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Some Approximation Results on $\lambda-$Szász-Mirakjan-Kantorovich Operators

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Abstract

In this article, we purpose to obtain several approximation properties of Szász-Mirakjan-Kantorovich operators with shape parameter $\lambda \in [-1,1]$. We compute some preliminaries results such as moments and central moments for these operators. Next, we derive the Korovkin type convergence theorem, estimate the degree of convergence with respect to the moduli of continuity, for the functions belong to Lipschitz-type class and Peetre's K-functional, respectively. Further, we investigate Voronovskaya type asymptotic theorem and give the comparison of the convergence of these newly defined operators to the certain functions with some graphics.

1. Introduction

In [1, 2], Szász and Mirakjan defined and introduced the following polynomials

$$S_m(\mu;y) = \sum_{j=0}^{\infty} \mu\left(\frac{j}{m}\right) s_{m,j}(y), \tag{1.1}$$

where $y \geq 0$, $m \in \mathbb{N}$, $\mu \in C[0,\infty)$ and Szász-Mirakjan basis functions $s_{m,j}(y)$ are given as below:

$$s_{m,j}(y) = e^{-my}\frac{(my)^j}{j!}.$$

A Kantorovich variant of (1.1) operators is presented by Ditzian and Totik [3] as follows:

$$K_m(\mu;y) = m\sum_{j=0}^{\infty} s_{m,j}(y) \int_{\frac{j}{m}}^{\frac{j+1}{m}} \mu(t)\,dt,\ y \geq 0. \tag{1.2}$$

Various approximation features of (1.1) and (1.2) operators have been introduced by many authors. More details on these directions, we refer the readers to [4]-[12].
Very recently, Qi et al. [13] defined a new generalization of $\lambda-$Szász-Mirakjan operators with shape parameter $\lambda \in [-1,1]$, as below:

$$S_{m,\lambda}(\mu;y) = \sum_{j=0}^{\infty} \mu\left(\frac{j}{m}\right) \widetilde{s}_{m,j}(\lambda;y),$$

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where Szász-Mirakjan bases functions $\widetilde{s}_{m,j}(\lambda;y)$ with shape parameter $\lambda\in[-1,1]$:

$$\widetilde{s}_{m,0}(\lambda;y)=s_{m,0}(y)-\frac{\lambda}{m+1}s_{m+1,1}(y);$$

$$\widetilde{s}_{m,i}(\lambda;y)=s_{m,i}(y)+\lambda\left(\frac{m-2i+1}{m^2-1}s_{m+1,i}(y)\ -\frac{m-2i-1}{m^2-1}s_{m+1,i+1}(y)\right),\ (i=1,2,...,\infty,\ y\in[0,\infty)). \tag{1.3}$$

They studied several theorems such as Korovkin approximation, local approximation, Lipschitz type convergence, Voronovskaja and Grüss-Voronovskaja type for these new form operators. In the literature, recently several researchers have obtained some approximation results for various linear positive operators with shape parameter $\lambda\in[-1,1]$, one can refer to [14]-[23].
Now, motivated by all above mentioned works, we propose the Kantorovich kind of $\lambda-$Szász-Mirakjan operators as follows:

$$R_{m,\lambda}(\mu;y)=m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_{\frac{j}{m}}^{\frac{j+1}{m}}\mu(t)\,dt,\quad y\in[0,\infty), \tag{1.4}$$

where $\widetilde{s}_{m,j}(\lambda;y)$ $(j=0,1,..\infty)$ given in (1.3) and $\lambda\in[-1,1]$.
The structure of this work is organized as follows: In section 2, we compute some moments and central moments. In section 3, we establish Korovkin type approximation theorem and discuss the order of convergence in terms of the usual moduli of continuity, for the function belongs to Lipschitz-type class and Peetre's K-functional, respectively. In section 4, we derive a Voronovskaya type asymptotic theorem. In the final section, we show the comparison of the convergence of operators (1.4) to the certain functions for the different values of m and λ. We also compare the convergence of operators (1.2) and (1.4) to the certain function to see the behaviour of λ parameter.

