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İkinci Mertebeden Fark Denklemlerin Schur Kararlılığı ve Salınımlılığı Üzerine

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ÖZET: Bu çalışmada, ikinci mertebeden fark denklemlerin çözümlerinin Schur kararlı ve salınımlı olup olmadığı durumlar incelendi. Ayrıca Schur kararlı ve salınımlı olan ikinci mertebeden fark denklemlerinin hangi bozunumlar altında Schur kararlı ve salınımlı kaldığı bölgeler belirlendi. Elde edilen sonuçlar nümerik örnekler ile desteklendi.

Anahtar Kelimeler: Fark denklemleri, Schur kararlılık, salınımlılık, hassasiyet, bozunum sistemleri

On Schur Stability and Oscillation of Second Order Difference Equations

ABSTRACT: In this study, the solutions of second order difference equations were examined with respect to whether they were Schur stable and oscillatory or not. The results determining under which perturbation the solutions retain their characteristics were given. The obtained results were analyzed with numerical examples.

Keywords: Difference equations, Schur stability, oscillation, sensitivity, perturbation systems

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Bu çalışma Ramazan ÇAKIROĞLU'nun yüksek lisans tezinden üretilmiştir.

INTRODUCTION

Difference equations have been used in modelling the problems given in many fields such as the study of number of living population in biology, in examining the stock maket movements in economy and in the study of cell movement in medical science (Daganzo, 1994; Elaydi, 2005; Neusser, 2019). Hence how the solutions of difference equations behave is important in field of application.

In recent years, numerous studies have been conducted on the behavior of solutions of difference equations, focusing on Schur stability or oscillation characteristics. For example, sensitivity analysis of Schur stability was studied depending on the stability parameter, which indicates the quality of Schur stability of linear difference equation systems and the stability regions of the given systems were determined with the results obtained from these studies (Duman and Aydın, 2011; Duman and Aydın, 2014; Duman et al., 2016; Duman et al., 2018). Additionally, the results focused on oscillation characteristics were obtained on the oscillation of the difference equations (Braverman and Karpuz, 2011; Asteris and Chatzarakis, 2017; Chatzarakis and Shaikhet, 2017).

However, to our knowledge, there has not been sufficient studies conducted in which oscillation and Schur stability of difference equations are provided together. Therefore, the conditions under which homogeneous second order difference equations with constant coefficients are both Schur stable and oscillatory (SSO) are examined depending on the spectral criterion in this study and the behavior of solutions is investigated as a result of perturbating such conditions.

Consider the second order difference equations with constant coefficients

$$x(n+2) + p_1 x(n+1) + p_2\, x(n) = 0, \tag{1}$$

where $p_1, p_2 \in \mathbb{R}$. Let roots of characteristic equation of (1) be r_1 and r_2. (1) equation can be written as;

$$x(n+2) - (r_1 + r_2)x(n+1) + r_1 . r_2 x(n) = 0. \tag{2}$$

Now let us give the definitions of Schur stability and oscillatory.

Definition 1. A non-trivial solution of (1) is said to be *oscillatory* (around zero) if for every positive integer n there exists $k \in \mathbb{N}$ such that $x(n)x(n+k) < 0$. Otherwise, the solution of (1) is said to be non oscillatory. In other words, the solution of (1) is oscillatory if it is neither eventually positive nor eventually negative (Györi and Ladas, 1991; Agarwal, 2000; Elaydi, 2005).

Definition 2. Under the initial condition $x(n_0)$, (1) equation can be given as;

$$x(n+2) + p_1 x(n+1) + p_2\, x(n) = 0, x(n_0) = a, n \geq n_0. \tag{3}$$

The system (3) is stable if any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|x(n_0)\| < \delta$ implies $\|x(n)\| < \varepsilon$ for all $n > n_0$. If the solution $x(n)$ of (3) is not stable, then it is called unstable. The system (3) is asymptotically (Schur) stable if it is stable and $\lim_{n\to\infty} \|x(n)\| = 0$ (Györi and Ladas, 1991; Akın and Bulgak, 1998; Agarwal, 2000; Elaydi, 2005).

MATERIALS AND METHODS

SSO second order linear homogenous difference equations (1) with constant coefficients are considered. It is determined under which perturbation these equations remain SSO. While doing examination, the roots of the characteristic equation of (1) are considered as real and then complex.

If The Roots of Characteristic Equation are Real

In this section when the roots $r_{1,2}$ of the characteristic equation of (1) are real, the conditions under which (1) is SSO are examined. In addition, with the help of iterative perturbation equations, sensitivity analysis is performed in order to preserve the properties of the given equations.

Now let us give the following theorem determining SSO of (1).

Theorem 1. Let $r_1, r_2 \in \mathbb{R}$. The following statements hold:

i. All solutions of (1) oscillate (about zero) if and only if the characteristic equation has no positive real roots ($r_1, r_2 < 0$),

ii. All solution of (1) converge to zero (i.e., all solutions are Schur stable) if and only if $|r_{1,2}| < 1$ (Elaydi, 2005).

Note 1. Theorem 1 (ii) is called Spectral criterion in the literature.

Remark 1. The equation (1) is SSO if and only if $-1 < r_{1,2} < 0$.

It is clear that Remark 1 is true when *i*) and *ii*) of Theorem 1 are considered together.

Definition 3. r_1 and r_2 are distinct real roots. $x_1(n) = {r_1}^n$ and $x_2(n) = {r_2}^n$ are linearly independent solutions of (2). If $|r_1| > |r_2|$, then we call $x_1(n)$ the dominant solution, r_1 the dominant characteristic root (Elaydi, 2005).

The general solution of (2) is $x(n) = a_1 {r_1}^n + a_2 {r_2}^n$. Let $|r_1| > |r_2|$. Then

$$x(n) = {r_1}^n \left[1 + \left(\frac{r_2}{r_1}\right)^n\right].$$

Since $\left|\frac{r_2}{r_1}\right| < 1$, it follows that $\left(\frac{r_2}{r_1}\right)^n \to 0$ as $n \to \infty$.

Consequently, $\lim_{n\to\infty} x(n) = \lim_{n\to\infty} {r_1}^n$ (Elaydi, 2005).

If The Roots of Characteristic Equation are Complex

Consider the second order difference equations with constant coefficients

$$x(n+2) - (r_1 + r_2)x(n+1) + r_1.r_2 x(n) = 0, \tag{4}$$

where $r_1, r_2 \in \mathbb{C}$. Suppose that $r_{1,2} = \alpha \pm i\beta$, where $\alpha, \beta \in \mathbb{R}, \beta \neq 0$. The solution of (4) is given by

$$x(n) = ar^n \cos(n\theta - b),$$

where $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = arc\tan\left(\frac{\beta}{\alpha}\right)$ and $a, b \in \mathbb{R}$ (Elaydi, 2005).

Now, consider the following perturbation equation of (4)

$$y(n+2) - (\hat{r}_1 + \hat{r}_2)y(n+1) + \hat{r}_1.\hat{r}_2 y(n) = 0. \tag{5}$$

The roots of the characteristic equation of (5) are $\hat{r}_1 = r_1 + h = (\alpha + x) + i(\beta + y) = \hat{\alpha} + i\hat{\beta}$, $\hat{r}_2 = r_2 + h = (\alpha + x) - i(\beta + y) = \hat{\alpha} - i\hat{\beta}$, where $h = x \pm iy$. The general solution of (5) is $y(n) = a\hat{r}^n \cos(n\hat{\theta} - b)$, where $\hat{r} = \sqrt{(\alpha + x)^2 + (\beta + y)^2}$, $\hat{\theta} = arc\tan\left(\frac{\beta + y}{\alpha + x}\right)$, $(\beta + y) \neq 0$ and $a, b \in \mathbb{R}$.

Let's assume that $a = 1, b = 0$ for the sake of convenience in our study. Thus the general solution of (4) and (5) are $x(n) = r^n \cos(n\theta)$ and $y(n) = \hat{r}^n \cos(n\hat{\theta})$, respectively. Note that here $|r_1| = |r_2| = r = \sqrt{\alpha^2 + \beta^2}$.

The general solution of (4) oscillates, since the cosine function oscillates. However, $x(n)$ oscillates in three different ways depending on the location of the conjugate characteristic roots:

1. $r < 1 \Rightarrow r_{1,2}$ lie inside unit disk. The solution $x(n)$ oscillates but converges to zero;
2. $r = 1 \Rightarrow r_{1,2}$ lie on the unit circle. The solution $x(n)$ is oscillating but constant in magnitude;
3. $r > 1 \Rightarrow r_{1,2}$ are outside the unit circle. $x(n)$ is oscillating but increasing in magnitude

(Elaydi, 2005).

Note 2. Considering Theorem 1 and three different cases given above, the solution of equation (4) is SSO if and only if $|r_{1,2}| = r < 1$.

