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On the Codes over a Family of Rings and Their Applications to DNA Codes

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Abstract
In this paper, the structures of the linear codes over a family of the rings $A_t = Z_4\left[u_1, \ldots, u_t\right]/\left\langle u_i^2 - u_i, u_i u_j - u_j u_i\right\rangle$ are given, where $i, j = 1, 2, \ldots, t$, $i \neq j$, $Z_4 = \{0, 1, 2, 3\}$. A map between the elements of the A_t and the alphabet $\{A, T, C, G\}^{2^t}$ is constructed. The DNA codes are obtained with three different methods, by using the cyclic, skew cyclic codes over a family of the rings A_t and θ_i-set, where θ_i is a non trivial automorphism on A_i, for $i = 1, 2, \ldots, t$.

Keywords: *DNA codes; cyclic codes; skew cyclic codes; reversibility.*

AMS Subject Classification (2020): *94B05; 94B15; 94B60.*

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1. Introduction

There are many methods in order to obtain DNA codes. In [1], it was used the cyclic codes over the finite ring $F_2[u]/\langle u^4 - 1\rangle$ in order to obtain DNA codes. The sufficient and necessary conditions of cyclic codes over the finite ring satisfying the reverse complement constraints was given. By introducing a map, the DNA codes were obtained from these types codes. In different method, it was used the skew cyclic codes over $Z_4[u, v]/\langle u^2 - u, v^2 - v, uv - vu\rangle$ in order to obtain reversible DNA codes, in [2]. Thanks to this, reversibility problem was solved for DNA 4-bases. This problem arises from the fact that the pairing of nucleotides in two different strands of a DNA sequence is done in opposite direction and reverse order. For example, take $t = 1$. Let $(\alpha_1, \alpha_2) \in A_1^2$ be a codeword corresponding to $CTCG$, where $A_1 = Z_4 + u_1 Z_4, u_1^2 = u_1$. The reverse of (α_1, α_2) is (α_2, α_1). The vector (α_2, α_1) corresponding to $CGCT$. It is not reverse of $CTCG$. The reverse of $CTCG$ is $GCTC$. In order to solve reversibility problem, there is a different approach. In [3], it was used θ-set, where θ is a non trivial automorphism on $F_2[u, v]/\langle u^2, v^2 - v, uv - vu\rangle$ in order to obtain reversible and reversible complement DNA codes.

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Moreover, there are similar papers in the literature, [4–6]. Motivated from all these works in which were considered the codes over one ring and were used one method in order to DNA codes, we decide to consider the codes over a family of rings and use three methods in order to obtain DNA codes.

In this paper, we use the cyclic, skew cyclic codes over a family of the rings $A_t = Z_4[u_1, \ldots, u_t]/\langle u_i^2 - u_i, u_iu_j - u_ju_i\rangle$, where $i, j = 1, 2, \ldots, t, i \neq j$ and $Z_4 = \{0, 1, 2, 3\}$ and θ_i-set, where θ_i is a non trivial automorphism on A_i, for $i = 1, 2, \ldots, t$ in order to obtain DNA codes. Section 2 includes some knowledge about a family of the rings A_t. A map ϕ_i is defined from A_i to A_{i-1}^2, for $i = 1, 2, \ldots, t$. A map ξ_i is defined from A_i to $\{A, T, C, G\}^{2^i}$, for $i = 1, 2, \ldots, t$. A Gray map is defined on A_i, for $i = 1, \ldots, t$. In the section 3 and 4, the structures of linear and cyclic codes over A_t are given, respectively. In the section 5.1 and 5.2 the sufficient and necessary conditions of cyclic codes over A_t satisfying the reverse and reverse complement constraints are given, respectively. The DNA codes are obtained with first method. In the section 6, by defining a non trivial automorphism on A_i for $i = 1, \ldots, t$, the skew cyclic codes over a family of the finite rings are introduced. By using the skew cyclic codes over A_t, the DNA codes are obtained with second method. In the section 7, by using the θ_i-set, where θ_i is a non trivial automorphism on A_i, for $i = 1, 2, \ldots, t$, the DNA codes are obtained with third method.

2. Preliminaries

A family of the finite rings $A_t = Z_4[u_1, \ldots, u_t]/\langle u_i^2 - u_i, u_iu_j - u_ju_i\rangle$, where $i, j = 1, 2, \ldots, t, i \neq j$ contains the commutative the finite rings with characteristic 4 and cardinality 4^{2^t}. The finite rings of the family are written as recursively

$$A_r = A_{r-1} + u_rA_{r-1}$$

where $r = 1, 2, \ldots, t$ and $A_1 = Z_4 + u_1Z_4, u_1^2 = u_1$, where $A_0 = Z_4 = \{0, 1, 2, 3\}$.

We define a map as follows for every $a_i = x_{i-1} + u_iy_{i-1} \in A_i$,

$$\begin{array}{rcl} \phi_i & : & A_i \longrightarrow A_{i-1}^2 \\ a_i & \longmapsto & \phi_i\left(a_i\right) = \left(x_{i-1}, x_{i-1} + y_{i-1}\right) \end{array}$$

where $i = 1, 2, \ldots, t$ and

$$\begin{array}{rcl} \phi_1 & : & A_1 \longrightarrow A_0^2 \\ a_1 & = & x_0 + u_1y_0 \longmapsto \phi_1\left(a_1\right) = \left(x_0, x_0 + y_0\right) \end{array}$$

where $A_0 = Z_4$.

The map ϕ_i can be extended to A_i^n naturally, for $i = 1, \ldots, t$.

