □

Example 4.15. Consider the $\mathbb{Z}$-module $M = \mathbb{Q}$. Then $\sigma[M] = \mathbb{Z}$-Mod. Since M has no maximal submodules, M satisfies the condition (2) of Corollary 4.14. Note that M is not a Kasch module and M does not generate the simple modules in $\sigma[M]$.

Corollary 4.16. *Let M be a module which generates every simple module in $\sigma[M]$. If every essential submodule of M is an M-annihilator, then M is Kasch.*

Proof. By Proposition 4.7 every maximal submodule of M is an M-annihilator. Then M is a Kasch module by Corollary 4.14. □

Definition 4.17. Let M and N be two modules. The module N is called *M-torsionless* if N can be embedded in a direct product of copies of M.

It is clear that every direct summand of a module M is M-torsionless. Also, if M is Kasch, then by definition every module in $\sigma[M]$ is $\widehat{M}$-torsionless.

Proposition 4.18. *The following conditions are equivalent for $N \leq M$:*

(a) $N = \mathbf{r}_M(\mathbf{l}_S(N))$.

(b) *M/N is M-torsionless.*

Proof. (a)⇒(b) If $N = \mathbf{r}_M(\mathbf{l}_S(N))$ then M/N can be embedded in $M^{\mathbf{l}_S(N)}$.

(b)⇒(a) Suppose that there exists a monomorphism $\alpha : M/N \to M^X$ for some index set X. Let $m \in \mathbf{r}_M(\mathbf{l}_S(N))$, $\rho : M \to M/N$ be the canonical projection and $\pi_x : M^X \to M$ be the canonical projections for every $x \in X$. Consider the composition $\pi_x \circ \alpha \circ \rho : M \to M$, then

$$0 = \pi_x(\alpha(\rho(m))) = \pi_x(\alpha(m + N))$$

for all $x \in X$, thus $\alpha(m + N) = 0$. Since α is a monomorphism $m + N = 0$, so $m \in N$. Thus $N = \mathbf{r}_M(\mathbf{l}_S(N))$. □

Corollary 4.19. *Let M be a module. Then $N = \mathbf{r}_M(\mathbf{l}_S(N))$ for every direct summand N of M.*

In [11, pp. 1072], a ring R is called *left annular* if every left ideal is a left annihilator. Following this, we introduce the next definition.

Definition 4.20. An R-module M is called *annular* if every submodule is an M-annihilator.

Corollary 4.21. *A module M is annular if and only if every factor module of M is M-torsionless.*

Corollary 4.22. *If M is a nonzero annular module, then M is a Kasch module. In particular,* $\mathrm{Soc}(M) \neq 0$.

Proof. Let $X < N \leq M$ with X maximal in N. By Corollary 4.21, M/X is M-torsionless. Hence by Proposition 4.18

$$X = \mathbf{r}_M(\mathbf{l}_S(X)).$$

Since $X \neq N$ there exists $0 \neq f \in \mathbf{l}_S(X)$ such that $f(N) \neq 0$, hence $f|_N \in \mathbf{l}_M^N(X)$. Thus, by Proposition 4.13(c) M is Kasch. □

Proposition 4.23. *Let N be a fully invariant submodule of a module M. If M is an annular module, then so is N.*

Proof. The proof is similar to that of Proposition 4.6. □

It follows from Proposition 4.23 that if R is a commutative annular ring, then every ideal of R is an annular module.

Proposition 4.24. *The following conditions are equivalent for an M-injective module M:*

(a) *M is a Kasch module.*
(b) *M is an annular module.*
(c) *$N = \mathbf{r}_M(\mathbf{l}_S(N))$ for all $N \leq M$.*

Proof. (a)⇒(b) Let $N \leq M$. Since M is a cogenerator of $\sigma[M]$ there exists a monomorphism $M/N \to M^X$ for some index set X. Then by Proposition 4.18 $N = \mathbf{r}_M(\mathbf{l}_S(N))$.

(b)⇒(c) This follows from Corollary 4.21 and Proposition 4.18.

(c)⇒(a) Let $X \leq N \leq M$ with X maximal in N. We have that

$$\mathbf{l}_S(X) = \{f : M \to M | f(X) = 0\} \subseteq \mathbf{l}_M^N(X) = \{f : N \to M | f(X) = 0\}.$$

Therefore,

$$X \subseteq \mathbf{r}_M(\mathbf{l}_M^N(X)) \subseteq \mathbf{r}_M(\mathbf{l}_S(X)) = X.$$

Then M is Kasch by Proposition 4.13. □

Corollary 4.25. *Let M be a Kasch module. Then $\widehat{M}$ is an annular module.*

Corollary 4.26. *The following conditions are equivalent for a left self-injective ring R:*

(a) *R is left Kasch.*
(b) *${}_RR$ is an annular module.*
(c) *$I = \mathbf{r}_R(\mathbf{l}_R(I))$ for every left ideal I of R.*

Example 4.27. (1) In $\mathbb{Z}$-Mod, given a prime number $p \in \mathbb{Z}$, the modules $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^\infty}$ are self-injective Kasch modules. Hence they are annular modules.