RESULTS AND DISCUSSION

Sensitivity of SSO of Equation (1) when The Roots of Characteristic Equation are Real

Let us determine under which perturbation SSO (1) equation remains SSO. In other words let us determine the sensitivity of SSO of equation (1).

Let $\hat{r}_1 = r_1 + h_1$,$\hat{r}_2 = r_2 + h_2$ ve $h_1, h_2 \in \mathbb{R}$. Consider the following difference equations

$$y(n+2) - (\hat{r}_1 + \hat{r}_2)y(n+1) + \hat{r}_1.\hat{r}_2 y(n) = 0. \quad (6)$$

The difference equation (6) is the perturbation difference equation of (2) (or (1)). The general solution of (6) is $y(n) = c_1\hat{r}_1{}^n + c_2\hat{r}_2{}^n$, where $c_1, c_2 \in \mathbb{R}$ (Elaydi, 2005). Let's assume that $c_1 = c_2 = 1$ for the sake of convenience in our study. Hence, the general solution of (2) and (6) are $x(n) = r_1{}^n + r_2{}^n$ and $y(n) = \hat{r}_1{}^n + \hat{r}_2{}^n$, respectively.

Theorem 2. Let (2) is SSO $(-1 < r_1, r_2 < 0)$. If h_1 and h_2 satisfy the following inequality

$$-1 - r_1 < h_1 < -r_1 \, , -1 - r_2 < h_2 < -r_2$$

then the equation (6) is SSO.

Proof. Let the roots of the characteristic equation of (6) are $\hat{r}_1 = r_1 + h_1$, $\hat{r}_2 = r_2 + h_2$, where $h_1, h_2 \in \mathbb{R}$. The equation (2) is SSO $(-1 < r_{1,2} < 0)$ hence in order for perturbation equation (6) to be SSO, it should satisfy the following inequality $-1 < h_1 + r_1 < 0$ or $-1 < h_2 + r_2 < 0$.

It is clearly seen that the equation (6) is SSO for h_1 and h_2 which satisfy the following inequality

$$-1 - r_1 < h_1 < -r_1 \, , -1 - r_2 < h_2 < -r_2.$$

Example 1. Let (2) be SSO i.e., $(-1 < r_1, r_2 < 0)$. According the Theorem 2, let us examine h_1 and h_2 numerically, which make perturbation equation (6) SSO.

$$\text{For } r_1 = -\tfrac{1}{2}; \; -1 - \left(-\tfrac{1}{2}\right) < h_1 < -\left(-\tfrac{1}{2}\right) \Rightarrow -\tfrac{1}{2} < h_1 < \tfrac{1}{2}.$$

$$\text{For } r_2 = -\tfrac{1}{3}; \; -1 - \left(-\tfrac{1}{3}\right) < h_2 < -\left(-\tfrac{1}{3}\right) \Rightarrow -\tfrac{2}{3} < h_2 < \tfrac{1}{3}.$$

Let $l_i = \frac{i}{10}$, $(i = 1,2,\ldots,9)$. The following table is given.

Table 1. The general and dominant solution of equation (6) with $\hat{r}_1 = r_1 + h_1$ and $\hat{r}_2 = r_2 + h_2$, for $r_1 = -\frac{1}{2}$ and $r_2 = -\frac{1}{3}$

h_1	$\hat{r}_1 = r_1 + h_1$	h_2	$\hat{r}_2 = r_2 + h_2$	$y(n) = \hat{r}_1 + \hat{r}_2$	Dominant Solution
$h_1 \to \left(-\frac{1}{2}\right)^+$	-1^+	$h_2 \to \left(\frac{1}{3}\right)^-$	0^-	$y(n) = (-1^+)^n + (0^-)^n$	$y(n) = (-1^+)^n$
$-\frac{1}{2} + l_1$	-0.9	$\frac{1}{3} - l_1$	-0.1	$y(n) = (-0.9)^n + (-0.1)^n$	$y(n) = (-0.9)^n$
$-\frac{1}{2} + l_2$	-0.8	$\frac{1}{3} - l_2$	-0.2	$y(n) = (-0.8)^n + (-0.2)^n$	$y(n) = (-0.8)^n$
$-\frac{1}{2} + l_3$	-0.7	$\frac{1}{3} - l_3$	-0.3	$y(n) = (-0.7)^n + (-0.3)^n$	$y(n) = (-0.7)^n$
$-\frac{1}{2} + l_4$	-0.6	$\frac{1}{3} - l_4$	-0.4	$y(n) = (-0.6)^n + (-0.4)^n$	$y(n) = (-0.6)^n$
$-\frac{1}{2} + l_5$	-0.5	$\frac{1}{3} - l_5$	-0.5	$y(n) = (-0.5)^n + (-0.5)^n$	$y(n) = (-0.5)^n$
$-\frac{1}{2} + l_6$	-0.4	$\frac{1}{3} - l_6$	-0.6	$y(n) = (-0.4)^n + (-0.6)^n$	$y(n) = (-0.6)^n$
$-\frac{1}{2} + l_7$	-0.3	$\frac{1}{3} - l_7$	-0.7	$y(n) = (-0.3)^n + (-0.7)^n$	$y(n) = (-0.7)^n$
$-\frac{1}{2} + l_8$	-0.2	$\frac{1}{3} - l_8$	-0.8	$y(n) = (-0.2)^n + (-0.8)^n$	$y(n) = (-0.8)^n$
$-\frac{1}{2} + l_9$	-0.1	$\frac{1}{3} - l_9$	-0.9	$y(n) = (-0.1)^n + (-0.9)^n$	$y(n) = (-0.9)^n$
$h_1 \to \left(\frac{1}{2}\right)^-$	0^-	$h_2 \to \left(-\frac{2}{3}\right)^+$	-1^+	$y(n) = (0^-)^n + (-1^+)^n$	$y(n) = (-1^+)^n$

When the graph which will be given below according to Table 1 are examined, it will be seen how SSO of equation (6) changes as the values h_1 and h_2 approach towards the left and right ends in their intervals.

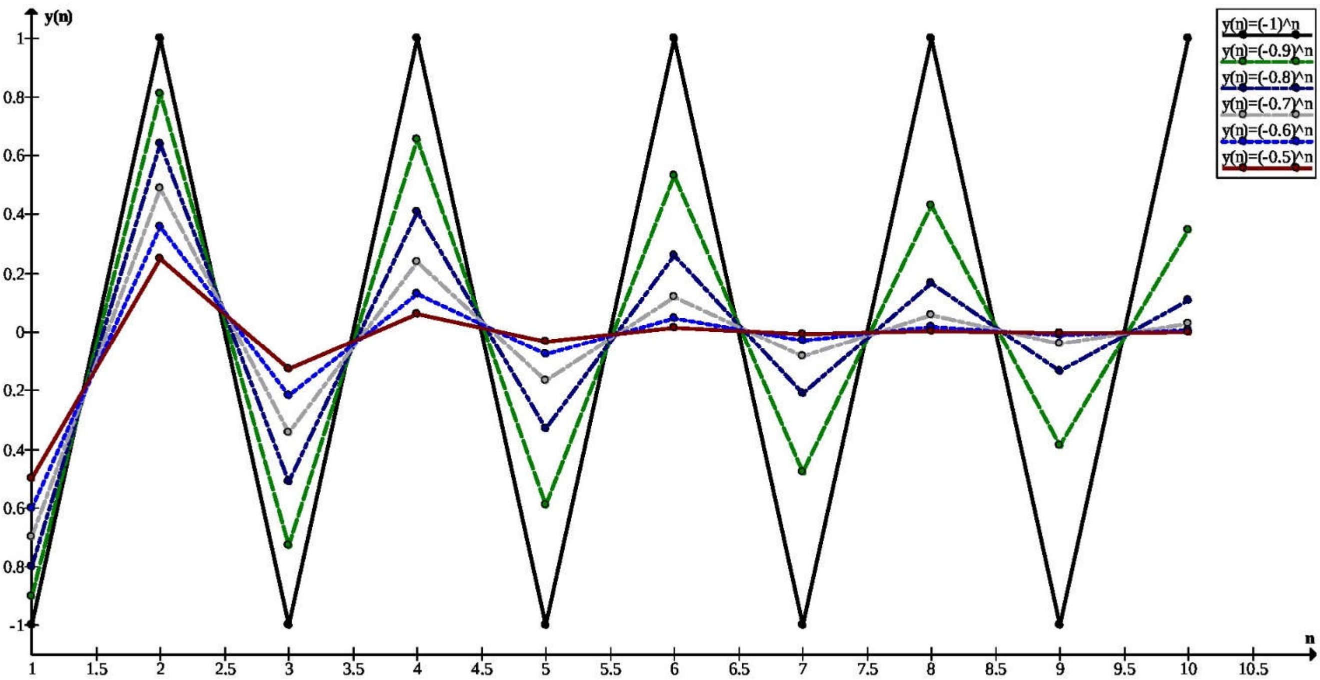

Figure 1. A graphical representation of dominant solutions of $y(n) = \hat{r}_1{}^n + \hat{r}_2{}^n$ for $r_1 = -0.5$ and $r_2 = -0.3333333333$.