2. Preliminaries

Lemma 2.1. *[13]. For the $\lambda-$Szász-Mirakjan operators $S_{m,\lambda}(\mu;y)$ following expressions are satisfied:*

$$S_{m,\lambda}(1;y)=1;$$

$$S_{m,\lambda}(t;y)=y+\left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda;$$

$$S_{m,\lambda}(t^2;y)=y^2+\frac{y}{m}+\left[\frac{2y+e^{-(m+1)y}-1-4(m+1)y^2}{m^2(m-1)}\right]\lambda;$$

$$S_{m,\lambda}(t^3;y)=y^3+\frac{3y^2}{m}+\frac{y}{m^2}+\left[\frac{1-e^{-(m+1)y}-2y+3(m-3)(m+1)y^2-6(m+1)y^3}{m^3(m-1)}\right]\lambda;$$

$$S_{m,\lambda}(t^4;y)=y^4+\frac{6y^3}{m}+\frac{7y^2}{m^2}+\frac{y}{m^3}+\left[\frac{e^{-(m+1)y}-1+2my+2(3m-11)(m+1)y^2+4(m-8)(m+1)^2y^3-8(m+1)^3y^4}{m^4(m-1)}\right]\lambda.$$

Lemma 2.2. *Let the operators $R_{m,\lambda}$ be defined by (1.4). Then, we have*

$$R_{m,\lambda}(1;y)=1; \tag{2.1}$$

$$R_{m,\lambda}(t;y)=y+\frac{1}{2m}+\left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda; \tag{2.2}$$

$$R_{m,\lambda}(t^2;y)=y^2+\frac{2y}{m}+\frac{1}{3m^2}+\left[\frac{-4(m+1)y^2}{m^2(m-1)}\right]\lambda; \tag{2.3}$$

$$R_{m,\lambda}(t^3;y)=y^3+\frac{9y^2}{2m}+\frac{7y}{2m^2}+\frac{1}{4m^3}+\left[\frac{3(m-5)(m+1)y^2-y-6(m+1)y^3+\frac{1}{2}-\frac{1}{2}e^{-(m+1)y}}{m^3(m-1)}\right]\lambda; \tag{2.4}$$

$$R_{m,\lambda}(t^4;y)=y^4+\frac{8y^3}{m}+\frac{15y^2}{m^2}+\frac{6y}{m^3}+\frac{1}{5m^4}+\left[\frac{2(m-1)y+12(m-4)(m+1)y^2+4(m-11)(m+1)^2y^3-8(m+1)^3y^4}{m^4(m-1)}\right]\lambda. \tag{2.5}$$

Proof. Taking Özger et al. [24] in to account and using (1.4), it is easy to see $\sum\limits_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)=1$, hence we get (2.1). Now, with the help of Lemma 2.1, we will compute expressions (2.2) and (2.3).

$$\begin{aligned}
R_{m,\lambda}(t;y) &= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_{\frac{j}{m}}^{\frac{j+1}{m}} t\,dt = \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{2j+1}{2m} \\
&= \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{j}{m}+\frac{1}{2m}\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y) \\
&= S_{m,\lambda}(t;y)+\frac{1}{2m}=y+\frac{1}{2m}+\left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda,
\end{aligned}$$

$$\begin{aligned}
R_{m,\lambda}(t^2;y) &= m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_{\frac{j}{m}}^{\frac{j+1}{m}} t^2dt = \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{3j^2+3j+1}{3m^2} \\
&= \sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{j^2}{m^2}+\frac{1}{m}\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\frac{j}{m}+\frac{1}{3m^2}\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y) \\
&= S_{m,\lambda}(t^2;y)+\frac{1}{m}S_{m,\lambda}(t;y)+\frac{1}{3m^2}=y^2+\frac{2y}{m}+\frac{1}{3m^2}+\left[\frac{-4(m+1)y^2}{m^2(m-1)}\right]\lambda.
\end{aligned}$$

Analogously, taking into consideration Lemma 2.1, hence we can arrive expressions (2.4) and (2.5) by simple computation, thus we omitted details. □