Let $S_{D_4} = \{A, T, C, G\}$ represent the DNA alphabet. The Watson Crick Complement is given $A^c = T, T^c = A, G^c = C, C^c = G$. We use the same notation for the set $S_{D_{16}} = \{AA, TT, \ldots, CG\}$ which was presented in [7]. It is extended the notation to the elements of $S_{D_{16}}$ such that $AA^c = TT, AT^c = TA, \ldots, GG^c = CC$. By using the matching the elements of A_0 and $S_{D_4} = \{A, T, C, G\}$ which is given as $\xi_0(0) = A, \xi_0(1) = T, \xi_0(3) = C, \xi_0(2) = G$ and by using the map ϕ_1 from $A_1 = Z_4 + u_1Z_4$ to Z_4^2, we defined a ξ_1 correspondence between the elements of the finite ring $A_1 = Z_4 + u_1Z_4$ and DNA double pairs by $a_1 = x_0 + u_1y_0 \mapsto (\xi_0(x_0), \xi_0(x_0 + y_0))$ in [7],

Elements a_1	DNA double pairs $\xi_1(a_1)$
0	AA
1	TT
2	GG
3	CC
u_1	AT
$1+u_1$	TG
u_1+2	GC
u_1+3	CA
$2u_1$	AG
$1+2u_1$	TC
$2+2u_1$	GA
$3+2u_1$	CT
$3u_1$	AC
$1+3u_1$	TA
$2+3u_1$	GT
$3+3u_1$	CG

Table 1. Identifying codons with the elements of the ring A_1.

By using the map ϕ_2 and ξ_1, we established ξ_2 correspondence between the elements of A_2 and DNA 4-bases by $a_2 = x_1 + u_1 y_1 \mapsto (\xi_1(x_1), \xi_1(x_1 + y_1))$ as follows in [2],

Elements a_2	DNA 4-bases $\xi_2(a_2)$
0	$AAAA$
1	$TTTT$
2	$GGGG$
3	$CCCC$
u_1	$ATAT$
u_2	$AATT$
$\vdots$	$\vdots$

Table 2. Identifying codons with the elements of the ring A_2.

By using the map ϕ_i and ξ_{i-1}, we can establish ξ_i correspondence between the element of A_i and DNA 2^i-bases for $i = 1, .., t$ as follows.

$$\xi_i : A_i \longrightarrow A_{i-1}^2 \longrightarrow \{A, T, C, G\}^{2^i}$$

$$a_i = x_{i-1} + u_i y_{i-1} \longmapsto \phi_i(a_i) = (x_{i-1}, x_{i-1} + y_{i-1}) \longmapsto \gamma_i(\phi_i(a_i)) = (\xi_{i-1}(x_{i-1}), \xi_{i-1}(x_{i-1} + y_{i-1}))$$

where $\xi_i = \gamma_i \phi_i$ and the map γ_i is defined from A_{i-1}^2 to DNA 2^i-bases as follows,

$$\gamma_i(s_{i-1}, t_{i-1}) = (\xi_{i-1}(s_{i-1}), \xi_{i-1}(t_{i-1}))$$

where $s_{i-1}, t_{i-1} \in A_{i-1}$, for $i = 1, ..., t$.

We established ξ_i correspondence between the elements of A_i and DNA 2^i-bases as follows

Elements a_i	DNA 2^i-bases $\xi_i(a_i)$
0	$\underbrace{AA\dots A}_{2^i \text{ times}}$
1	$\underbrace{TT\dots T}_{2^i \text{ times}}$
2	$\underbrace{GG\dots G}_{2^i \text{ times}}$
3	$\underbrace{CC\dots C}_{2^i \text{ times}}$
u_1	$\underbrace{ATAT\dots AT}_{2^i \text{ times}}$
$\vdots$	$\vdots$

Table 3. Identifying codons with the elements of the ring A_i.

for $i = 1, \dots, t$.

We can also express an element of A_t as follows uniquely.

Let $B \subseteq \{1, 2, \dots, t\}$ and $u_B = \prod_{i\in B} u_i$. In particular $u_\emptyset = 1$. Each element of A_t is of the form $\sum_{B\in P_t} \alpha_B u_B$, where $\alpha_B \in Z_4, P_t$ is the power set of the set $\{1, 2, \dots, t\}$. For $A, B \subseteq \{1, 2, \dots, t\}$, we have that $u_A u_B = u_{A\cup B}$ which gives that $\sum_{B\in P_t} \alpha_B u_B . \sum_{C\in P_t} \beta_C u_C = \sum_{D\in P_t} \left(\sum_{B\cup C=D} \alpha_B \beta_C \right) u_D$. Moreover,

$$e_{u_\emptyset} = 1 + (-1)^{|B|} \sum_{B\in P_t} u_B$$

and the number of $e_{u_\emptyset}$ is $\binom{t}{0}$.

$$e_{u_i} = u_i + (-1)^{|B|+1} \sum_{\substack{i\in B\in P_t,\\ |B|\geq 2}} u_B$$

for $i = 1, 2, \dots, t$ and the number of e_{u_i} is $\binom{t}{1}$.

$$e_{u_i u_j} = \underset{i<j}{u_i u_j} + (-1)^{|B|+2} \sum_{\substack{i,j\in B\in P_t,\\ |B|\geq 3}} u_B$$

for $i, j = 1, 2, \dots, t$ and the number of $e_{u_i u_j}$ is $\binom{t}{2}$.

$$e_{u_i u_j u_s} = \underset{i<j<s}{u_i u_j u_s} + (-1)^{|B|+3} \sum_{\substack{i,j,s\in B\in P_t,\\ |B|\geq 4}} u_B$$

for $i, j, s = 1, 2, \dots, t$ and the number of $e_{u_i u_j u_s}$ is $\binom{t}{3}$

$$\vdots$$

$$e_{u_1 u_2 \dots u_t} = u_1 u_2 \dots u_t$$

and the number of $e_{u_1 u_2 \dots u_t}$ is $\binom{t}{t}$.

Then we have $\sum_{B\in P_t} e_{u_B} = 1, (e_{u_B})^2 = e_{u_B}$ and $e_{u_B} e_{u_A} = 0$ if $A \neq B$ for any $A, B \subseteq \{1, 2, \dots, t\}$. Hence $A_t = \bigoplus_{B\in P_t} A_t e_{u_B} \cong \bigoplus_{B\in P_t} Z_4 e_{u_B}$. So every element z of A_t can be uniquely expressed as $z = \sum_{B\in P_t} a_{u_B} e_{u_B}$, where $a_{u_B} \in Z_4$.