(2) Let R be a left local ring with S the unique simple R-module, up to isomorphism. Then $E(S)$, the injective hull of S in R-Mod, is an annular module.

If R is not left local and S is a simple module, then $E(S)$ might not be an annular module, as the following example shows.

Example 4.28. Let K be a field and let R be the 2×2 lower triangular matrix ring with coefficients in K. Then, R has the following decomposition

$$R = \left(\begin{smallmatrix} K & 0 \\ K & K \end{smallmatrix}\right) = \left(\begin{smallmatrix} K & 0 \\ K & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 0 \\ 0 & K \end{smallmatrix}\right).$$

Let S be the minimal ideal $\left(\begin{smallmatrix} 0 & 0 \\ K & 0 \end{smallmatrix}\right)$ and let $M = E(S) = \left(\begin{smallmatrix} K & 0 \\ K & 0 \end{smallmatrix}\right)$. Then, M/S is a simple R-module which cannot be embedded in M. Therefore, M is neither Kasch nor annular. However, the module $E(S) \oplus M/S$ is an injective Kasch module and hence an annular module by Proposition 4.24. The module $E(S) \oplus M/S$ also shows that a direct summand of a annular module might not inherit the property.

To finish this section we will present some results on M-annihilators which will be applied in the next section.

Proposition 4.29. *Let $N = \mathbf{r}_M(X)$ be an M-annihilator. Then, N is fully invariant if and only if $N = \mathbf{r}_M(I)$ for some ideal I of $S = \mathrm{End}_R(M)$.*

Proof. Suppose that $N = \mathbf{r}_M(X)$ is fully invariant. Let I denote the ideal generated by X in S. It is clear that $\mathbf{r}_M(I) \subseteq \mathbf{r}_M(X)$. Let $g \in I$ and $n \in \mathbf{r}_M(X)$. There exist $h, h' \in S$ such that $g = h'fh$. Since N is fully invariant, $h(n) \in N = \mathbf{r}_M(X)$. Hence, $g(n) = h'fh(n) = 0$. Thus, $\mathbf{r}_M(I) = \mathbf{r}_M(X)$. □

Corollary 4.30. *Let M be a finitely generated quasi-projective module. Then N is a fully invariant M-annihilator if and only if $N = \mathrm{Ann}_M(K)$ for some $K \leq M$.*

Proof. Suppose N is a fully invariant M-annihilator. Then, $N = \mathbf{r}_M(I)$ for some ideal I of $\mathrm{End}_R(M)$ by Proposition 4.29. Since M is finitely generated and quasi-projective, $I = \mathrm{Hom}_R(M, IM)$ by [19, 18.4]. Thus, $N = \mathbf{r}_M(\mathrm{Hom}_R(M, IM)) = \mathrm{Ann}_M(IM)$. The converse is clear. $\square$

Lemma 4.31. *Let $\Gamma = \{\mathbf{l}_S(Y) \mid Y \subseteq M\}$. If $\mathbf{l}_S(Y)$ is a minimal element of Γ then $\mathbf{r}_M\mathbf{l}_S(Y)$ is a maximal M-annihilator. Conversely, if $\mathbf{r}_M(X)$ is a maximal M-annihilator then $\mathbf{l}_S(\mathbf{r}_M(X))$ is a minimal element in Γ.*

Proof. Suppose that $\mathbf{l}_S(Y)$ is a minimal element in Γ. Let $\mathbf{r}_M(X) \neq M$ be an M-annihilator such that $\mathbf{r}_M\mathbf{l}_S(Y) \leq \mathbf{r}_M(X)$. So $\mathbf{l}_S(\mathbf{r}_M(X)) \leq \mathbf{l}_S(\mathbf{r}_M\mathbf{l}_S(Y)) = \mathbf{l}_S(Y)$. Since $\mathbf{l}_S(Y)$ is minimal then $\mathbf{l}_S(\mathbf{r}_M(X)) = 0$ or $\mathbf{l}_S(\mathbf{r}_M(X)) = \mathbf{l}_S(Y)$. We have that $X \subseteq \mathbf{l}_S(\mathbf{r}_M(X))$ and $\mathbf{r}_M(X) \neq M$ hence $\mathbf{l}_S(\mathbf{r}_M(X)) \neq 0$. Therefore $\mathbf{l}_S(Y) = \mathbf{l}_S(\mathbf{r}_M(X))$. Thus $\mathbf{r}_M\mathbf{l}_S(Y) = \mathbf{r}_M(X)$.