Corollary 2.3. *Let $y\in[0,\infty)$, $m>1$ and $\lambda\in[-1,1]$. As a consequence of Lemma 2.2, we obtain the following relations:*

$$\begin{aligned}
(i)\ R_{m,\lambda}(t-y;y) &= \frac{1}{2m}+\left[\frac{1-e^{-(m+1)y}-2y}{m(m-1)}\right]\lambda \\
&\leq \frac{m+1+2e^{-(m+1)y}+4y}{2m(m-1)} := \beta_m(y);
\end{aligned}$$

$$\begin{aligned}
(ii)\ R_{m,\lambda}((t-y)^2;y) &= \frac{y}{m}+\frac{1}{3m^2}+\left[\frac{2(e^{-(m+1)y}-1)y}{m(m-1)}-\frac{4y^2}{m^2(m-1)}\right]\lambda \\
&\leq \frac{y}{m}+\frac{1}{3m^2}+\frac{2(e^{-(m+1)y}+1)y}{m(m-1)}+\frac{4y^2}{m^2(m-1)} := \gamma_m(y);
\end{aligned}$$

$$\begin{aligned}
(iii)\ R_{m,\lambda}((t-y)^4;y) = \frac{3y^2}{m^2}+\frac{5y}{m^3}+\frac{1}{5m^4}+\Bigg(&\frac{2(me^{-(m+1)y}-1)y}{m^4(m-1)}+\frac{4(3m^2-8m-12)y^2}{m^4(m-1)} \\
&-\frac{4(3m^3+3m^2-6m-11)+4m^3e^{-(m+1)y})y^3}{m^4(m-1)}-\frac{8y^4}{m^4(m-1)}\Bigg)\lambda.
\end{aligned}$$

Lemma 2.4. *Let $y\in[0,\infty)$ and $\lambda\in[-1,1]$. Then, the following expressions holds true:*

$$\begin{aligned}
(i)\lim_{m\to\infty} mR_{m,\lambda}(t-y;y) &= \frac{1}{2}; \\
(ii)\lim_{m\to\infty} mR_{m,\lambda}((t-y)^2;y) &= y; \\
(iii)\lim_{m\to\infty} m^2R_{m,\lambda}((t-y)^4;y) &= 3y^2.
\end{aligned}$$

3. Direct theorems of $R_{m,\lambda}$

In the next theorem, we introduce a Korovkin type approximation theorem. As it is known, the space $C[0,\infty)$ denotes the all continuous and bounded functions on $[0,\infty)$ and it is equipped with the sup-norm for a function μ as follows:

$$\|\mu\|_{[0,\infty)} = \sup_{y\in[0,\infty)}|\mu(y)|.$$

Theorem 3.1. *Let $\mu \in C[0,\infty)$, then $R_{m,\lambda}(\mu;y)$ converge uniformly to μ on $[0,\infty)$.*

Proof. According to the Bohman-Korovkin theorem [25], it is sufficient to verify

$$\lim_{m\to\infty}\sup_{y\in[0,\infty)}\left|R_{m,\lambda}(t^s;y)-y^s\right|=0, \text{ for } s=0,1,2.$$

Using (2.1), for $s=0$, it can be seen that above expression is clear.
For $s=1$, in view of (2.2), we have

$$\begin{aligned}\lim_{m\to\infty}\sup_{y\in[0,\infty)}\left|R_{m,\lambda}(t;y)-y\right| &= \lim_{m\to\infty}\sup_{y\in[0,\infty)}\left|\frac{1}{2m}+\left(\frac{1-e^{-(m+1)y}}{m(m-1)}-\frac{2y}{m(m-1)}\right)\lambda\right| \\ &\leq \lim_{m\to\infty}\sup_{y\in[0,\infty)}\left(\frac{m+1+2e^{-(m+1)y}+4y}{2m(m-1)}\right)=0.\end{aligned}$$