Example 2.1. Let t be 3. Then $A_3 = Z_4 + u_1 Z_4 + u_2 Z_4 + u_3 Z_4 + u_1 u_2 Z_4 + u_1 u_3 Z_4 + u_2 u_3 Z_4 + u_1 u_2 u_3 Z_4$. Consider the elements of A_3 below

$$e_{u_\emptyset} = e_1 = 1 - u_1 - u_2 - u_3 + u_1u_2 + u_1u_3 + u_2u_3 - u_1u_2u_3$$

$$e_{u_1} = u_1 - u_1u_2 - u_1u_3 + u_1u_2u_3$$

$$e_{u_2} = u_2 - u_1u_2 - u_2u_3 + u_1u_2u_3$$

$$e_{u_3} = u_3 - u_1u_3 - u_2u_3 + u_1u_2u_3$$

$$e_{u_1u_2} = u_1u_2 - u_1u_2u_3$$

$$e_{u_1u_3} = u_1u_3 - u_1u_2u_3$$

$$e_{u_2u_3} = u_2u_3 - u_1u_2u_3$$

$$e_{u_1u_2u_3} = u_1u_2u_3$$

We can also define Gray map as follows,

$$\begin{array}{rcl} \Psi_t & : & A_t \longrightarrow Z_4^{2^t} \\ z = \sum\limits_{B \in P_t} a_{u_B} e_{u_B} & \longmapsto & \Psi_t(z) = \gamma \end{array}$$

where $\gamma = \begin{pmatrix} \sum\limits_{B=\emptyset} a_{u_B}, \sum\limits_{B \subseteq \{1\}} a_{u_B}, \ldots, \sum\limits_{B \subseteq \{t\}} a_{u_B}, \sum\limits_{B \subseteq \{1,2\}} a_{u_B}, \sum\limits_{B \subseteq \{1,3\}} a_{u_B}, \ldots, \\ \sum\limits_{\substack{B \subseteq \{i,j\}, \\ i<j}} a_{u_B}, \sum\limits_{B \subseteq \{1,2,3\}} a_{u_B}, \ldots, \sum\limits_{\substack{B \subseteq \{i,j,s\}, \\ i<j<s}} a_{u_B}, \ldots, \sum\limits_{B \subseteq \{1,2,\ldots,t\}} a_{u_B} \end{pmatrix}$ and $a_{u_B} \in Z_4$, for $i, j, s, \ldots \in \{1, 2, \ldots, t\}$.

The map Ψ_t can be extended from A_t^n, naturally.

Example 2.2. Let $t = 3$. Then

$$\begin{array}{rcl} \Psi_3 & : & A_3 \longrightarrow Z_4^8 \\ z = \sum\limits_{B \in P_3} a_{u_B} e_{u_B} & \longmapsto & \Psi_3(z) = \gamma \end{array}$$

where $\gamma = (a_1, a_1 + a_{u_1}, a_1 + a_{u_2}, a_1 + a_{u_3}, a_1 + a_{u_1} + a_{u_2} + a_{u_1u_2}, a_1 + a_{u_1} + a_{u_3} + a_{u_1u_3}, a_1 + a_{u_2} + a_{u_3} + a_{u_2u_3}, a_1 + a_{u_1} + a_{u_2} + a_{u_3} + a_{u_1u_2} + a_{u_2u_3} + a_{u_1u_3} + a_{u_1u_2u_3})$.

The Lee weight on Z_4, denoted w_L, is defined as $w_L(p) = 0$ if $p = 0$, $w_L(p) = 1$ if $p = 1$ or $p = 3$, $w_L(p) = 2$ if $p = 2$. For any $x = \sum\limits_{B \in P_t} a_{u_B} e_{u_B} \in A_t$, the Gray weight of x is defined as

$$w_G(x) = w_L(\Psi_t(x)) = \sum_{i=1}^{2^t} w_L(x_i)$$

where $\Psi_t(x) = (x_1, \ldots, x_{2^t})$ and $x_i \in Z_4$ for $i = 1, 2, \ldots, 2^t$. The Gray weight of a vector $\mathbf{a} = (a_1, \ldots, a_n) \in A_t^n$ is defined to be a rational sum of the Gray weight of its components. Moreover, for any $\mathbf{c}, \mathbf{d} \in A_t^n$, the Gray distance between $\mathbf{c}$ and $\mathbf{d}$ is defined as $d_G(\mathbf{c}, \mathbf{d}) = w_G(\mathbf{c} - \mathbf{d})$.

Theorem 2.1. *The map Ψ_i is a linear and distance preserving map, for $i = 1, \ldots, t$.*

3. Linear codes over A_t

A non empty subset $C \subseteq A_t^n$ is called linear code over A_t if C is a submodule of A_t.

Let $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1})$ be two vectors in A_t^n. The Euclidean inner product of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=0}^{n-1} x_j y_j$$

where the operations are performed in the ring A_t.

Dual of the code $C \subseteq A_t^n$ is the code

$$C^{\perp} = \{\mathbf{x} \in A_t^n : \langle \mathbf{x}, \mathbf{y} \rangle = 0, \forall \mathbf{y} \in C\}.$$

Clearly, $C^{\perp}$ is also linear.

Denote $\mathbf{r} = (r^{(0)}, \ldots, r^{(n-1)}) \in A_t^n$, where $r^{(i)} = \sum_{B \in P_t} a_{iu_B} e_{u_B}$ for $i = 0, 1, 2, \ldots, n-1$. Then $\mathbf{r}$ can be uniquely expressed as $\mathbf{r} = \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B}$, where $\mathbf{a}_{u_B} = (a_{0u_B}, a_{1u_B}, \ldots, a_{n-1u_B})$, each $B \in P_t$.