Reciprocally, let $\mathbf{r}_M(X)$ be a maximal M-annihilator. Let $0 \neq \mathbf{l}_S(Y) \subseteq \mathbf{l}_S(\mathbf{r}_M(X))$. Hence,

$$\mathbf{r}_M(X) = \mathbf{r}_M(\mathbf{l}_S(\mathbf{r}_M(X))) \subseteq \mathbf{r}_M(\mathbf{l}_S(Y)).$$

Since $\mathbf{r}_M(X)$ is a maximal M-annihilator and $0 \neq \mathbf{l}_S(Y)$, $\mathbf{r}_M(X) = \mathbf{r}_M(\mathbf{l}_S(Y))$. This implies $\mathbf{l}_S(Y) = \mathbf{l}_S(\mathbf{r}_M(X))$. $\square$

Lemma 4.32. *Let $\Gamma = \{\mathbf{l}_S(Y) | Y \subseteq M\}$. Let $\bigoplus_{i=1}^{n} \mathbf{l}_S(Y_i)$ be a maximal direct sum of minimal elements in Γ. Then, there exist $f_i \in \mathbf{l}_S(Y_i)$ $1 \leq i \leq n$ such that $\mathbf{r}_M(\{f_1, \ldots, f_n\})$ is the intersection of all maximal M-annihilators.*

Proof. By Lemma 4.31, $\mathbf{r}_M\mathbf{l}_S(Y_i)$ is a maximal M-annihilator for all $1 \leq i \leq n$. Hence $\mathrm{Ker}(f) = \mathbf{r}_M\mathbf{l}_S(Y_i)$ for all $f \in \mathbf{l}_S(Y_i)$ and for all $1 \leq i \leq n$. Let $\mathbf{r}_M(X)$ be a maximal M-annihilator. By hypothesis

$$\bigoplus_{i=1}^{n} \mathbf{l}_S(Y_i) \cap \mathbf{l}_S(\mathbf{r}_M(X)) \neq 0,$$

so there exists $f \in \mathbf{l}_S(\mathbf{r}_M(X))$ such that $f = f_1 + \cdots + f_n$ with $f_i \in \mathbf{l}_S(Y_i)$. We have that $\mathbf{r}_M(X) \leq \mathrm{Ker}(f)$. Since $\mathbf{r}_M(X)$ is a maximal M-annihilator, $\mathbf{r}_M(X) = \mathrm{Ker}(f)$. Thus

$$\mathrm{Ker}(f_1) \cap \cdots \cap \mathrm{Ker}(f_n) \leq \mathrm{Ker}(f_1 + \cdots + f_n) = \mathrm{Ker}(f) = \mathbf{r}_M(X).$$

This implies that $\bigcap_{i=1}^{n} \mathbf{r}_M(\mathbf{l}_S(Y_i)) = \mathrm{Ker}(f_1) \cap \cdots \cap \mathrm{Ker}(f_n) \subseteq \mathbf{r}_M(X)$ for any maximal M-annihilator. $\square$

5. Johns and quasi-Johns modules

Definition 5.1. A module M is called *quasi-Johns* if any essential submodule of M is an M-annihilator and the set of essential submodules of M satisfies the ascending chain condition (acc).

It follows that the ring R is right quasi-Johns if and only if ${}_RR$ is a quasi-Johns module [18]. It is clear that any semisimple module is quasi-Johns. Next proposition provides more examples.

Proposition 5.2. *If M is a quasi-Johns module and $\bigoplus_{i\in I} S_i$ is any semisimple module, then $M \oplus \bigoplus_{i\in I} S_i$ is quasi-Johns.*

Proof. Let N be an essential submodule of $L = M \oplus \bigoplus_{i\in I} S_i$. Since N is essential, $S_i \leq N$ for all $i \in I$. Therefore $\bigoplus_{i\in I} S_i \leq N$. This implies that $N = K \oplus \bigoplus_{i\in I} S_i$ for some $K \leq M$. It follows that K must be essential in M because N is essential in L. Now, it is clear that, if M satisfies acc on essential submodules, so does L. Since M is quasi-Johns, M/K is M-torsionless. Therefore, $L/N \cong M/K$ is L-torsionless. By Proposition 4.18, N is an L-annihilator. $\square$

Proposition 5.3. *Let N be an essential fully invariant submodule of a module M. If M is a quasi-Johns module, then so is N.*

Proof. Since N is essential in M, any essential submodule of N is essential in M. This implies that the essential submodules of N satisfies acc. Now the proof follows from Proposition 4.6. □

Proposition 5.4. *Let M be projective in $\sigma[M]$. If M is a quasi-Johns module such that* $\mathrm{Soc}(M) \leq^{ess} M$*, then $\mathcal{Z}(M)$ is nilpotent.*

Proof. Note that if $\mathrm{Hom}_R(M, \mathcal{Z}(M)) = 0$, then $\mathcal{Z}(M)_M\mathcal{Z}(M) = 0$. Suppose that $\mathrm{Hom}_R(M, \mathcal{Z}(M)) \neq 0$. Consider the descending chain $\mathcal{Z}(M) \supseteq \mathcal{Z}(M)^2 \supseteq \cdots$. Suppose that $\mathcal{Z}(M)^n \neq 0$ for all $n > 0$. Then, there is an ascending chain $\mathrm{Ann}_M(\mathcal{Z}(M)) \subseteq \mathrm{Ann}_M(\mathcal{Z}(M)^2) \subseteq \cdots$. By Proposition 4.10 and the hypothesis, $\mathrm{Ann}_M(\mathcal{Z}(M)^n) \leq^{\mathrm{ess}} M$ for all $n > 0$. Hence there exists n such that $\mathrm{Ann}_M(\mathcal{Z}(M)^n) = \mathrm{Ann}_M(\mathcal{Z}(M)^{n+i})$ for all $i > 0$. Since $(\mathcal{Z}(M)_M\mathcal{Z}(M))_M\,\mathcal{Z}(M)^n = \mathcal{Z}(M)^{n+2} \neq 0$, it follows that there exists $0 \neq f \in \mathrm{Hom}_R(M, \mathcal{Z}(M))$ such that $f(\mathcal{Z}(M))_M\mathcal{Z}(M)^n \neq 0$. This implies that $f(M)_M\mathcal{Z}(M)^n \neq 0$. Consider the set