Similarly, by (2.3), one has

$$\begin{aligned}\lim_{m\to\infty}\sup_{y\in[0,\infty)}\left|R_{m,\lambda}(t^2;y)-y^2\right| &= \lim_{m\to\infty}\sup_{y\in[0,\infty)}\left|\frac{2y}{m}+\frac{1}{3m^2}+\left(\frac{-4(m+1)}{m^2(m-1)}y^2\right)\lambda\right| \\ &\leq \lim_{m\to\infty}\sup_{y\in[0,\infty)}\left(\frac{2y}{m}+\frac{1}{3m^2}+\frac{4(m+1)}{m^2(m-1)}y^2\right)=0.\end{aligned}$$

Hence, we get the required sequel. □

Further, we discuss the order of convergence in connection with the usual moduli of continuity, for the function belong to Lipschitz type continuous and Peetre's K-functional. The Peetre's K-functional is defined by

$$K_2(\mu,\eta)=\inf_{\nu\in C^2[0,\infty)}\left\{\|\mu-\nu\|+\eta\left\|\nu''\right\|\right\},$$

where $\eta>0$ and $C^2[0,\infty)=\{\nu\in C[0,\infty):\nu',\nu''\in C[0,\infty)\}$.
Taking into account [26], there exist an absolute constant $C>0$ such that

$$K_2(\mu;\eta)\leq C\omega_2(\mu;\sqrt{\eta}),\qquad \eta>0, \tag{3.1}$$

where

$$\omega_2(\mu;\eta)=\sup_{0<\alpha\leq\eta}\sup_{y\in[0,\infty)}|\mu(y+2\alpha)-2\mu(y+\alpha)+\mu(y)|,$$

is the second order modulus of smoothness of the function $\mu\in C[0,\infty)$. Further, by

$$\omega(\mu;\eta):=\sup_{0<\alpha\leq\eta}\sup_{y\in[0,\infty)}|\mu(y+\alpha)-\mu(y)|,$$

we denote the usual moduli of continuity of $\mu\in C[0,\infty)$. Since $\eta>0$, $\omega(\mu;\eta)$ has some useful properties see details: [27]. Also, we give an element of Lipschitz continuous function with $Lip_L(\zeta)$, where $L>0$ and $0<\zeta\leq 1$. If the expression below:

$$|\mu(t)-\mu(y)|\leq L|t-y|^{\zeta},\qquad (t,y\in\mathbb{R}),$$

holds, then one can say a function μ is belong to $Lip_L(\zeta)$.

Theorem 3.2. *Let $\mu\in C[0,\infty)$, $y\in[0,\infty)$ and $\lambda\in[-1,1]$. Then, we have following inequality verify*

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right|\leq 2\omega(\mu;\sqrt{\gamma_m(y)}),$$

where $\gamma_m(y)$ given as in Corollary 2.3.

Proof. Using the well-known property of moduli of continuity $|\mu(t)-\mu(y)|\leq\left(1+\frac{|t-y|}{\delta}\right)\omega(\mu;\delta)$ and after operating $R_{m,\lambda}(.;y)$, it becomes

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right|\leq\left(1+\frac{1}{\delta}R_{m,\lambda}(|t-y|;y)\right)\omega(\mu;\delta).$$

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality and from Corollary 2.3, we get

$$\begin{aligned}\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| &\leq \left(1+\frac{1}{\delta}\sqrt{R_{m,\lambda}((t-y)^2;y)}\right)\omega(\mu;\delta) \\ &\leq \left(1+\frac{1}{\delta}\sqrt{\gamma_m(y)}\right)\omega(\mu;\delta).\end{aligned}$$

Taking $\delta=\sqrt{\gamma_m(y)}$, hence we obtain the proof of Theorem 3.2. □

Theorem 3.3. *Let $\mu \in Lip_L(\zeta)$, $y \in [0,\infty)$ and $\lambda \in [-1,1]$. Then, we obtain*

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| \leq L(\gamma_m(y))^{\frac{\zeta}{2}}.$$

Proof. By the linearity and monotonicity of the operators (1.4), it follows

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| \leq R_{m,\lambda}(|\mu(t)-\mu(y)|;y) \leq m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_{\frac{j}{m}}^{\frac{j+1}{m}}|\mu(t)-\mu(y)|\,dt \leq Lm\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_{\frac{j}{m}}^{\frac{j+1}{m}}|t-y|^{\zeta}\,dt.$$