Let

$$R_1 \oplus \ldots \oplus R_{2^t} = \{r_1 + \ldots + r_{2^t} | r_i \in R_i, i = 1, \ldots, 2^t\},$$
$$R_1 \oplus \ldots \oplus R_{2^t} = \{(r_1, \ldots, r_{2^t}) | r_i \in R_i, i = 1, \ldots, 2^t\}.$$

Define the codes C_{u_B} as follows

$$C_{u_\emptyset} = C_1 = \{\mathbf{a}_{u_\emptyset} \in Z_4^n | \exists \mathbf{a}_{u_B, B \neq \emptyset} \in Z_4^n, \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B} \in C\}$$

$$C_{u_1} = \{\mathbf{a}_{u_1} \in Z_4^n | \exists \mathbf{a}_{u_B, B \neq \{1\}} \in Z_4^n, \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B} \in C\}$$

$$C_{u_2} = \{\mathbf{a}_{u_2} \in Z_4^n | \exists \mathbf{a}_{u_B, B \neq \{2\}} \in Z_4^n, \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B} \in C\}$$

$$\vdots$$

$$C_{u_t} = \{\mathbf{a}_{u_t} \in Z_4^n | \exists \mathbf{a}_{u_B, B \neq \{t\}} \in Z_4^n, \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B} \in C\}$$

$$C_{u_1 u_2} = \{\mathbf{a}_{u_1 u_2} \in Z_4^n | \exists \mathbf{a}_{u_B, B \neq \{1,2\}} \in Z_4^n, \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B} \in C\}$$

$$\vdots$$

$$C_{u_1 u_2 \ldots u_t} = \{\mathbf{a}_{u_1 u_2 \ldots u_t} \in Z_4^n | \exists \mathbf{a}_{u_B, B \neq \{1, \ldots, t\}} \in Z_4^n, \sum_{B \in P_t} \mathbf{a}_{u_B} e_{u_B} \in C\}.$$

The number of C_{u_B} is 2^t. Clearly C_{u_B} is a linear code of length n over Z_4. C can be uniquely decomposed into

$$C = \bigoplus_{B \in P_t} C_{u_B} e_{u_B}$$

and hence we have $|C| = \prod_{B \in P_t} |C_{u_B}|$.

The following theorems can be proved as in [8].

Theorem 3.1. *Let $C = \bigoplus_{B \in P_t} C_{u_B} e_{u_B}$ be a linear code of length n over A_t. Then the dual $C^{\perp} = \bigoplus_{B \in P_t} C_{u_B}{}^{\perp} e_{u_B}$ is also a linear code of length n over A_t.*

Theorem 3.2. *If C is a (n, M, d_G) linear code over A_i, then $\Psi_i(C)$ is a $(2^i n, M, d_L)$ linear code over Z_4 for $i = 1, \ldots, t$, where $d_G = d_L$.*

Theorem 3.3. *Let C be a linear code of length n over A_i. Then $\Psi_i(C) = \bigotimes_{B \in P_i} C_{u_B}$, for $i = 1, \ldots, t$.*

4. Cyclic codes over A_t

In [9], the structures of cyclic codes of length n over Z_4 were determined as follows. By using this, we will obtain the structures of cyclic codes over A_i for $i = 1, \ldots, t$.

Theorem 4.1. *[9] Let C be a cyclic code of length n over $R_n = Z_4[x]/\langle x^n - 1\rangle$.*
1. If n is odd, then R_n is a principal ideal ring and $C = \langle g(x), 2a(x)\rangle = \langle g(x) + 2a(x)\rangle$, where $g(x)$ and $a(x)$ are polynomials with $a(x)|g(x)|x^n - 1 \pmod 4$.
2. If n is not odd, then

i. If $g(x) = a(x)$, then $C = \langle g(x) + 2a(x)\rangle$, where $g(x)|x^n - 1 \pmod 2$, $g(x) + 2a(x)|x^n - 1 \pmod 4$,

ii. $C = \langle g(x) + 2p(x), 2a(x)\rangle$, where $g(x), a(x)$ and $p(x)$ are polynomials with $g(x)|x^n - 1 \pmod 2$ and $a(x)|p(x)\,(x^n - 1/g(x)) \pmod 2$, deg $a(x) >$ deg $p(x)$.

Theorem 4.2. *Let $C = \bigoplus_{B \in P_t} C_{u_B} e_{u_B}$ be a linear code over A_t. Then C is a cyclic code over A_t if and only if C_{u_B} are cyclic codes over Z_4 for all $B \in P_t$. Moreover, if C is a cyclic code over A_t, then*

$$C = \langle f_1(x)e_1, f_{u_1}(x)e_{u_1}, \ldots, f_{u_t}(x)e_{u_t}, f_{u_1u_2}(x)e_{u_1u_2}, \ldots, f_{u_1u_2\ldots u_t}(x)e_{u_1u_2\ldots u_t}\rangle$$

where $f_{u_B}(x)$ are generator polynomials of C_{u_B}, for all $B \in P_t$, respectively.

Proof. This can be proven similarly to [7]. □

5. The reversible codes and reversible complement codes

In [7], the sufficient and necessary conditions of cyclic codes over A_1 satisfying the reverse constraint and reverse complement constraint were given. In this section, the sufficient and necessary conditions of cyclic codes over A_i satisfying the reverse constraint and reverse complement constraint are given for $i = 2, \ldots, t$.

Definition 5.1. A cyclic code C of length n over A_t is said to be reversible if $\mathbf{x}^r = (x_{n-1}, \ldots, x_0) \in C$, for all $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in C$.

Definition 5.2. For each polynomial $c(x) = c_0 + c_1x + \ldots + c_mx^m$ with $c_m \neq 0$, the reciprocal polynomial of $c(x)$ is defined to be the polynomial $c^*(x) = x^m c(x^{-1})$. The polynomial $c(x)$ and $c^*(x)$ always have the same degree. The polynomial $c(x)$ is called reciprocal if and only if $c(x) = c^*(x)$.

Lemma 5.1. *Let $f(x)$ and $g(x)$ be polynomials in $A_t[x]$. Suppose that deg $f(x)$-deg $g(x) = m$, then*

$$(f(x).g(x))^* = f^*(x)g^*(x)$$

and

$$(f(x) + g(x))^* = f^*(x) + x^m g^*(x).$$

5.1 The reversible codes

In [9], the author studied the reversible codes over Z_4 as follows, by using this, the sufficient and necessary conditions of cyclic codes over A_i satisfying the reverse constraint are given for $i = 2, \ldots, t$.