Now, let us examine the results that we have obtained by accepting the characteristic roots of second order difference equation as real, considering the characteristic roots as complex.

Sensitivity of SSO of Equation (1) when The Roots of Characteristic Equation are Complex

Now, when the equation (4) is oscillatory, let us give the sets that show the cases where the oscillation of equation (5) is provided.

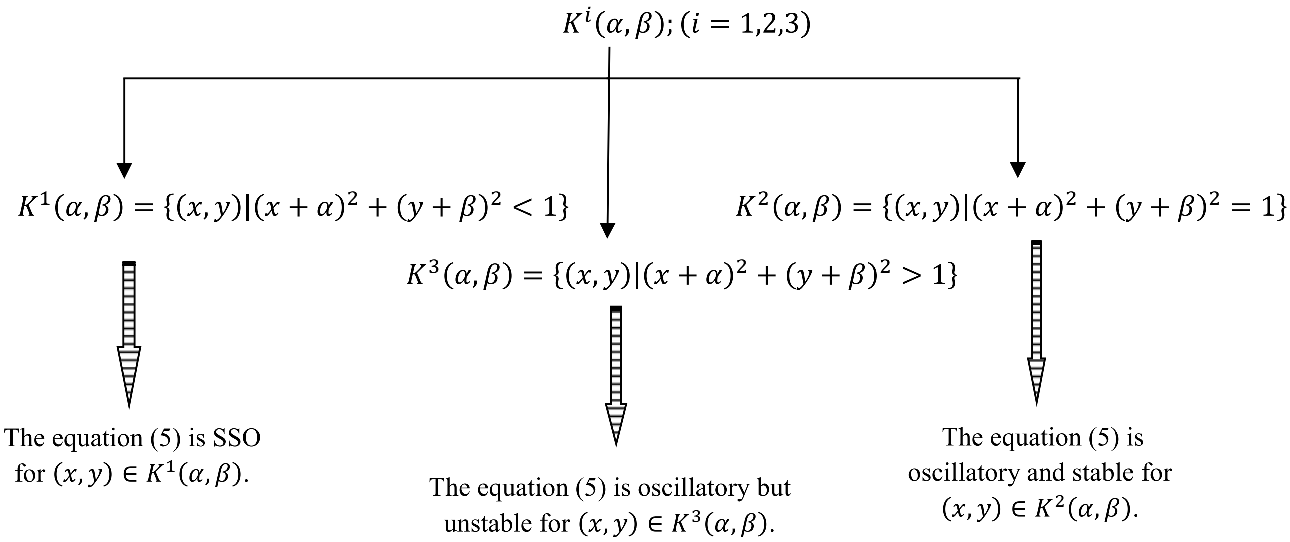

Figure 2. The sets giving the oscillation of the equation (5) are $K^i(\alpha,\beta); (i = 1,2,3)$.

The following theorem shows that for which perturbation equation (5) is SSO when equation (4) is SSO.

Theorem 3. Let the equation (4) be SSO. For $(x, y) \in K^1(\alpha, \beta)$, the perturbation equation (5) is SSO too.

Proof. Let the roots of the characteristic equation of (4) are $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$ and $(x, y) \in K^1(\alpha, \beta)$, where $\alpha, \beta \in \mathbb{R}$ $(\beta \neq 0)$. The equation (4) is SSO, hence $|r_{1,2}| = \alpha^2 + \beta^2 < 1$. The roots of the characteristic equation of (5) are $\hat{r}_1 = (\alpha + x) + i(\beta + y)$, $\hat{r}_2 = (\alpha + x) - i(\beta + y)$, for $(x, y) \in K^1(\alpha, \beta)$

$$(x + \alpha)^2 + (y + \beta)^2 < 1 \ \Rightarrow \ |\hat{r}_{1,2}| < 1.$$

Thus, according the spectral criterion, the perturbation equation (5) is SSO.

Now, according to Theorem 3, let us examine $(x, y) \in K^1(\alpha, \beta)$ numerically for $\left(|r_{1,2}| = r < 1\right)$.

Example 2. Let $r_1 = 0.5 + 0.5i,\ r_2 = 0.5 - 0.5i$. Therefore

$K^1(0.5,0.5) = \{(x, y)|(x + 0.5)^2 + (y + 0.5)^2 < 1\}$.

Let show $K^1(0.5,0.5)$ region below:

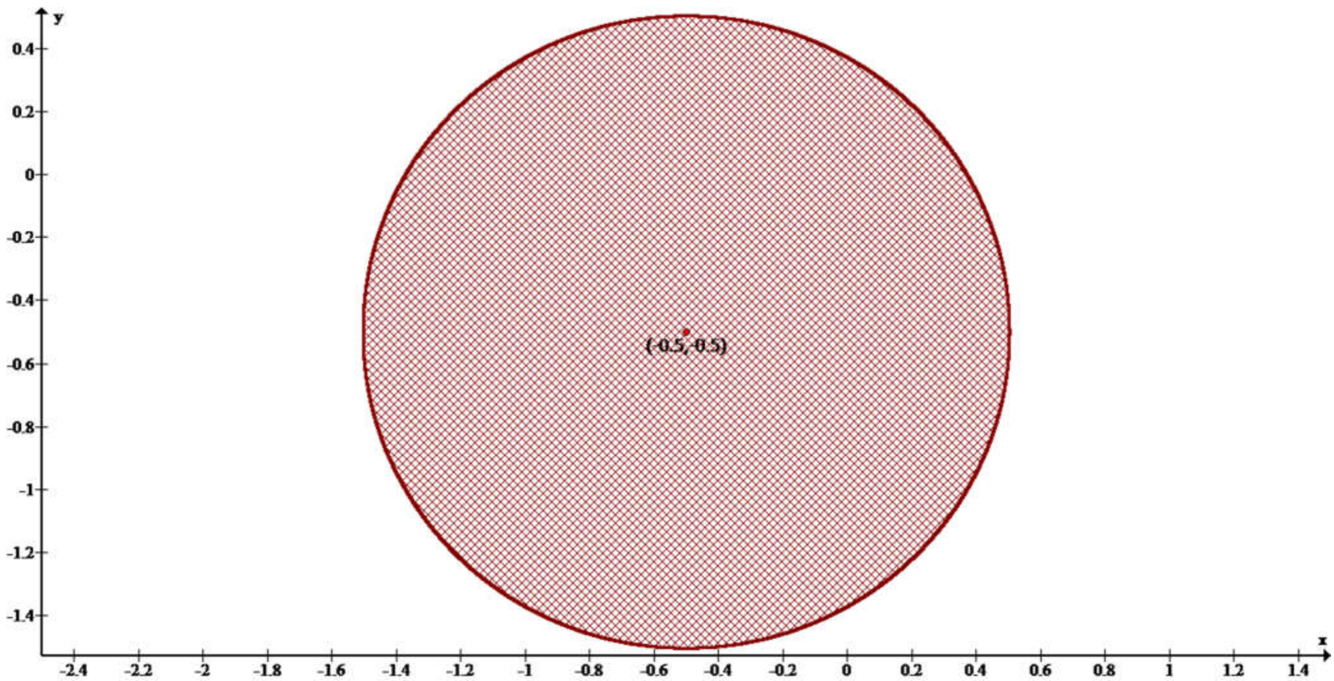

Figure 3. The graph of $K^1(0.5,0.5) = \{(x, y)|(x + 0.5)^2 + (y + 0.5)^2 < 1\}$ for $r_{1,2} = 0.5 \pm 0.5i$.

According to Theorem 3; while the equation (4) is SSO, the perturbation equation (5) is SSO for $(x, y) \in K^1(\alpha, \beta)$.

Now, while the equation is SSO, the $K^1(\alpha, \beta)$ regions are determined for different α and β that make the perturbation equation is SSO.

- Let $r_1 = \alpha_j + i\beta_j, r_2 = \alpha_j - i\beta_j$ for $\alpha_j = \frac{j}{10}, \beta_j = \frac{j}{10}; (j = 1,2,\ldots,9)$. The following graph $K^1(\alpha, \beta) = \left\{(x, y)\left|\left(x + \alpha_j\right)^2 + \left(y + \beta_j\right)^2 < 1\right.\right\}$ shows that under which perturbation the equation (5) is SSO when the equation (4) is SSO.