Utilizing the Hölder's inequality with $p_1=\frac{2}{\zeta}$ and $p_2=\frac{2}{2-\zeta}$ and in view of Corollary 2.3 and Lemma 2.2, we arrive

$$\begin{aligned}\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| &\leq L\left\{m\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\int_{\frac{j}{m}}^{\frac{j+1}{m}}(t-y)^2\,dt\right\}^{\frac{\zeta}{2}}\left\{\sum_{j=0}^{\infty}\widetilde{s}_{m,j}(\lambda;y)\right\}^{\frac{2-\zeta}{2}}\\ &= L\left\{R_{m,\lambda}((t-y)^2;y)\right\}^{\frac{\zeta}{2}}\left\{R_{m,\lambda}(1;y)\right\}^{\frac{2-\zeta}{2}} \leq L(\gamma_m(y))^{\frac{\zeta}{2}}.\end{aligned}$$

Thus, we get the proof of this theorem. □

Theorem 3.4. *For all $\mu \in C[0,\infty)$, $y \in [0,\infty)$ and $\lambda \in [-1,1]$, the following inequality holds:*

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| \leq C\omega_2(\mu;\frac{1}{2}\sqrt{\gamma_m(y)+(\beta_m(y))^2}+\omega(\mu;\beta_m(y)),$$

where $C>0$ is a constant, $\beta_m(y)$, $\gamma_m(y)$ defined as in Corollary 2.3.

Proof. Let $\mu \in C[0,\infty)$. We denote $\alpha_{m,\lambda}(y) := y+\frac{1}{2m}+\left[\frac{1-2y-e^{-(m+1)y}}{m(m-1)}\right]\lambda$, it is obvious that $\alpha_{m,\lambda}(y)\in[0,\infty)$ for suffcently large m. We define the following auxiliary operators:

$$\widehat{R}_{m,\lambda}(\mu;y)=R_{m,\lambda}(\mu;y)-\mu(\alpha_{m,\lambda}(y))+\mu(y). \tag{3.2}$$

In view of (2.1) and (2.2), it follows that

$$\widehat{R}_{m,\lambda}(t-y;y)=0.$$

By Taylor's formula, one has

$$\xi(t)=\xi(y)+(t-y)\xi'(y)+\int_y^t(t-u)\xi''(u)du, \quad (\xi\in C^2[0,\infty)). \tag{3.3}$$

After operating $\widehat{R}_{m,\lambda}(.;y)$ to (3.3), yields

$$\begin{aligned}\widehat{R}_{m,\lambda}(\xi;y)-\xi(y) &= \widehat{R}_{m,\lambda}((t-y)\xi'(y);y)+\widehat{R}_{m,\lambda}(\int_y^t(t-u)\xi''(u)du;y)\\ &= \xi'(y)\widehat{R}_{m,\lambda}(t-y;y)+R_{m,\lambda}(\int_y^t(t-u)\xi''(u)du;y)-\int_y^{\alpha_{m,\lambda}(y)}(\alpha_{m,\lambda}(y)-u)\xi''(u)du\\ &= R_{m,\lambda}(\int_y^t(t-u)\xi''(u)du;y)-\int_y^{\alpha_{m,\lambda}(y)}(\alpha_{m,\lambda}(y)-u)\xi''(u)du.\end{aligned}$$

Taking Lemma 2.2 and (3.2) into the account, we get

$$\begin{aligned}\left|\widehat{R}_{m,\lambda}(\xi;y)-\xi(y)\right| &\leq \left|R_{m,\lambda}(\int_y^t(t-u)\xi''(u)du;y)\right|+\left|\int_y^{\alpha_{m,\lambda}(y)}(\alpha_{m,\lambda}(y)-u)\xi''(u)du\right|\\ &\leq R_{m,\lambda}(\left|\int_y^t(t-u)\right|\left|\xi''(u)\right||du|;y)+\int_y^{\alpha_{m,\lambda}(y)}\left|\alpha_{m,\lambda}(y)-u\right|\left|\xi''(u)\right||du|\\ &\leq \|\xi''\|\left\{R_{m,\lambda}((t-y)^2;y)+\left(\alpha_{m,\lambda}(y)-y\right)^2\right\}\leq\left\{\gamma_m(y)+(\beta_m(y))^2\right\}\|\xi''\|.\end{aligned}$$