$$\Gamma = \{\mathrm{Ker}\, f \mid f \in \mathrm{Hom}_R(M, \mathcal{Z}(M)) \text{ and } f(M)_M\mathcal{Z}(M)^n \neq 0\}.$$

It follows from [7, Proposition 1.2] that $\mathrm{Ker}\, f \leq^{\mathrm{ess}} M$ for every $f \in \mathrm{Hom}_R(M, \mathcal{Z}(M))$. Hence, Γ has maximal elements because M is quasi-Johns. Now the proof follows as in [4, Proposition 2.29]. □

Corollary 5.5. *If R is a left quasi-Johns ring such that* $\mathrm{Soc}({}_RR) \leq^{ess} R$*, then the left singular ideal of R is nilpotent.*

Definition 5.6. An R-module M is called *Johns* if M is a Noetherian annular module.

Example 5.7. (1) A ring R is left Johns if and only if ${}_RR$ is a Johns module.

(2) Every finitely generated semisimple module is Johns.
(3) The $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ is self-injective and Kasch for every prime p and all $n > 0$, then $\mathbb{Z}_{p^n}$ is a Johns module (Proposition 4.24).
(4) Let $R = K\mathcal{Q}$ be the path algebra of the finite quiver $\mathcal{Q}$ [2]. Let $E = \bigoplus_{i=1}^n E_i$ be the direct sum of indecomposable injective R-modules up to isomorphism. Then E is a Noetherian Kasch module. By Proposition 4.24, E is Johns.
(5) Every uniform quasi-Johns module is Johns.
(6) Let $p, q, n \in \mathbb{Z}$ with $p \neq q$ prime numbers and let X be any infinite set. It follows from Proposition 5.2 that the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^n} \oplus \mathbb{Z}_q^{(X)}$ is a quasi-Johns module. Since X is an infinite set, M is not a Johns module.

Remark 5.8. Notice that by Corollary 4.22 every Johns module is a Kasch module.

Proposition 5.9. *Let M be a Johns module and N a fully invariant submodule of M. Then N is a Johns module.*

Proof. It follows from Proposition 4.23. □

The following lemma is an exercise given in [19], we prove it here for the reader's convenience.

Lemma 5.10 ([19, 27.7(1)])**.** *Let M be a finitely generated, quasi-projective module. Then,* $\mathrm{End}_R(M)$ *is right Noetherian if and only if M satisfies acc on M-generated submodules.*

Proof. Let I_S be a right ideal of $S = \mathrm{End}_R(M)$. Then IM is an M-generated submodule of M and $I = \mathrm{Hom}_R(M, IM)$ by [19, 18.4]. On the other hand, let N be an M-generated submodule of M. Hence, $\mathrm{Hom}_R(M, N)$ is right ideal of S. Since N is M-generated, $\mathrm{Hom}_R(M, N)N = \mathrm{tr}^M(N) = N$ where $\mathrm{tr}^M(N)$ is the trace of M in N. Now, the lemma's proof is straightforward. □

Corollary 5.11. *Let M be a quasi-projective module with $S = \mathrm{End}_R(M)$. If M is Johns, then S is right Noetherian.*

Lemma 5.12. *Let M be an annular module. Then $\mathbf{r}_M(\mathbf{l}_S(N) \cap \mathbf{l}_S(L)) = \mathbf{r}_M(\mathbf{l}_S(N)) + \mathbf{r}_M(\mathbf{l}_S(L))$ for all $N, L \leq M$.*

Proof. Let $N, L \leq M$. Then,

$$N + L = \mathbf{r}_M(\mathbf{l}_S(N)) + \mathbf{r}_M(\mathbf{l}_S(L)) \subseteq \mathbf{r}_M(\mathbf{l}_S(N) \cap \mathbf{l}_S(L)) = \mathbf{r}_M(\mathbf{l}_S(N + L)) = N + L.$$

□

Proposition 5.13. *The following conditions are equivalent for a Noetherian module M with $S = \mathrm{End}_R(M)$:*

(a) *M is an annular module;*

(b) *Every cyclic submodule of M is an M-annihilator, and*

$$\mathbf{r}_M(\mathbf{l}_S(N) \cap \mathbf{l}_S(L)) = \mathbf{r}_M(\mathbf{l}_S(N)) + \mathbf{r}_M(\mathbf{l}_S(L))$$

for all submodules $N, L \leq M$.