Lemma 5.2. *[9] Let $C = \langle g(x), 2a(x)\rangle = \langle g(x) + 2a(x)\rangle$ be a cyclic code of odd length n over Z_4. Then C is reversible if and only if both $g(x)$ and $a(x)$ are self reciprocal.*

Theorem 5.1. *[9] Let $C = \langle g(x) + 2p(x)\rangle$ be a cyclic code of even length n over Z_4. Then C is reversible if and only if*

i. $g(x)$ is self reciprocal,

ii. $a(x)|\left(x^i p^*(x) + p(x)\right)$*, where* $i =$*deg* $g(x)$*- deg* $p(x)$*.*

Theorem 5.2. *[9] Let* $C = \langle g(x) + 2p(x), 2a(x)\rangle$ *with* $g(x)|x^n-1 \pmod 2$*,* $a(x)|g(x) \pmod 2$*,* $a(x)|p(x)|\left(x^n - 1/g(x)\right) \pmod 2$ *and deg* $a(x) >$ *deg* $p(x)$ *be a cyclic code of even length* n *over* Z_4*. Then* C *is reversible if and only if*

i. $g(x)$ *and* $a(x)$ *are self reciprocal,*

ii. $a(x)|(x^i p^*(x) + p(x))$*, where* $i =$*deg* $g(x)$*-deg* $p(x)$*.*

Theorem 5.3. *Let* $C = \bigoplus_{B\in P_t} C_{u_B} e_{u_B}$ *be a cyclic code of length* n *over* A_t*. Then* C *is reversible if and only if* C_{u_B} *are reversible, where* C_{u_B} *are cyclic codes over* Z_4*, for all* $B \in P_t$*.*

Proof. This can be proven similarly to [7]. □

5.2 The reversible complement codes

In this section, the sufficient and necessary conditions of cyclic codes over A_i satisfying the reverse complement constraint are given for $i = 2, \ldots, t$ and DNA codes are obtained by using cyclic DNA codes over A_t.

Definition 5.3. A cyclic code C of length n over A_t is said to be complement if $\mathbf{x}^c = (x_0^c, \ldots, x_{n-1}^c) \in C$, for all $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in C$.

A cyclic code C of length n over A_t is said to be reversible complement if $\mathbf{x}^{rc} = (x_{n-1}^c, \ldots, x_0^c) \in C$, for all $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in C$.

A cyclic code C of length n over A_t that has reversible complement property is said to be cyclic DNA code.

Lemma 5.3. *The following conditions hold,*

i. For any element $a_i \in A_i$*,* $a_i^c = (x_{i-1} + u_i y_{i-1})^c = x_{i-1}^c + 3u_i y_{i-1}$*, where* $x_{i-1}, y_{i-1} \in A_{i-1}$*,* $i = 1, 2, \ldots, t$*.*

ii. For all $a \in A_t$*, we have* $a + a^c = 1$*.*

iii. For all $a, b \in A_t$*, we have* $(a + b)^c = a^c + b^c + 3$*.*

Proof. i., ii. According the tables, the computations are easy.
iii. Let $a, b \in A_t$. From ii., $(a + b)^c = 1 - (a + b) = (1 - a) + (1 - b) - 1 = a^c + b^c + 3$. □

Theorem 5.4. *Let* $C = \bigoplus_{B\in P_t} C_{u_B} e_{u_B}$ *be a cyclic code of length* n *over* A_t*. Then* C *is reversible complement if and only if* C *is reversible and* $(0^c, \ldots, 0^c) \in C$*, where* C_{u_B} *are cyclic codes over* Z_4*, for all* $B \in P_t$*.*

Proof. This can be proven similarly to [7]. □

Corollary 5.1. *Let* C *be a cyclic DNA code of length* n *over* A_t *and minimum Hamming distance* d*. Then* $\xi_t(C)$ *is a DNA code of length* $2^t n$ *over the alphabet* $\{A, C, G, T\}$ *with minimum Hamming distance at least* d*.*

6. Skew cyclic codes over A_t

For $i = 2$, the reversibility problem was solved in [2]. In this section, by using the skew cyclic codes over A_i, the reversibility problem for DNA 2^i-mers is solved for $i = 1, 3, \ldots, t$.

Definition 6.1. Let B be a finite ring and θ be a non trivial automorphism over B. A subset C of B^n is called a skew cyclic code of length n if C satisfies the following conditions,

i. C is a submodule of B^n

ii. If $c = (c_0, \ldots, c_{n-1}) \in C$, then $\sigma_\theta(c) = (\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C$,

where σ_θ is the skew cyclic shift operator.

By defining a non trivial automorphism on A_t as follows, we can define the skew cyclic codes over A_t.

$$\begin{array}{rcl} \theta_i & : & A_i \longrightarrow A_i \\ x_{i-1}+u_iy_{i-1} & \longmapsto & \theta_{i-1}(x_{i-1}+y_{i-1})-u_i\theta_{i-1}(y_{i-1}) \end{array}$$

and

$$\begin{array}{rcl} \theta_1 & : & A_1 \longrightarrow A_1 \\ x_0+u_1y_0 & \longmapsto & (x_0+y_0)-u_1y_0 \end{array}$$

where $i=2,3,\ldots,t$. The order of θ_i is 2, where $i=1,2,\ldots,t$.

The rings

$$A_i[x,\theta_i]=\{b_0^i+b_1^ix+\ldots+b_{n-1}^ix^{n-1}:b_j^i\in A_i,n\in N,i=1,\ldots,t,j=0,\ldots,n-1\}$$

are called skew polynomial rings with the usual polynomial addition and the multiplication as follows

$$(\varrho x^s)(\eta x^v)=\varrho\theta_i^s(\eta)x^{s+v}$$

where $i=1,\ldots,t$. They are non commutative rings.