Also from (2.1), (2.2) and (3.2), it deduce the following

$$\left|\widehat{R}_{m,\lambda}(\mu;y)\right| \leq \left|R_{m,\lambda}(\mu;y)\right| + 2\|\mu\| \leq \|\mu\| R_{m,\lambda}(1;y) + 2\|\mu\| \leq 3\|\mu\|. \tag{3.4}$$

On the other hand, by (3.3) and (3.4) imply

$$\begin{aligned} \left|R_{m,\lambda}(\mu;y)-\mu(y)\right| &\leq \left|\widehat{R}_{m,\lambda}(\mu-\xi;y)-(\mu-\xi)(y)\right| + \left|\widehat{R}_{m,\lambda}(\xi;y)-\xi(y)\right| + \left|\mu(y)-\mu(\alpha_{m,\lambda}(y))\right| \\ &\leq 4\|\mu-\xi\| + \left\{\gamma_m(y)+(\beta_m(y))^2\right\}\|\xi''\| + \omega(\mu;\beta_m(y)). \end{aligned}$$

On account of this, if we take the infimum on the right hand side over all $\xi \in C^2[0,\infty)$ and by (3.1), we arrive

$$\begin{aligned} \left|R_{m,\lambda}(\mu;y)-\mu(y)\right| &\leq 4K_2(\mu;\frac{\left\{\gamma_m(y)+(\beta_m(y))^2\right\}}{4}) + \omega(\mu;\beta_{m,\lambda}(y)) \\ &\leq C\omega_2(\mu;\frac{1}{2}\sqrt{\gamma_m(y)+(\beta_m(y))^2}) + \omega(\mu;\beta_m(y)). \end{aligned}$$

Hence, we obtain the proof of this theorem. □

Theorem 3.5. *If $\mu \in C^1[0,\infty) := \{\mu : \mu'$ is continuous and bounded on $[0,\infty)\}$, then for all $y \in [0,\infty)$ and $\lambda \in [-1,1]$, we arrive*

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| \leq \beta_m(y)\left|\mu'(y)\right| + 2\sqrt{\gamma_m(y)}\,\omega(\mu';\sqrt{\gamma_m(y)}),$$

where $\beta_m(y)$, $\gamma_m(y)$ defined as in Corollary 2.3.

Proof. Let $\mu \in C^1[0,\infty)$. For any $y,t \in [0,\infty)$, we get

$$\mu(t)-\mu(y) = \mu'(y)(t-y) + \int_y^t (\mu'(u)-\mu'(y))du.$$

After operating $R_{m,\lambda}(.;y)$ to the both sides of above expression, it gives

$$R_{m,\lambda}(\mu(t)-\mu(y);y) = \mu'(y)R_{m,\lambda}(t-y;y) + R_{m,\lambda}(\int_y^t (\mu'(u)-\mu'(y))du;y).$$

Taking into consideration the following well-known property

$$|\mu(u)-\mu(y)| \leq \left(1+\frac{|u-y|}{\delta}\right)\omega(\mu;\delta), \quad \delta>0,$$

then

$$\left|\int_y^t \left|\mu'(u)-\mu'(y)\right|du\right| \leq \left(\frac{(t-y)^2}{\delta}+|t-y|\right)\omega(\mu';\delta).$$

Hence,

$$\left|R_{m,\lambda}(\mu;y)-\mu(y)\right| \leq \left|R_{m,\lambda}(t-y;y)\right|\left|\mu'(y)\right| + \left[\frac{R_{m,\lambda}((t-y)^2;y)}{\delta} + R_{m,\lambda}(|t-y|;y)\right]\omega(\mu';\delta).$$