Proof. (a)⇒(b) follows from Lemma 5.12.

(b)⇒(a) Let $K \leq M$. Since M is Noetherian, $K = Rx_1 + Rx_2 + \cdots + Rx_n$. We proceed by induction on n. By hypothesis, $K = Rx_1$ is an M-annihilator. Now, suppose the result is true for submodules with less than n generators and $K = Rx_1 + Rx_2 + \cdots + Rx_n$. Write $N = Rx_1 + \cdots + Rx_{n-1}$ and $L = Rx_n$, so $K = N + L$. It follows that

$$\begin{aligned} \mathbf{r}_M(\mathbf{l}_S(K)) &= \mathbf{r}_M(\mathbf{l}_S(N + L)) = \mathbf{r}_M(\mathbf{l}_S(N) \cap \mathbf{l}_S(L)) \\ &= \mathbf{r}_M(\mathbf{l}_S(N)) + \mathbf{r}_M(\mathbf{l}_S(L)) = N + L = K. \end{aligned}$$

Thus, K is an M-annihilator. □

Definition 5.14. Let M be a module and let $N \leq M$. The *right annihilator of N in M* is given by the submodule

$$\mathrm{Ann}^r_M(N) = \sum\{K \leq M \mid N_M K = 0\}.$$

Remark 5.15. The right annihilator $\mathrm{Ann}^r_M(N)$, was defined in [4]. If M is projective in $\sigma[M]$, it is not difficult to see that $\mathrm{Ann}^r_M(N)$ is fully invariant and is the largest submodule of M such that $N_M \mathrm{Ann}^r_M(N) = 0$. Also, it can be proved that $\mathrm{Ann}_M(\mathrm{Ann}^r_M(\mathrm{Ann}_M(N))) = \mathrm{Ann}_M(N)$ for all $N \leq M$.

Proposition 5.16. *Let M be a quasi-projective module. If M is Johns, then the lattice of fully invariant submodules of M satisfies acc and dcc, that is, the lattice has finite length.*

Proof. Let $N_1 \geq N_2 \geq \cdots$ be a descending chain of fully invariant submodules of M. Hence, there is an ascending chain $\mathrm{Ann}^r_M(N_1) \leq \mathrm{Ann}^r_M(N_2) \leq \cdots$. Since M is Noetherian, there exists $\ell > 0$ such that $\mathrm{Ann}^r_M(N_\ell) = \mathrm{Ann}^r_M(N_{\ell+i})$ for all $i > 0$. Hence $\mathrm{Ann}_M(\mathrm{Ann}^r_M(N_\ell)) = \mathrm{Ann}_M(\mathrm{Ann}^r_M(N_{\ell+i}))$. It follows from Corollary 4.30 that $N_\ell = N_{\ell+i}$ for all $i > 0$. The other chain condition holds because M is Noetherian. □

Corollary 5.17. *Let M be a quasi-projective module. If M is Johns, then $\mathrm{Rad}(M)$ is nilpotent.*

Proof. Let $J := \mathrm{Rad}(M)$. We have a descending chain $J \geq J^2 \geq J^3 \geq \ldots$. By Proposition 5.16 $J^n = J^{n+1}$, thus by Lemma 3.12 $J = 0$. □

Recall that a module M is *duo* if every submodule is fully invariant in M [15].

Corollary 5.18. *Let M be a quasi-projective duo module. If M is a Johns module, then M is Artinian.*

Corollary 5.19. *Let R be a commutative ring. If R is Johns, then R is Artinian.*

The ring R given in [12, Example 6.2] is a commutative annular ring which is not Artinian. In fact, $J^2 = J$ where J is the Jacobson radical of R.

Lemma 5.20. *If M is a quasi-projective Johns module then $\operatorname{Ann}^r_M(\operatorname{Rad}(M))$ is essential in M.*

Proof. Let $0 \neq K \leq {}_RM$. By Corollary 5.17, $\operatorname{Rad}(M)_MK = 0$ or there exists $n > 0$ such that $\operatorname{Rad}(M)^n{}_MK = 0$ but $\operatorname{Rad}(M)^{n-1}{}_MK \neq 0$. Any of those two conditions implies $K \cap \operatorname{Ann}^r_M(\operatorname{Rad}(M)) \neq 0$. $\square$

Remark 5.21. Suppose that ${}_RM$ is quasi-projective and generates all its submodules. If $A_MB = 0$ with $A \leq^{\text{ess}} M$, then $B \leq \mathcal{Z}(M)$. In fact, since $A_MB = 0$, $A \leq \operatorname{Ker} f$ for all $f \in \operatorname{Hom}_R(M, B)$, so $\operatorname{Ker} f \leq^{\text{ess}} M$ and $f(B)$ is M-singular. Thus B is M-singular.