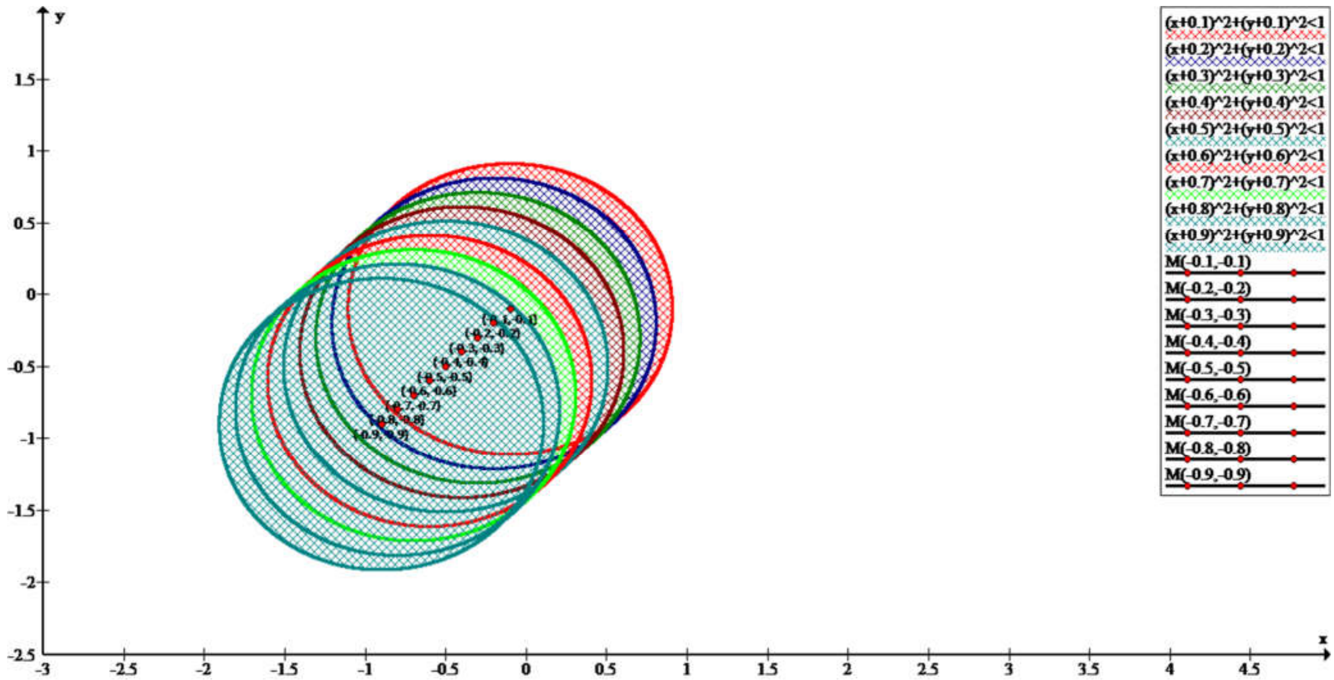

Figure 4. The graph of the unit disk $K^1(\alpha,\beta) = \left\{(x,y)\,\middle|\,(x+\alpha_j)^2 + (y+\beta_j)^2 < 1\right\}$ for $\alpha_j,\beta_j = \frac{j}{10}(j = 1,2,\dots,9)$.

- Let $r_1 = \alpha_j + i\beta_j, r_2 = \alpha_j - i\beta_j$ for $\alpha_j = -\frac{j}{10}, \beta_j = -\frac{j}{10}; (j = 1,2,\dots,9)$. The following graph $K^1(\alpha,\beta) = \left\{(x,y)\,\middle|\,(x+\alpha_j)^2 + (y+\beta_j)^2 < 1\right\}$ shows that under which perturbation the equation (5) is SSO when the equation (4) is SSO.

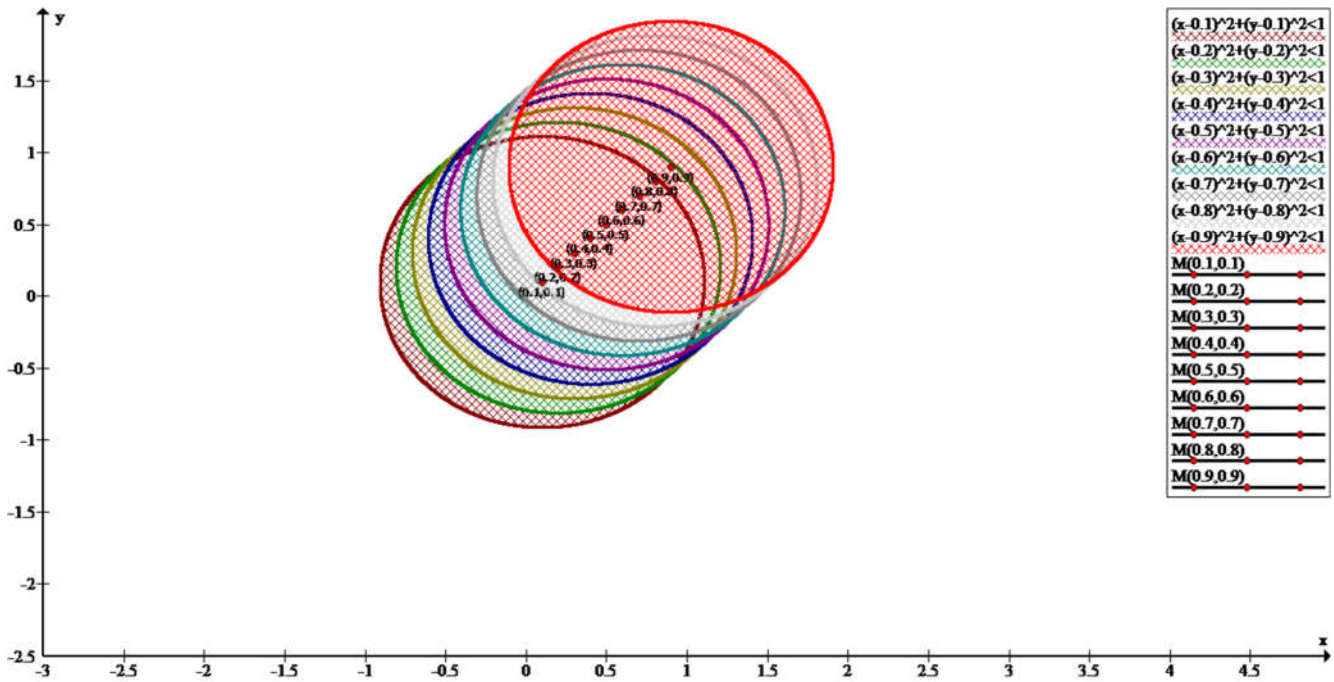

Figure 5. The graph of the unit disk $K^1(\alpha,\beta) = \left\{(x,y)\,\middle|\,(x+\alpha_j)^2 + (y+\beta_j)^2 < 1\right\}$ for $\alpha_j,\beta_j = -\frac{j}{10}(j = 1,2,\dots,9)$.

- Let $r_1 = \alpha_j + i\beta_j, r_2 = \alpha_j - i\beta_j$ for $\alpha_j = -\frac{j}{10}, \beta_j = \frac{j}{10}; (j = 1,2,\ldots,9)$. The following graph $K^1(\alpha,\beta) = \left\{(x,y) \left| \left(x+\alpha_j\right)^2 + \left(y+\beta_j\right)^2 < 1\right.\right\}$ shows that under which perturbation the equation (5) is SSO when the equation (4) is SSO.

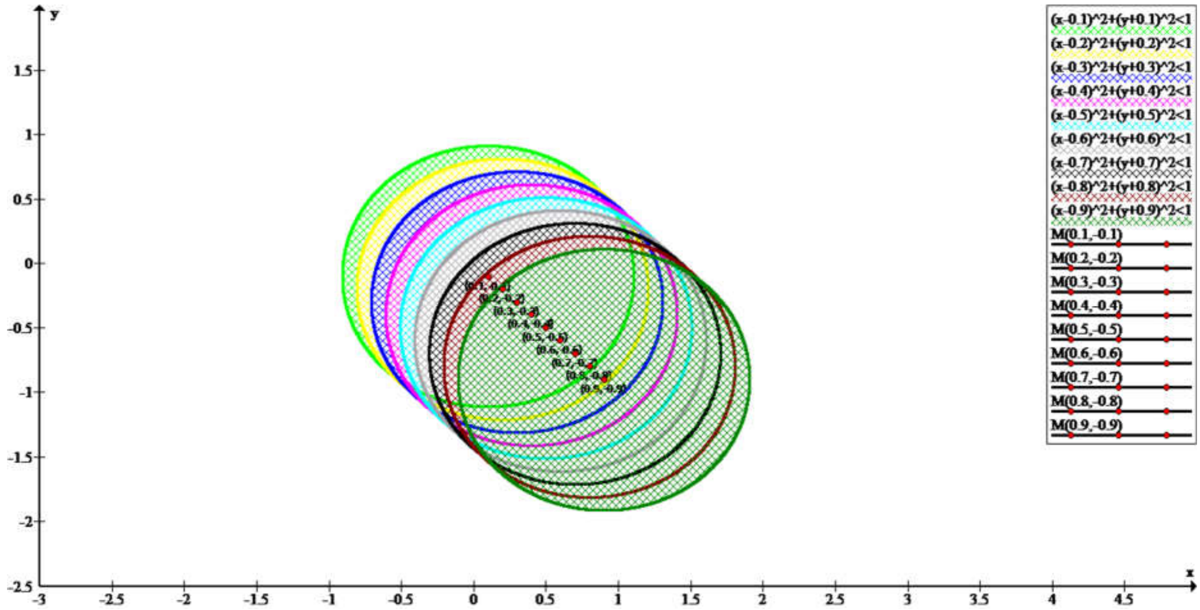

Figure 6. The graph of the unit disk $K^1(\alpha,\beta) = \left\{(x,y) \left| \left(x+\alpha_j\right)^2 + \left(y+\beta_j\right)^2 < 1\right.\right\}$ for $\alpha_j = -\frac{j}{10}$, $\beta_j = \frac{j}{10}; (j = 1,2,\ldots,9)$.