The set $A_{\theta_i,n}=A_i[x,\theta_i]/\langle x^n-1\rangle=\{f_i(x)+\langle x^n-1\rangle : f_i(x)\in A_i[x,\theta_i]\}$ is a left $A_i[x,\theta_i]$-module with the multiplication from left as follows,

$$r_i(x)(f_i(x)+\langle x^n-1\rangle)=r_i(x)f_i(x)+\langle x^n-1\rangle$$

where for any $r_i(x)\in A_i[x,\theta_i]$, for $i=1,\ldots,t$.

A code C_i over A_i of length n is a skew cyclic code if and only if C_i is a left $A_i[x,\theta_i]$-submodule of $A_{\theta_i,n}$, where $i=1,\ldots,t$. Let $f_i(x)$ be a polynomial in C_i of minimal degree. If the leading cofficient of $f_i(x)$ is a unit in A_i, then $C_i=\langle f_i(x)\rangle$, where $f_i(x)$ is a right divisor of x^n-1.

We can express the matching the elements A_1 and $S_{D_{16}}=\{AA,TT,\ldots,GG\}$ by means of the automorphism θ_1 as follows.

Each element $\alpha_1=x_0+u_1y_0\in A_1$ and $\theta_1(\alpha_1)$ are mapped to DNA 2-bases which are reverse of each other. Let ξ_1 be a correspondence the elements of the finite ring A_1 and DNA 2-bases. For example

$$\xi_1(u_1)=AT,\text{ while }\xi_1\left(\theta_1(u_1)\right)=TA$$

By using a map $\xi_i=\gamma_i\circ\phi_i$, where the map γ_i is defined from A_{i-1}^2 to DNA 2^i-bases as foolows

$$\gamma_i(s_{i-1},t_{i-1})=(\xi_{i-1}(s_{i-1}),\xi_{i-1}(t_{i-1}))$$

where $s_{i-1},t_{i-1}\in A_{i-1}$, for $i=1,\ldots,t$, we can explain a relationship between skew cyclic codes and DNA codes. Actually, $\xi_i(r_i)$ and $\xi_i\left(\theta_i(r_i)\right)$ are DNA reverse of each other, where $r_i=a_{i-1}+u_ib_{i-1}$, $a_{i-1},b_{i-1}\in A_{i-1}$ for $i=1,\ldots,t$.

For $r_i=a_{i-1}+u_ib_{i-1}\in A_i$, we have

$$\begin{aligned} \xi_i(r_i) &= \gamma_i\left(\phi_i(a_{i-1}+u_ib_{i-1})\right)=\gamma_i\left(a_{i-1},a_{i-1}+b_{i-1}\right) \\ &= (\xi_{i-1}(a_{i-1}),\xi_{i-1}(a_{i-1}+b_{i-1})) \end{aligned}$$

On the other hand,

$$\begin{aligned} \xi_i\left(\theta_i(r_i)\right) &= \xi_i\left(\theta_{i-1}(a_{i-1}+b_{i-1})-u_i\theta_{i-1}(b_{i-1})\right) \\ &= \gamma_i\left(\phi_i\left(\theta_{i-1}(a_{i-1}+b_{i-1})-u_i\theta_{i-1}(b_{i-1})\right)\right) \\ &= \gamma_i\left(\theta_{i-1}(a_{i-1}+b_{i-1}),\theta_{i-1}(a_{i-1})\right) \\ &= \left(\xi_{i-1}\left(\theta_{i-1}(a_{i-1}+b_{i-1})\right),\xi_{i-1}\left(\theta_{i-1}(a_{i-1})\right)\right) \end{aligned}$$

where $i=1,\ldots,t$.

This map can be extended as follows. For any $\mathrm{s}_i=(s_0^i,\ldots,s_{n-1}^i)\in A_i^n$,

$$\left(\xi_i\left(s_0^i\right),\xi_i\left(s_1^i\right),\ldots,\xi_i\left(s_{n-1}^i\right)\right)^r=\left(\xi_i\left(\theta_i\left(s_{n-1}^i\right)\right),\ldots,\xi_i\left(\theta_i\left(s_1^i\right)\right),\xi_i\left(\theta_i\left(s_0^i\right)\right)\right)$$

where $i=1,2,\ldots,t$.

Example 6.1. If $r_2 = 1 + u_1 + u_2(2 + 3u_1) \in A_2$, then we have

$$\begin{aligned}\xi_2(r_2) &= \gamma_2(\phi_2(r_3)) = \gamma_2(1 + u_1, 3)\\ &= (\xi_1(1 + u_1), \xi_1(3))) = (TG, CC)\end{aligned}$$

On the other hand,

$$\begin{aligned}\xi_2(\theta_2(r_2)) &= \xi_2(\theta_1(3) - u_2\theta_1(2 + 3u_1))\\ &= \gamma_2(\theta_1(3), \theta_1(1 + u_1))\\ &= (\xi_1(\theta_1(3)), \xi_1(\theta_1(1 + u_1)))\\ &= (CC, GT)\end{aligned}$$

Definition 6.2. Let C_i be a code of length n over A_i, for $i = 1, \ldots, t$. If $\xi_i(\mathbf{c})^r \in \xi_i(C_i)$ for all $\mathbf{c} \in C_i$, then C_i or equivalently $\xi_i(C_i)$ is called a reversible DNA code, for $i = 1, \ldots, t$.

The skew cyclic code of odd length over A_i with respect to θ_i is a cyclic code, as the order of θ_i is 2 for $i = 1, \ldots, t$. So we will take the length n to be even.

Definition 6.3. Let $g_i(x) = b_0^i + b_1^i x + b_2^i x^2 + \ldots + b_s^i x^s$ be a polynomial of degree s over A_i, for $i = 1, \ldots, t$. $g_i(x)$ is called a palindromic polynomial if $b_j^i = b_{s-j}^i$ for all $j \in \{0, 1, \ldots, s\}$. $g_i(x)$ is called a θ_i-palindromic polynomial if $b_j^i = \theta_i(b_{s-j}^i)$ for all $j \in \{0, 1, \ldots, s\}$, for $i = 1, \ldots, t$.