Lemma 5.22. *Suppose M is quasi-projective and generates all its submodules. If M is a Johns module, then $\operatorname{Ann}^r_M(\operatorname{Rad}(M)) = \operatorname{Ann}_M(\operatorname{Rad}(M))$.*

Proof. Since $\operatorname{Ann}^r_M(\operatorname{Rad}(M)) \leq^{\text{ess}} M$ then $\operatorname{Ann}^r_M(\operatorname{Ann}^r_M(\operatorname{Rad}(M))) \leq \mathcal{Z}(M)$. We have that $\mathcal{Z}(M)$ is nilpotent by [4, Proposition 2.29]. Since $\operatorname{Rad}(M)$ is a semiprime submodule, $\operatorname{Ann}^r_M(\operatorname{Ann}^r_M(\operatorname{Rad}(M))) \leq \mathcal{Z}(M) \leq \operatorname{Rad}(M)$. Hence, applying Ann^r_M, we get an ascending chain

$$(\operatorname{Ann}^r_M)^3(\operatorname{Rad}(M)) \subseteq (\operatorname{Ann}^r_M)^5(\operatorname{Rad}(M)) \subseteq \cdots$$

Since M is Noetherian, there exists $\ell > 0$ such that

$$(\operatorname{Ann}^r_M)^\ell(\operatorname{Rad}(M)) = (\operatorname{Ann}^r_M)^{\ell+2}(\operatorname{Rad}(M)).$$

Applying Ann_M, and using Corollary 4.30, we have that

$$\begin{aligned}(\operatorname{Ann}^r_M)^{\ell-1}(\operatorname{Rad}(M)) &= \operatorname{Ann}_M(\operatorname{Ann}^r_M)^\ell(\operatorname{Rad}(M)) \\ &= \operatorname{Ann}_M(\operatorname{Ann}^r_M)^{\ell+2}(\operatorname{Rad}(M)) \\ &= (\operatorname{Ann}^r_M)^{\ell+1}(\operatorname{Rad}(M))\end{aligned}$$

Continuing in this way, $\operatorname{Ann}^r_M(\operatorname{Ann}^r_M(\operatorname{Rad}(M))) = \operatorname{Rad}(M)$ which implies that $\operatorname{Ann}^r_M(\operatorname{Rad}(M)) = \operatorname{Ann}_M(\operatorname{Rad}(M))$. $\square$

Proposition 5.23. *Suppose M is quasi-projective and generates all its submodules. If M is a Johns module, then*

(1) $\operatorname{Ann}_M(\mathcal{Z}(M)) = \operatorname{Ann}_M(\operatorname{Rad}(M)) = \operatorname{Soc}(M)$.
(2) $\operatorname{Soc}(M) \leq^{ess} M$.

Proof. *(1)* We have that $\mathcal{Z}(M) \leq \operatorname{Rad}(M)$. Therefore,

$$\operatorname{Soc}(M) \leq \operatorname{Ann}^r_M(\operatorname{Rad}(M)) = \operatorname{Ann}_M(\operatorname{Rad}(M)) \leq \operatorname{Ann}_M(\mathcal{Z}(M)) = \operatorname{Soc}(M),$$

by Proposition 4.10 and Lemma 5.22.

(2) It follows from Lemma 5.20 and Lemma 5.22. $\square$

Recall that a module M is said to be *cosemisimple* if every proper submodule of M is an intersection of maximal submodules [19, Sec. 23]. This is equivalent to say that every simple module in $\sigma[M]$ is M-injective. For the case of a ring R, the ring R is a *left V-ring* if and only if ${}_RR$ is cosemisimple.

Proposition 5.24. *Suppose M is quasi-projective and generates all its submodules. If M is a Johns module, then $M/Rad(M)$ is a cosemisimple module.*

Proof. Let $\overline{M}$ denote the factor module $M/\operatorname{Rad}(M)$. Let $S \in \sigma[\overline{M}]$ be a simple module and consider $E^{[\overline{M}]}(S)$ the $\overline{M}$-injective hull of S. Suppose $E^{[\overline{M}]}(S) \neq S$. Since $E^{[\overline{M}]}(S)$ is $\overline{M}$-generated, there exists $h : \overline{M} \to E^{[\overline{M}]}(S)$ such that $h(M)$ contains properly S. Let A denote $\operatorname{Ker} h\pi$ where $\pi : M \to \overline{M}$ is the canonical projection. Hence $\operatorname{Rad}(M) \leq A$. We claim that $\mathbf{l}_S(A) = \{f : M \to \mathrm{Soc}(M) \mid f(A) = 0\}$. Let $f \in \mathbf{l}_S(A)$. Then $f(\mathrm{Rad}(M)) = 0$. Since M is quasi-projective, $\mathrm{Rad}(M)_M f(M) = 0$. This implies that $f(M) \subseteq \mathrm{Soc}(M)$ by Lemma 5.22 and Proposition 5.23. Thus, $\mathbf{l}_S(A) \subseteq \{f : M \to \mathrm{Soc}(M) \mid f(A) = 0\}$. The other inclusion is clear. Now, since M is Johns, $A = \mathbf{r}_M(\mathbf{l}_S(A)) = \mathbf{r}_M(\{f : M \to \mathrm{Soc}(M) \mid f(A) = 0\})$. This implies that $h(\overline{M}) \cong M/A$ embeds in a direct product of simple modules. Since $\mathrm{Soc}(h(\overline{M})) \leq^{\mathrm{ess}} h(\overline{M})$ and $\mathrm{Soc}(h(\overline{M})) = S$, $h(\overline{M})$ embeds in a finite product of simple modules, and therefore $h(\overline{M})$ is semisimple. Hence $h(\overline{M}) = S$ which is a contradiction. Thus, $S = E^{[\overline{M}]}(S)$ and so, S is $\overline{M}$-injective. □