- Let $r_1 = \alpha_j + i\beta_j, r_2 = \alpha_j - i\beta_j$ for $\alpha_j = \frac{j}{10}, \beta_j = -\frac{j}{10}; (j = 1,2,\ldots,9)$. The following graph $K^1(\alpha,\beta) = \left\{(x,y) \left| \left(x+\alpha_j\right)^2 + \left(y+\beta_j\right)^2 < 1\right.\right\}$ shows that under which perturbation the equation (5) is SSO when the equation (4) is SSO.

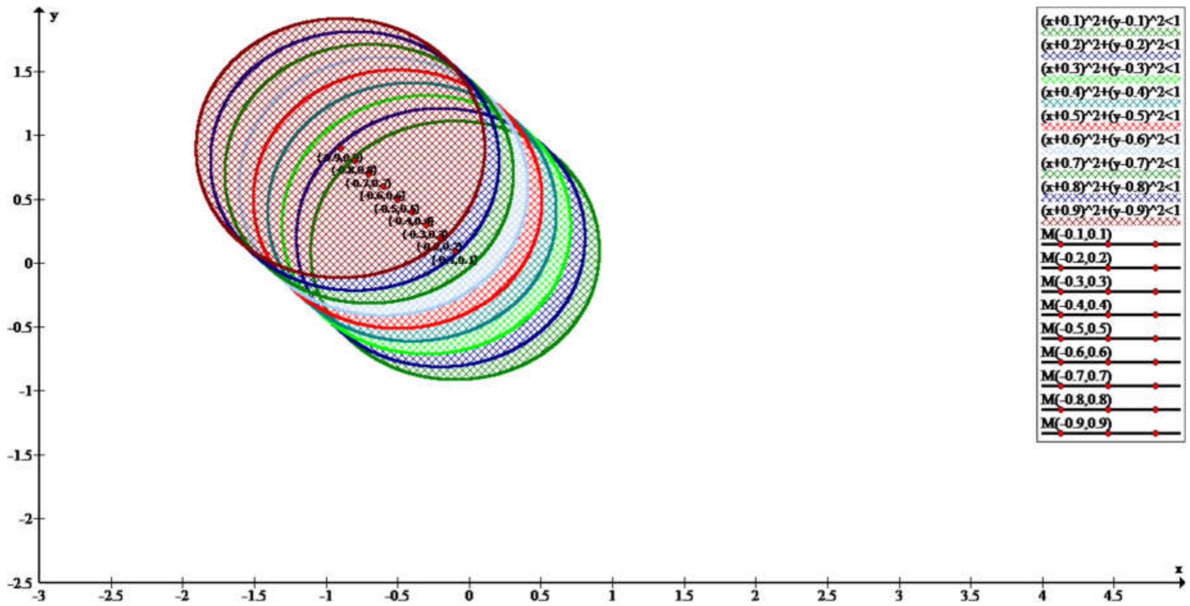

Figure 7. The graph of the unit disk $K^1(\alpha,\beta) = \left\{(x,y) \left| \left(x+\alpha_j\right)^2 + \left(y+\beta_j\right)^2 < 1\right.\right\}$ for $\alpha_j = \frac{j}{10}$, $\beta_j = -\frac{j}{10}; (j = 1,2,\ldots,9)$.

The obtained unit discs given in Figure 4-7 are shown in the same graph in Figure 8. The equation (5) is SSO for the points that are taken from the inside of each unit discs.

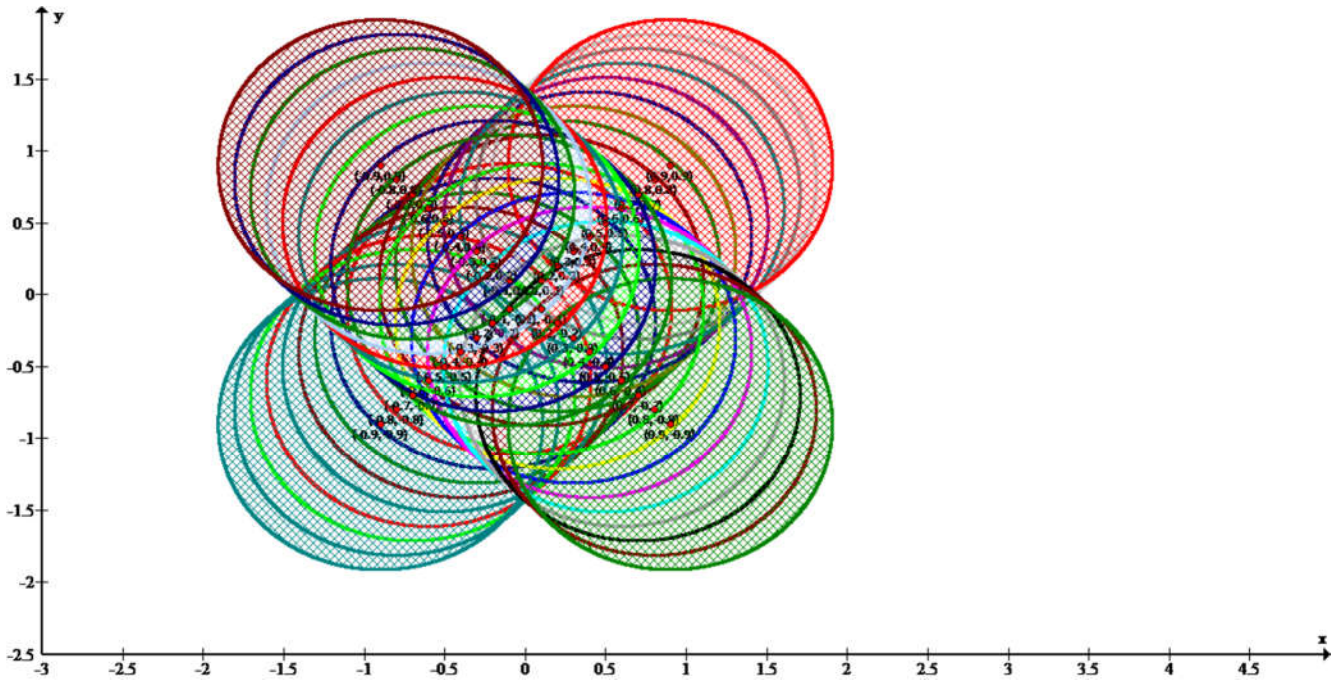

Figure 8. A graphical representation of the unit discs $K^1(\alpha,\beta)$ given in Figure 4-7 for $r_1 = \alpha_j + i\beta_j$, $r_2 = \alpha_j - i\beta_j$. According to Figure 3, the roots $\hat{r}_1, \hat{r}_2$ of the characteristic equation of (5) are given in Table 2, where $z_i = \frac{i}{10}, (i = 1,2\ldots,5)$.

Table 2. The values $\hat{r}_1$, $\hat{r}_2$, $\hat{r}$ and $\hat{\theta}$ for $r_1 = 0.5 + 0.5i$ and $r_2 = 0.5 - 0.5i$.

$(x,y) = (z_i, -z_i)$	$\hat{r}_1 = (\alpha + x) + i(\beta + y)$	$\hat{r}_2 = (\alpha + x) - i(\beta + y)$	$\hat{r} = \sqrt{(\alpha + x)^2 + (\beta + y)^2}$	$\hat{\theta} = arc\,tan\left(\frac{\beta + y}{\alpha + x}\right)$
$(z_1, -z_1)$	$0.6 + i0.4$	$0.6 - i0.4$	0,7211102551	33,690067526
$(z_2, -z_2)$	$0.7 + i0.3$	$0.7 - i0.3$	0,7615773106	23,1985905136
$(z_3, -z_3)$	$0.8 + i0.2$	$0.8 - i0.2$	0,8246211251	14,0362434679
$(z_4, -z_4)$	$0.9 + i0.1$	$0.9 - i0.1$	0,9055385138	6,3401917459
$(z_5, -z_5)$	$1 + i0$	$1 - i0$	1	0

In Figure 9, SSO variation of the equation (5) is given according to points, where $(x,y) \in K^1(\alpha,\beta)$ that are take in from the inside of unit discs whose center is M(-0.5,-0.5).