Applying Cauchy-Bunyakovsky-Schwarz inequality on the right hand side of foregoing inequality and taking into consideration Corollary 2.3, we find

$$\begin{aligned} \left|R_{m,\lambda}(\mu;y)-\mu(y)\right| &\leq \left|R_{m,\lambda}(t-y;y)\right|\left|\mu'(y)\right| + \omega(\mu';\delta)\left[\frac{\sqrt{R_{m,\lambda}((t-y)^2;y)}}{\delta}+1\right]\sqrt{R_{m,\lambda}((t-y)^2;y)} \\ &\leq \beta_m(y)\left|\mu'(y)\right| + \omega(\mu';\delta)\left[\frac{\sqrt{\gamma_m(y)}}{\delta}+1\right]\sqrt{\gamma_m(y)}. \end{aligned}$$

By taking $\delta = \sqrt{\gamma_m(y)}$, the required result is obtained. □

4. Voronovskaya type asymptotic theorem

Theorem 4.1. *Let $\mu \in C[0,\infty)$ such that $\mu', \mu'' \in C[0,\infty)$ and $\lambda \in [-1,1]$, then we have for any $y \in [0,\infty)$ that*

$$\lim_{m\to\infty} m\left[R_{m,\lambda}(\mu;y)-\mu(y)\right] = \frac{\mu'(y)+y\mu''(y)}{2}.$$

Proof. Suppose that $y \in [0,\infty)$ and $\mu', \mu'' \in C[0,\infty)$. From Taylor's formula, one has

$$\mu(t) = \mu(y)+(t-y)\mu'(y)+\frac{1}{2}(t-y)^2\mu''(y)+(t-y)^2\phi(t;y). \tag{4.1}$$

In (4.1), $\phi(t;y)$ is a Peano of the remainder term and by the fact that $\phi(.;y) \in C[0,\infty)$, we have $\lim\limits_{t\to y}\phi(t;y) = 0$.
After operating $R_{m,\lambda}(.;y)$ to (4.1), hence

$$R_{m,\lambda}(\mu;y)-\mu(y) = R_{m,\lambda}((t-y);y)\mu'(y)+\frac{1}{2}R_{m,\lambda}((t-y)^2;y)\mu''(y)+R_{m,\lambda}((t-y)^2\phi(t;y);y).$$

If we take the limit of the both sides of above relation as $m\to\infty$, then

$$\lim_{m\to\infty} m(R_{m,\lambda}(\mu;y)-\mu(y) = \lim_{m\to\infty} m\left(R_{m,\lambda}((t-y);y)\mu'(y)+\frac{1}{2}R_{m,\lambda}((t-y)^2;y)\mu''(y)+R_{m,\lambda}((t-y)^2\phi(t;y);y)\right). \tag{4.2}$$

Utilizing the Cauchy-Bunyakovsky-Schwarz inequality to the last term on the right hand side of the above expression, it becomes

$$\lim_{m\to\infty} mR_{m,\lambda}((t-y)^2\phi(t;y);y) \leq \sqrt{\lim_{m\to\infty} R_{m,\lambda}(\phi^2(t;y);y)}\sqrt{\lim_{m\to\infty} m^2R_{m,\lambda}((t-y)^4;y)}.$$

It is observed that as $\phi(t;y) \in C[0,\infty)$, thus by Theorem 3.1, $\lim\limits_{t\to y}\phi(t;y) = 0$. It follows that

$$\lim_{m\to\infty} R_{m,\lambda}(\phi^2(t;y);y) = \phi^2(y;y) = 0. \tag{4.3}$$

If we combine (4.2)-(4.3) and in view of Lemma 2.4 (iii), we arrive

$$\lim_{m\to\infty} mR_{m,\lambda}((t-y)^2\phi(t;y);y) = 0.$$

Thus, we obtain the desired sequel as follows:

$$\lim_{m\to\infty} m\left[R_{m,\lambda}(\mu;y)-\mu(y)\right] = \frac{\mu'(y)+y\mu''(y)}{2}.$$