Corollary 5.25. *Suppose M is quasi-projective and generates all its submodules, and let $S = \mathrm{End}_R(M)$. If M is a Johns module, then $S/J(S)$ is a right Noetherian right V-ring where $J(S)$ is the Jacobson radical of S.*

Proof. Denote by $\overline{M}$ the factor module $M/\mathrm{Rad}(M)$. By Proposition 5.24, $\overline{M}$ is cosemisimple. Hence $\overline{M}$ is a finitely generated progenerator of $\sigma[\overline{M}]$. Therefore $\sigma[\overline{M}] \cong \mathrm{End}_R(\overline{M})^{op}$-Mod. It follows from [19, 22.2] that $\mathrm{End}_R(\overline{M}) \cong S/J(S)$. Thus, $S/J(S)$ is a right Noetherian right V-ring. □

Remark 5.26. Let M be a quasi-projective Johns module. Let N denote the intersection of all prime submodules of M. By Corollary 5.17, $\mathrm{Rad}(M) \subseteq N$. On the other hand, since $\mathrm{Rad}(M)$ is a semiprime submodule, it is an intersection of prime submodules. Hence $N \subseteq \mathrm{Rad}(M)$. Thus, $\mathrm{Rad}(M) = N$. Therefore, $\mathrm{Rad}(M)$ is contained in each prime submodule of M.

Corollary 5.27. *Suppose M is quasi-projective and generates all its submodules. If M is Johns, then*

(1) there are finitely many simple modules in $\sigma[M]$, $S_1, \dots, S_k$ up to isomorphism.
(2) $Spec(M) = \{P_1, ..., P_\ell\}$ with $\ell \leq k$.
(3) $P_i = \mathrm{Ann}_M(S_j)$ for some $1 \leq j \leq k$.
(4) $P_i \nsubseteq P_j$ for $1 \leq i \neq j \leq \ell$.

Proof. Since M is Johns, M is a Kasch module. Therefore, there is an embedding $S \hookrightarrow M$ for every simple module S in $\sigma[M]$. On the other hand, $\mathrm{Soc}(M)$ is finitely generated because M is Noetherian. This implies that there are only finitely many simple modules, up to isomorphism in $\sigma[M]$. By Proposition 5.24, $M/\mathrm{Rad}(M)$ is a cosemisimple Noetherian module, it follows from [6, Corollary 5.9] that $Spec(M/\mathrm{Rad}(M)) = \{P_1/\mathrm{Rad}(M), ..., P_\ell/\mathrm{Rad}(M)\}$ and each $P_i/\mathrm{Rad}(M)$ is a maximal fully invariant submodule of $M/\mathrm{Rad}(M)$. This implies, by [16, Proposition 18] and Remark 5.26, that $Spec(M) = \{P_1, ..., P_\ell\}$. Let P_i be a prime submodule of M. There exists a maximal submodule L of M such that $P_i \leq L$. Since M is Johns, there is an endomorphism $f : M \to M$ such that $L = \mathrm{Ker}\, f$. It follows that $P_{iM}(M/L) = 0$ because M is quasi-projective. Therefore, $P_i \subseteq \mathrm{Ann}_M(M/L)$. But $\mathrm{Ann}_M(M/L)$ is a prime submodule beacause M/L is simple (see [14, Proposition 3.4]). Hence $P_i = \mathrm{Ann}_M(M/L)$. □

Corollary 5.28. *Suppose M is quasi-projective and generates all its submodules. If M is a Johns module, then the following conditions are equivalent:*

(a) *$\mathrm{Soc}(M/P) \neq 0$ for all $P \in Spec(M)$;*
(b) *M/P is Artinian for all $P \in Spec(M)$;*
(c) *M is semiprimary*

Proof. Note that $M/\mathrm{Rad}(M) \cong M/P_1 \oplus \cdots \oplus M/P_\ell$ where $Spec(M) = \{P_1, ..., P_\ell\}$ by Proposition 5.24 and [6, Corollary 5.11].

(a)⇒(b) We have that M/P is a prime Noetherian module. Since $\mathrm{Soc}(M/P) \neq 0$, M/P is semisimple Artinian by [6, Proposition 2.8].

(b)⇒(c) Since $M/\operatorname{Rad}(M) \cong M/P_1 \oplus \cdots \oplus M/P_\ell$. By hypothesis, $M/\operatorname{Rad}(M)$ must be Artinian and hence semisimple by [5, Theorem 1.17]. Since M is Johns, $\operatorname{Rad}(M)$ is nilpotent (Corollary 5.17).