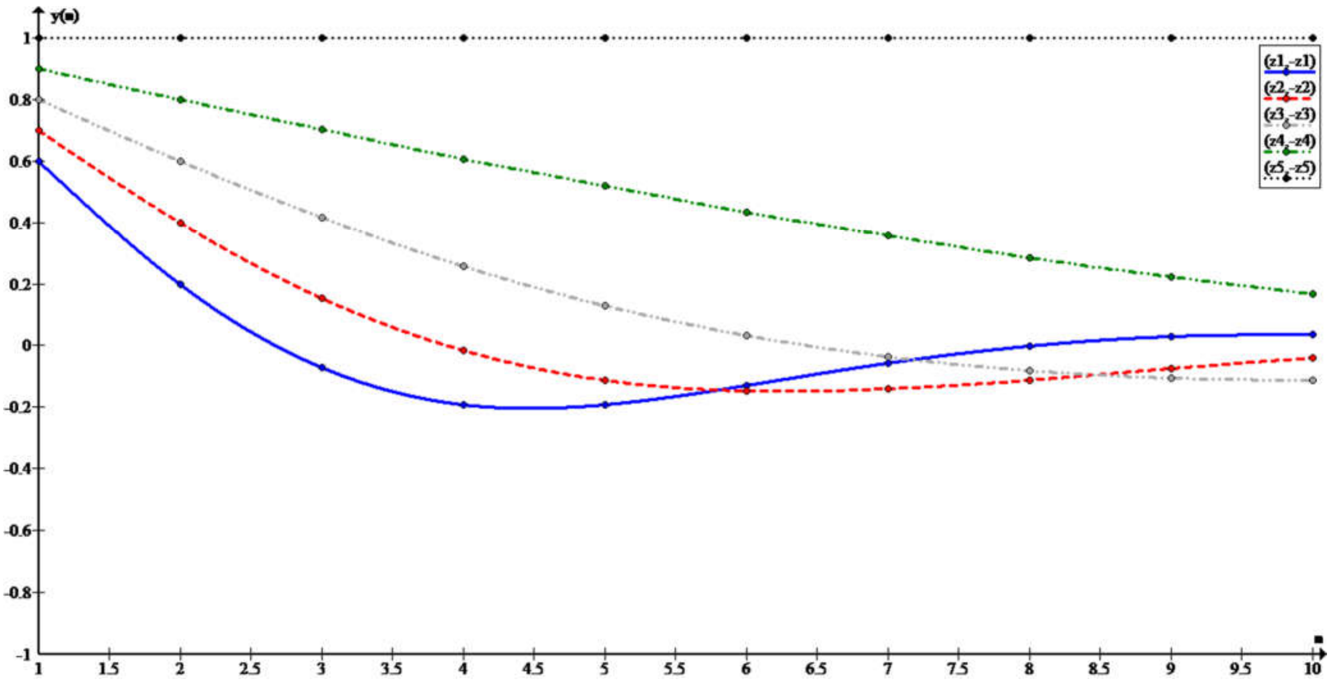

Figure 9. The representation of $y(n) = \hat{r}^n \cos(n\hat{\theta})$ for $r_1 = 0.5 + 0.5i, r_2 = 0.5 - 0.5i$, $x \in [1,10]$ and $y \in [-1,1]$.

Iterative Perturbation Equations of Second Order Difference Equations

In this section, let the difference equation that is SSO be given. Let us determine under which conditions each perturbation equation remains SSO when this equation is subjected to iterative perturbations.

Consider the following perturbation equation of (5)

$$\tilde{y}(n+2) - (\tilde{r}_1 + \tilde{r}_2)\tilde{y}(n+1) + \tilde{r}_1.\tilde{r}_2\tilde{y}(n) = 0. \quad (7)$$

The roots of the characteristic equation of (7) are $\tilde{r}_1 = \hat{r}_1 + h = (\hat{\alpha} + x) + i(\hat{\beta} + y) = \tilde{\alpha} + i\tilde{\beta}$, $\tilde{r}_2 = \hat{r}_2 + h = (\hat{\alpha} + x) - i(\hat{\beta} + y) = \tilde{\alpha} - i\tilde{\beta}$, where $h = x \pm iy$. The general solution of (7) is $\tilde{y}(n) = \tilde{r}^n \cos(n\tilde{\theta})$, where $\tilde{r} = \sqrt{(\hat{\alpha} + x)^2 + (\hat{\beta} + y)^2}, \tilde{\theta} = \tan^{-1}\left(\frac{\hat{\beta}+y}{\hat{\alpha}+x}\right), (\hat{\beta} + y) \neq 0$.

Note 3. Perturbation of equation (4) gives equation (5) and perturbation of equation (5) gives equation (7). If the perturbation is repeated in the same way, iterative perturbation equations are obtained.

The following theorem shows that for which perturbation equation (7) is SSO when equation (5) is SSO.

Theorem 4. Let the equation (5) be SSO. For $(x, y) \in \widehat{K}^1(\hat{\alpha}, \hat{\beta}) = \left\{(x, y) \middle| (x + \hat{\alpha})^2 + (y + \hat{\beta})^2 < 1\right\}$, the perturbation equation (7) is SSO too.

Proof. Let the roots of the characteristic equation of (5) are $\hat{r}_1 = \hat{\alpha} + i\hat{\beta}$, $\hat{r}_2 = \hat{\alpha} - i\hat{\beta}$ and $(x, y) \in \widehat{K}^1(\hat{\alpha}, \hat{\beta})$, where $\alpha, \beta \in \mathbb{R}$ $(\beta \neq 0)$. The equation (5) is SSO, hence $|\hat{r}_{1,2}| = \hat{\alpha}^2 + \hat{\beta}^2 < 1$. The roots of the characteristic equation of (7) are $\tilde{r}_1 = (\hat{\alpha} + x) + i(\hat{\beta} + y)$,$\tilde{r}_2 = (\hat{\alpha} + x) - i(\hat{\beta} + y)$, for $(x, y) \in \widehat{K}^1(\hat{\alpha}, \hat{\beta})$

$$(x + \hat{\alpha})^2 + (y + \hat{\beta})^2 < 1 \Rightarrow |\tilde{r}_{1,2}| < 1.$$

Thus, according the spectral criterion, the perturbation equation (7) is SSO.

Let us examine $(x, y) \in \widehat{K}^1(\hat{\alpha}, \hat{\beta})$ numerically for $(|r_{1,2}| = r < 1)$.

Example 3. Let $r_1 = 0.5 + 0.5i, r_2 = 0.5 - 0.5i$, $\alpha = 0.5$ and $\beta = 0.5$. Therefore

$$\widehat{K}^1(0.5 + x, 0.5 + y) = \{(x, y)|(2x + 0.5)^2 + (2y + 0.5)^2 < 1\}.$$

Let us show $\widehat{K}^1(0.5+x, 0.5+y)$ region below:

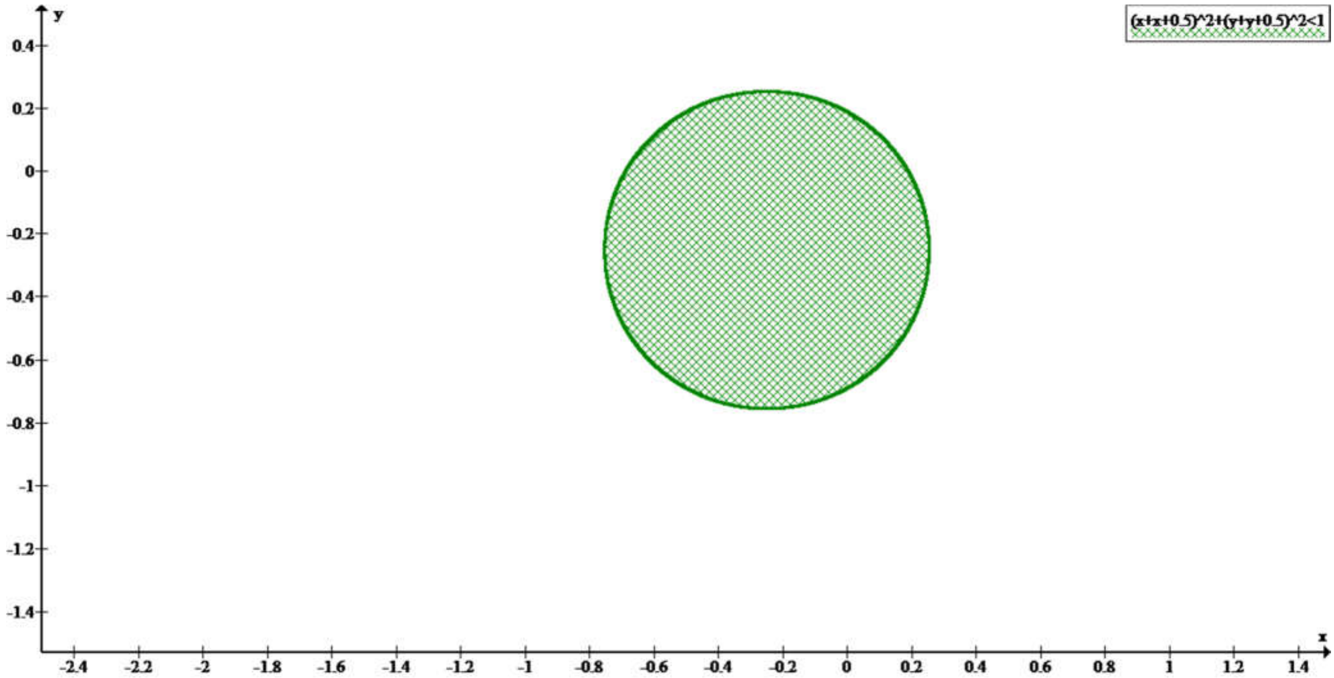

Figure 10. The graph of the unit disk $\widehat{K}^1(0.5+x, 0.5+y) = \{(x,y)|(x+x+0.5)^2 + (y+y+0.5)^2 < 1\}$ for $r_1 = 0.5+0.5i, r_2 = 0.5-0.5i$.

Now, the unit discs obtained in Figure 3 and Figure 10 are given in the same graph in Figure 11.

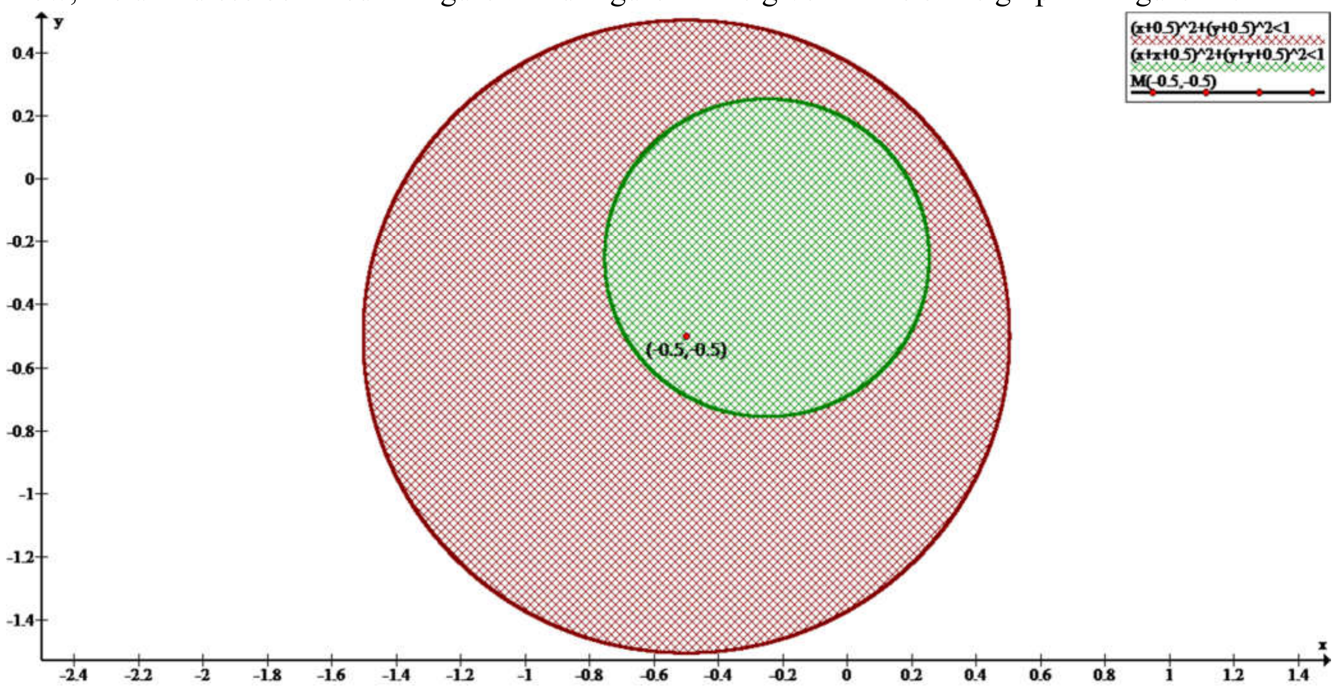

Figure 11. The unit discs obtained in Figure 3 and Figure 10 for $r_1 = 0.5+0.5i, r_2 = 0.5-0.5i$ are given in the same figure.

It is clearly seen from Figure 11, $\widehat{K}^1(\hat{\alpha}, \hat{\beta}) \subseteq K^1(\alpha, \beta)$.
By repeating perturbation of the equation (4), let's obtain the equations which are perturbation of each other and show them in Table 3-4 and Figure 12.

Table 3. The equations and perturbation equations of these equations.

	Equation	*Perturbation Equation*
1.	$x(n+2)-(r_1+r_2)x(n+1)+r_1.r_2x(n)=0$	$y(n+2)-(\hat{r}_1+\hat{r}_2)y(n+1)+\hat{r}_1.\hat{r}_2y(n)=0$
2.	$y(n+2)-(\hat{r}_1+\hat{r}_2)y(n+1)+\hat{r}_1.\hat{r}_2y(n)=0$	$\tilde{y}(n+2)-(\tilde{r}_1+\tilde{r}_2)\tilde{y}(n+1)+\tilde{r}_1.\tilde{r}_2\tilde{y}(n)=0$
3.	$\tilde{y}(n+2)-(\tilde{r}_1+\tilde{r}_2)\tilde{y}(n+1)+\tilde{r}_1.\tilde{r}_2\tilde{y}(n)=0$	$\breve{y}(n+2)-(\breve{r}_1+\breve{r}_2)\breve{y}(n+1)+\breve{r}_1.\breve{r}_2\breve{y}(n)=0$
4.	$\breve{y}(n+2)-(\breve{r}_1+\breve{r}_2)\breve{y}(n+1)+\breve{r}_1.\breve{r}_2\breve{y}(n)=0$	$\bar{y}(n+2)-(\bar{r}_1+\bar{r}_2)\bar{y}(n+1)+\bar{r}_1.\bar{r}_2\bar{y}(n)=0$
5.	$\bar{y}(n+2)-(\bar{r}_1+\bar{r}_2)\bar{y}(n+1)+\bar{r}_1.\bar{r}_2\bar{y}(n)=0$	$\dot{y}(n+2)-(\dot{r}_1+\dot{r}_2)\dot{y}(n+1)+\dot{r}_1.\dot{r}_2\dot{y}(n)=0$
6.	$\dot{y}(n+2)-(\dot{r}_1+\dot{r}_2)\dot{y}(n+1)+\dot{r}_1.\dot{r}_2\dot{y}(n)=0$	$\ddot{y}(n+2)-(\ddot{r}_1+\ddot{r}_2)\ddot{y}(n+1)+\ddot{r}_1.\ddot{r}_2\ddot{y}(n)=0$
7.	$\ddot{y}(n+2)-(\ddot{r}_1+\ddot{r}_2)\ddot{y}(n+1)+\ddot{r}_1.\ddot{r}_2\ddot{y}(n)=0$	$\dddot{y}(n+2)-(\dddot{r}_1+\dddot{r}_2)\dddot{y}(n+1)+\dddot{r}_1.\dddot{r}_2\dddot{y}(n)=0$
8.	$\dddot{y}(n+2)-(\dddot{r}_1+\dddot{r}_2)\dddot{y}(n+1)+\dddot{r}_1.\dddot{r}_2\dddot{y}(n)=0$	$\hat{y}(n+2)-(\hat{r}_1+\hat{r}_2)\hat{y}(n+1)+\hat{r}_1.\hat{r}_2\hat{y}(n)=0$
9.	$\hat{y}(n+2)-(\hat{r}_1+\hat{r}_2)\hat{y}(n+1)+\hat{r}_1.\hat{r}_2\hat{y}(n)=0$	$\overleftrightarrow{y}(n+2)-(\overleftrightarrow{r}_1+\overleftrightarrow{r}_2)\overleftrightarrow{y}(n+1)+\overleftrightarrow{r}_1.\overleftrightarrow{r}_2\overleftrightarrow{y}(n)=0$

Table 4. The characteristic roots and the unit discs regions of perturbation equations given in Table 3.