Theorem 6.1. *Let $C_i = \langle f_i(x) \rangle$ be a skew cyclic code of length n over A_i, for $i = 1, 3, \ldots, t$, where $f_i(x)$ is a right divisor of $x^n - 1$ and $\deg(f_i(x))$ is odd. If $f_i(x)$ is a θ_i-palindromic polynomial then $\xi_i(C_i)$ is a reversible DNA code.*

Proof. Let $f_i(x)$ be a θ_i-palindromic polynomial and $f_i(x) = a_0^i + a_1^i x + \ldots + a_{2s-1}^i x^{2s-1}$. So $a_j^i = \theta_i(a_{2s-1-j}^i)$, for all $j = 0, 1, \ldots, s-1$, $i = 1, 3, \ldots, t$. Let $h_i(x) = h_0^i + h_1^i x + \ldots + h_{2k-1}^i x^{2k-1}$. Let b_j^i be the coefficient of x^j in $h_i(x)f_i(x)$. For any $\kappa < n/2$, the coefficient of x^κ in $h_i(x)f_i(x)$ is

$$b_\kappa^i = \sum_{j=0}^{\kappa} h_j^i \theta_i^j(a_{\kappa-j}^i)$$

and the coefficient of $x^{(n-1)-\kappa}$ is $b_{(n-1)-\kappa}^i = \sum_{j=0}^{\kappa} h_{2k-1-j}^i \theta_i^{2k-1-j}(a_{2s-1-(\kappa-j)}^i)$, for $i = 1, 3, \ldots, t$.

The polynomial $h_i(x)f_i(x) = \sum_{p=0}^{2k-1} h_p^i x^p f_i(x)$ corresponds a vector $\mathbf{b} = (b_0^i, b_1^i, \ldots, b_{n-1}^i) \in C_i$, for $i = 1, 3, \ldots, t$. The vector $\xi_i(\mathbf{b})^r = \left(\left(\xi_i(b_0^i), \xi_i(b_1^i), \ldots, \xi_i(b_{n-1}^i)\right)\right)^r$ is equal to the vector $\xi_i(\mathbf{z})$, where the vector $\mathbf{z}$ corresponds the polynomial $\sum_{p=0}^{2k-1} \theta_i(h_p^i) x^{2k-1-p} f_i(x)$,for $i = 1, 3, \ldots, t$. So $\xi_i(C_i)$ is a reversible DNA code. □

Theorem 6.2. *Let $C_i = \langle f_i(x) \rangle$ be a skew cyclic code of length n over A_i, for $i = 1, 3, \ldots, t$, where $f_i(x)$ is a right divisor of $x^n - 1$ and $\deg(f_i(x))$ is even. If $f_i(x)$ is a palindromic polynomial then $\xi_i(C_i)$ is a reversible DNA code.*

Proof. Let $f_i(x)$ be a palindromic polynomial with even degree. $f_i(x) = a_0^i + a_1^i x + \ldots + a_{2s}^i x^{2s}$ and $a_p^i = a_{2s-p}^i$, for all $p = 0, 1, \ldots, s$, for $i = 1, 3, \ldots, t$. Let $h_i(x) = h_0^i + h_1^i x + \ldots + h_{2k}^i x^{2k}$. Let b_p^i be the coefficient of x^p in $h_i(x)f_i(x)$. For any $\kappa < n/2$, the coefficient of x^κ in $h_i(x)f_i(x)$ is

$$b_\kappa^i = \sum_{j=0}^{\kappa} h_j^i \theta_i^j(a_{\kappa-j}^i)$$

and the coefficient of $x^{(n-1)-\kappa}$ is $b_{(n-1)-\kappa}^i = \sum_{j=0}^{\kappa} h_{(2k)-j}^i \theta_i^{(2k)-j}(a_{2s-(\kappa-j)}^i)$, for $i = 1, 3, \ldots, t$.

The polynomial $h_i(x)f_i(x) = \sum_{p=0}^{2k} h_p^i x^p f_i(x)$ corresponds a vector $\mathbf{b} = (b_0^i, b_1^i, \ldots, b_{n-1}^i) \in C_i$, for $i = 1, 3, \ldots, t$. The vector $\xi_i(\mathbf{b})^r = \left(\left(\xi_i(b_0^i), \xi_i(b_1^i), \ldots, \xi_i(b_{n-1}^i)\right)\right)^r$ is equal to the vector $\xi_i(\mathbf{z})$, where the vector $\mathbf{z}$ corresponds the polynomial $\sum_{p=0}^{2k} \theta_i(h_p^i) x^{2k-p} f_i(x)$. So $\xi_i(C_i)$ is a reversible DNA code. □

7. θ_i-set

In this section, we will obtain DNA codes by using θ_i-set, where θ_i is a non trivial automorphism on A_i for $i=1,...,t$.

Definition 7.1. Let $f_{0,1},\dots,f_{0,2^i}$ be polynomials dividing x^n-1 over Z_4 and let $f_{i-1,1},f_{i-1,2}$ be polynomials with $\deg f_{i-1,1}=d_{i-1,1},\deg f_{i-1,2}=d_{i-1,2}$ and both are over A_{i-1} for $i=1,2,\dots,t$. Let

$$f_i=u_if_{i-1,1}+(1+u_i)f_{i-1,2}\in A_i[x]$$

and

$$\begin{aligned}
f_{i-1,1} &= u_{i-1}f_{i-2,1}+(1+u_{i-1})f_{i-2,2}\\
f_{i-1,2} &= u_{i-1}f_{i-2,3}+(1+u_{i-1})f_{i-2,4}\\
\\
f_{i-2,1} &= u_{i-2}f_{i-3,1}+(1+u_{i-2})f_{i-3,2}\\
f_{i-2,2} &= u_{i-2}f_{i-3,3}+(1+u_{i-2})f_{i-3,4}\\
f_{i-2,3} &= u_{i-2}f_{i-3,5}+(1+u_{i-2})f_{i-3,6}\\
f_{i-2,4} &= u_{i-2}f_{i-3,7}+(1+u_{i-2})f_{i-3,8}\\
&\vdots\\
f_{1,1} &= u_1f_{0,1}+(1+u_1)f_{0,2}\\
f_{1,2} &= u_1f_{0,3}+(1+u_1)f_{0,4}\\
&\vdots
\end{aligned}$$