□

5. Graphical analysis

In this section, we give some graphics to see the convergence of operators (1.4) to the certain functions. Also, we compare the convergence of our newly defined operators (1.4) with the operators (1.2) with the different values of m and λ.
In Figure 5.1, for $\lambda = 0.5$ and $m = 10, 40, 70$ respectively, we demonstrate the convergence of operators (1.4) to $\mu(y) = e^y$. In Figure 5.2, for $\lambda = 0.9$ and $m = 10, 40, 70$ respectively, we show the convergence of operators (1.4) to $\mu(y) = cos(\pi y)$. In Figure 5.3, we denote with LKMS:= λ-Szász-Mirakjan-Kantorovich operators defined by (1.4) and KMS:= Szász-Mirakjan-Kantorovich operators defined by (1.2). We compare the convergence of operators (1.4) with (1.2) for $\lambda = 0.5$, $m = 10$ to $\mu(y) = e^y$. We can conclude from Figure 5.1 and Figure 5.2 that, as the values of m increases than the convergence of operators (1.4) to the functions becomes better. Moreover, in Figure 5.3 it can be seen that for $\lambda = 0.5$ and $m = 10$ operators (1.4) have better approximation than operators (1.2).

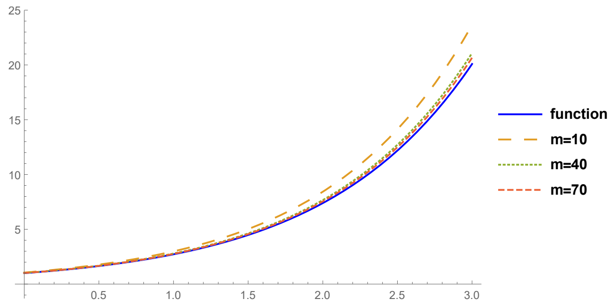

Figure 5.1: The convergence of $R_{m,\lambda}(\mu;y)$ to $\mu(y)=e^{y}$ for $\lambda=0.5$ and $m=10,40,70$.

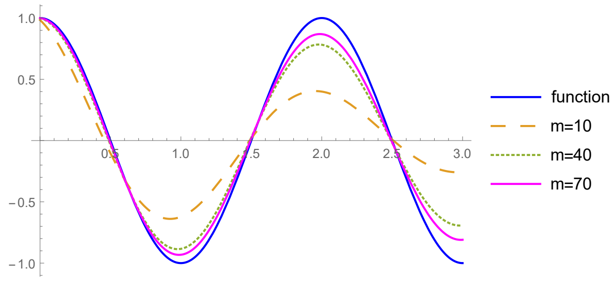

Figure 5.2: The convergence of $R_{m,\lambda}(\mu;y)$ to $\mu(y)=cos(\pi y)$ for $\lambda=0.9$ and $m=10,40,70$.

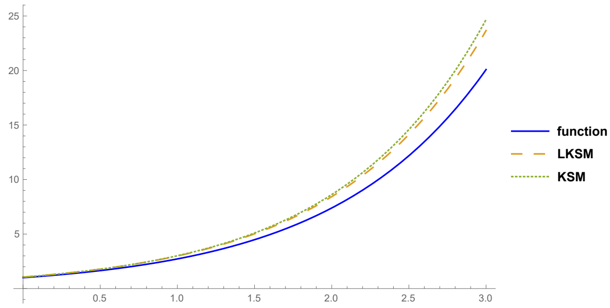

Figure 5.3: The convergence of $R_{m,\lambda}(\mu;y)$ and $K_m(\mu;y)$ to $\mu(y)=e^{y}$ for $\lambda=0.5$ and $m=10$.

6. Conclusion

In the present paper, we introduced Szász-Mirakjan-Kantorovich operators based on shape parameter $\lambda\in[-1,1]$. The importance of parameter λ, give us more flexibility in modeling. We derived a Korovkin type convergence theorem, estimated the degree of convergence in terms of the moduli of continuity, for the functions belong to Lipschitz class and Peetre's K-functional, respectively. We also discussed Voronovskaya type asymptotic theorem. Moreover, we gave the comparison of the convergence of our newly constructed operators (1.4) to the certain functions with some graphics and also we compared the convergence of (1.4) between (1.2). As future works, we will consider the Stancu, Durrmeyer and Baskakov type λ-Szász-Mirakjan operators.

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Author's contributions

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