(c)⇒(a) is clear. □

Theorem 5.29. *Suppose M is quasi-projective and generates all its submodules, and let $S = \operatorname{End}_R(M)$. If M is Johns, then the following conditions are equivalent:*

(a) $\operatorname{Rad}(M) = \operatorname{Ker} f_1 \cap \cdots \cap \operatorname{Ker} f_\ell$ *for some* $f_1, ..., f_\ell \in S$.
(b) $M/\operatorname{Rad}(M)$ *is semisimple Artinian.*
(c) M *is semiprimary.*

In addition, if M is a generator of $\sigma[M]$, the three conditions are equivalent to

(d) M *is Artinian.*

Proof. (a)⇒(b). By hypothesis, there is a monomorphism $M/\operatorname{Rad}(M) \to M^\ell$. It follows from Proposition 5.23 that $\operatorname{Soc}(M^\ell) \leq^{\text{ess}} M^\ell$. Hence $\operatorname{Soc}(M/\operatorname{Rad}(M)) \leq^{\text{ess}} M/\operatorname{Rad}(M)$. Since $M/\operatorname{Rad}(M)$ is Noetherian and semiprime with essential socle then it is semisimple Artinian by [6, Corollary 2.9].

(b)⇒(a). If $M/\operatorname{Rad}(M)$ is Artinian, there are $N_1, ..., N_\ell$ maximal submodules of M such that $\operatorname{Rad}(M) = N_1 \cap \cdots \cap N_\ell$. Since M is Johns, there exist $f_1, ..., f_\ell \in S$ such that $N_i = \operatorname{Ker} f_i$.

(c)⇔(b). One implication is by definition and the converse follows from Corollary 5.17.

(c)⇔(d) follows from Proposition 3.13 and Proposition 3.15. □

Corollary 5.30. *Suppose M is quasi-projective and generates all its submodules. If M is Johns and every maximal submodule is fully invariant then M is semiprimary. In addition, if M is a generator of $\sigma[M]$ then M is Artinian.*

Proof. By Corollary 5.27, $Spec(M) = \{P_1, ..., P_\ell\}$ with each P_i a maximal fully invariant submodule of M. By the hypothesis, $P_1, ..., P_\ell$ must be the maximal submodules of M. Hence M/P_i is simple. It follows from Corollary 5.28 that M is semiprimary. Now, if M is a generator of $\sigma[M]$, then M is Artinian by Theorem 5.29. □

It was proved in Corollary 5.19 that every commutative Johns ring is Artinian. The next corollary generalizes this fact.

Corollary 5.31. *Let R be a ring such that every maximal left ideal is a two-sided ideal. If R is left Johns, then R is left Artinian.*

In [10] is given an example of a Johns ring which is not Artinian. The next corollary shows that every fully bounded Johns ring is Artinian. Recall that a module M is said to be *bounded* if every essential submodule contains a fully invariant submodule which is essential. The module M is *fully bounded* if for every prime submodule P of M, the factor module M/P is bounded [14]. Hence, a ring R is left fully bounded if and only if ${}_RR$ is a fully bounded module

Corollary 5.32. *Suppose M is quasi-projective and generates all its submodules. If M is a fully bounded Johns module, then M is semiprimary.*

Proof. Since M is Johns, M is a Kasch module. It follows from [14, Proposition 4.14] that $M/\operatorname{Rad}(M)$ is semisimple Artinian. □

Corollary 5.33. *If R is a left fully bounded left Johns ring, then R is Artinian.*

Proposition 5.34. *Suppose M is quasi-projective and generates all its submodules, and let $S = \operatorname{End}_R(M)$. Suppose that M is Johns and ${}_SS$ has finite uniform dimension. Then M is semiprimary. In addition, if M is a generator of $\sigma[M]$, M is Artinian.*

Proof. Since M is Johns then it is Kasch, so the maximal submodules of M are maximal M-annihilators. This implies that the M-annihilators are the maximal submodules. Note that, if $\mathcal{M}$ is a maximal submodule of M then $\mathbf{l}_S(\mathcal{M})$ is a minimal element in $\Gamma = \{\mathbf{l}_S(X) | X \subseteq M\}$. Thus since ${}_SS$ has finite uniform dimension and by Lemma 4.32 we have that $\mathrm{Rad}(M) = \mathrm{Ker}(f_1) \cap \cdots \cap \mathrm{Ker}(f_n)$ for some $f_1, ..., f_n \in S$. By Theorem 5.29 $M/\,\mathrm{Rad}(M)$ is semisimple Artinian. Now, suppose M is a generator of $\sigma[M]$. By Theorem 5.29, M is Artinian. □

As the last results show, there are many conditions on a module which makes a Johns module to be Artinian. Here there is not assumption on the ring base but it would be nice to know for which rings R the Johns R-modules are Artinian. Also, as Proposition 4.12 shows, for a Noetherian M-injective module M, M is Kasch if and only if M is Johns without any projectivity condition on M. This raises the question that for which other modules, with no projectivity conditions, these two concepts coincide.

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