The perturbated roots of characteristic equation for $h=x\pm iy$	***The unit disk $\widehat{K}^1(\widehat{\alpha},\widehat{\beta})$***
$\hat{r}_1=r_1+h=(\alpha+x)+i(\beta+y)=\hat{\alpha}+i\hat{\beta}$, $\hat{r}_2=r_2+h=(\alpha+x)-i(\beta+y)=\hat{\alpha}-i\hat{\beta}$	$\hat{K}^1(\hat{\alpha},\hat{\beta})=\{(x,y)\,\lvert\,(x+\hat{\alpha})^2+(y+\hat{\beta})^2<1\}$
$\tilde{r}_1=\hat{r}_1+h=(\hat{\alpha}+x)+i(\hat{\beta}+y)=\tilde{\alpha}+i\tilde{\beta}$, $\tilde{r}_2=\hat{r}_2+h=(\hat{\alpha}+x)-i(\hat{\beta}+y)=\tilde{\alpha}-i\tilde{\beta}$	$\tilde{K}^1(\tilde{\alpha},\tilde{\beta})=\{(x,y)\,\lvert\,(x+\tilde{\alpha})^2+(y+\tilde{\beta})^2<1\}$
$\breve{r}_1=\tilde{r}_1+h=(\tilde{\alpha}+x)+i(\tilde{\beta}+y)=\breve{\alpha}+i\breve{\beta}$, $\breve{r}_2=\tilde{r}_2+h=(\tilde{\alpha}+x)-i(\tilde{\beta}+y)=\breve{\alpha}-i\breve{\beta}$	$\breve{K}^1(\breve{\alpha},\breve{\beta})=\{(x,y)\,\lvert\,(x+\breve{\alpha})^2+(y+\breve{\beta})^2<1\}$
$\bar{r}_1=\breve{r}_1+h=(\breve{\alpha}+x)+i(\breve{\beta}+y)=\bar{\alpha}+i\bar{\beta}$, $\bar{r}_2=\breve{r}_2+h=(\breve{\alpha}+x)-i(\breve{\beta}+y)=\bar{\alpha}-i\bar{\beta}$	$\bar{K}^1(\bar{\alpha},\bar{\beta})=\{(x,y)\,\lvert\,(x+\bar{\alpha})^2+(y+\bar{\beta})^2<1\}$
$\dot{r}_1=\bar{r}_1+h=(\bar{\alpha}+x)+i(\bar{\beta}+y)=\dot{\alpha}+i\dot{\beta}$, $\dot{r}_2=\bar{r}_2+h=(\bar{\alpha}+x)-i(\bar{\beta}+y)=\dot{\alpha}-i\dot{\beta}$	$\dot{K}^1(\dot{\alpha},\dot{\beta})=\{(x,y)\,\lvert\,(x+\dot{\alpha})^2+(y+\dot{\beta})^2<1\}$
$\ddot{r}_1=\dot{r}_1+h=(\dot{\alpha}+x)+i(\dot{\beta}+y)=\ddot{\alpha}+i\ddot{\beta}$, $\ddot{r}_2=\dot{r}_2+h=(\dot{\alpha}+x)-i(\dot{\beta}+y)=\ddot{\alpha}-i\ddot{\beta}$	$\ddot{K}^1(\ddot{\alpha},\ddot{\beta})=\{(x,y)\,\lvert\,(x+\ddot{\alpha})^2+(y+\ddot{\beta})^2<1\}$
$\dddot{r}_1=\ddot{r}_1+h=(\ddot{\alpha}+x)+i(\ddot{\beta}+y)=\dddot{\alpha}+i\dddot{\beta}$, $\dddot{r}_2=\ddot{r}_2+h=(\ddot{\alpha}+x)-i(\ddot{\beta}+y)=\dddot{\alpha}-i\dddot{\beta}$	$\dddot{K}^1(\dddot{\alpha},\dddot{\beta})=\{(x,y)\,\lvert\,(x+\dddot{\alpha})^2+(y+\dddot{\beta})^2<1\}$
$\hat{r}_1=\dddot{r}_1+h=(\dddot{\alpha}+x)+i(\dddot{\beta}+y)=\hat{\alpha}+i\hat{\beta}$, $\hat{r}_2=\dddot{r}_2+h=(\dddot{\alpha}+x)-i(\dddot{\beta}+y)=\hat{\alpha}-i\hat{\beta}$	$\hat{K}^1(\hat{\alpha},\hat{\beta})=\{(x,y)\,\lvert\,(x+\hat{\alpha})^2+(y+\hat{\beta})^2<1\}$
$\overleftrightarrow{r}_1=\hat{r}_1+h=(\hat{\alpha}+x)+i(\hat{\beta}+y)=\overleftrightarrow{\alpha}+i\overleftrightarrow{\beta}$, $\overleftrightarrow{r}_2=\hat{r}_2+h=(\hat{\alpha}+x)-i(\hat{\beta}+y)=\overleftrightarrow{\alpha}-i\overleftrightarrow{\beta}$	$\overleftrightarrow{K}^1(\overleftrightarrow{\alpha},\overleftrightarrow{\beta})=\{(x,y)\,\lvert\,(x+\overleftrightarrow{\alpha})^2+(y+\overleftrightarrow{\beta})^2<1\}$

Let's show the unit discs regions obtained from Table 3 and Table 4 in the graph below.

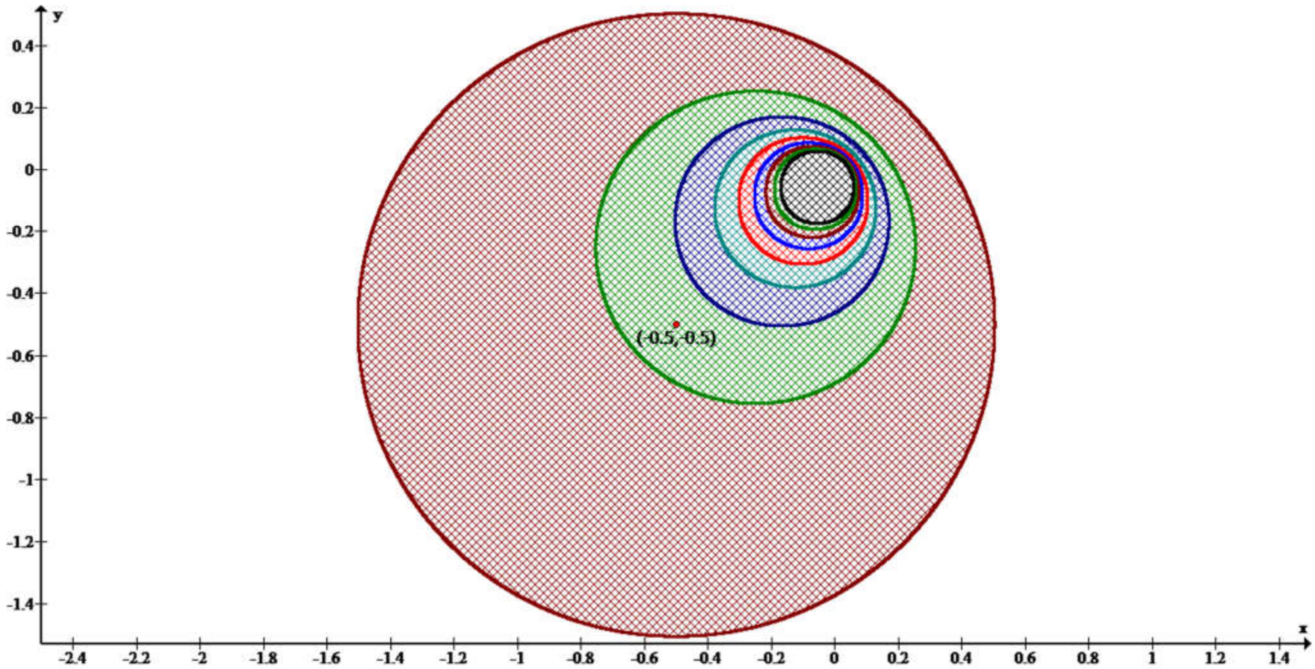

Figure 12. The unit discs regions formed by the perturbation equations obtained in Table 3 and Table 4.

It is clearly seen from Figure 12, $\overleftrightarrow{K}^1(\overleftrightarrow{\alpha},\overleftrightarrow{\beta})\subseteq\cdots\subseteq\widetilde{K}^1(\tilde{\alpha},\tilde{\beta})\subseteq\widehat{K}^1(\hat{\alpha},\hat{\beta})\subseteq K^1(\alpha,\beta)$.

CONCLUSION

In this study, the conditions under which second order difference equations are both Schur stable and oscillatory (SSO) are examined. Since it is easier to calculate the roots of the characteristic equation for systems of second-order difference equations than to calculate the Schur stability parameter, the results in this study are examined depending on the spectral criteria. Additionally, the stability regions of the second-order difference equations are determined. The obtained results are analyzed with numerical examples.

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Conflict of Interest

The article authors declare that there is no conflict of interest between them.

Author's Contributions

The authors declare that they have contributed equally to the article.

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