$$f_{1,2^{i-1}}=u_1f_{0,2^i-1}+(1+u_1)f_{0,2^i}$$

Let $m_i=\min\{n-d_{i-1,1},n-d_{i-1,2}\}$. The set $L(f_i)$ is called a θ_i-set and is defined as

$$L(f_i)=\{E_0,E_1,\dots,E_{m_i-1},F_0,F_1,\dots,F_{m_i-1}\}$$

where $E_j=x^jf_i,F_j=x^j\theta_i(h_i),0\le j\le m_i-1,i=1,2,\dots,t.$

If $\deg f_{0,2s}\ge\deg f_{0,2s-1}$,

$$h_{i,1,s}=u_1x^{\deg f_{0,2s}-\deg f_{0,2s-1}}f_{0,2s-1}+(1+u_1)\,f_{0,2s}$$

otherwise

$$h_{i,1,s}=u_1f_{0,2s-1}+(1+u_1)\,x^{\deg f_{0,2s-1}-\deg f_{0,2s}}f_{0,2s}$$

where $s=1,2,\dots,2^{i-1}$ and

If $\deg h_{i,1,2t}\ge\deg h_{i,1,2t-1}$,

$$h_{i,2,t}=u_2x^{\deg f_{ii,1,2t}-\deg f_{i,1,2t-1}}h_{i,1,2t-1}+(1+u_2)\,h_{i,1,2t}$$

otherwise

$$h_{i,2,t}=u_2h_{i,1,2t-1}+(1+u_2)\,x^{\deg f_{i,1,2t-1}-\deg f_{ii,1,2t}}h_{i,1,2t}$$

where $t=1,2,\dots,2^{i-2}$ and

$$\vdots$$

If $\deg h_{i,i-2,2v}\ge\deg h_{i,i-2,2v-1}$,

$$h_{i,i-1,v}=u_{i-1}x^{\deg h_{i,i-2,2v}-\deg h_{i,i-2,2v-1}}h_{i,i-2,2v-1}+(1+u_{i-1})\,h_{i,i-2,2v}$$

otherwise

$$h_{i,i-1,v}=u_{i-1}h_{i,i-2,2v-1}+(1+u_{i-1})\,x^{\deg h_{i,i-2,2v-1}-\deg h_{i,i-2,2v}}h_{i,i-2,2v}$$

where $v=1,2$ and

If $\deg h_{i,i-1,2} \geq \deg h_{i,i-1,1}$,

$$h_i = u_i x^{\deg h_{i,i-1,2} - \deg h_{i,i-1,1}} h_{i,i-1,1} + (1+u_i)\, h_{i,i-1,2}$$

otherwise

$$h_i = u_i h_{i,i-1,1} + (1+u_i)\, x^{\deg h_{i,i-1,1} - \deg h_{i,i-1,2}} h_{i,i-1,2}.$$

$L(f_i)$ generates a linear code C_i over A_i, where $i = 1, 2, \ldots, t$. It will be denoted by $C_i = \langle f_i \rangle_{\theta_i}$ or $C_i = \langle L(f_i) \rangle$. It means that it is A_i-submodule generated by the set $L(f_i)$, where $i = 1, 2, \ldots, t$. Let $f_i = a_0^i + a_1^i x + \ldots + a_p^i x^p \in A_i[x], \theta_i(h_i) = b_0^i + b_1^i x + \ldots + b_s^i x^s$, where $i = 1, 2, \ldots, t$. The A_i-submodule can be considered to be generated by the rows of the following matrix

$$L(f_i) = \begin{bmatrix} E_0 \\ F_0 \\ E_1 \\ F_1 \\ E_2 \\ F_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_0^i & a_1^i & a_2^i & \cdots & a_p^i & 0 & \cdots & \cdots & & \cdots & 0 \\ b_0^i & b_1^i & \cdots & \cdots & b_p^i & b_{p+1}^i & \cdots & b_s^i & 0 & \cdots & 0 \\ 0 & a_0^i & a_1^i & a_2^i & \cdots & a_p^i & 0 & 0 & & \cdots & 0 \\ 0 & b_0^i & b_1^i & \cdots & \cdots & \cdots & \cdots & \cdots & b_s^i & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & & \cdots & \vdots \end{bmatrix}$$

Theorem 7.1. *Let $f_{0,1}, \ldots, f_{0,2^i}$ be self reciprocal polynomials dividing $x^n - 1$ over Z_4. Then $C_i = \langle L(f_i) \rangle$ is a linear code over A_i and $\xi_i(C_i)$ is a reversible DNA code, where the map ξ_i is from C_i to $S_{D_4}^{2^i n}$, for $i = 1, 2, \ldots, t$.*

Proof. It is proved as in the proof of the Theorem 4.3 in [3]. □

Corollary 7.1. *Let $f_{0,1}, \ldots, f_{0,2^i}$ be self reciprocal polynomials dividing $x^n - 1$ over Z_4 and $C_i = \langle L(f_i) \rangle$ be a cyclic code over A_i for $i = 1, \ldots, t$. If $\frac{x^n-1}{x-1} \in C_i$, then $\xi_i(C_i)$ is a reversible complement DNA code.*

Example 7.1.

$$\begin{aligned} f_{0,1}(x) &= 2\,(x+1) \\ f_{0,2}(x) &= x^4 - x^3 + x^2 - x + 1 \end{aligned}$$

where all of them divide $x^{10} - 1$ over Z_4. Hence,

$$f_1 = u_1 f_{0,1} + (1+u_1)\, f_{0,2}$$

over A_1. That is

$$f_2 = (1+u_1)\, x^4 - (1+u_1)\, x^3 + (1+u_1)\, x^2 - (1-u_1)\, x + 1 + 3u_1.$$

We get $h_1 = u_1 x^3 h_{1,0,1} + (1+u_1) h_{1,0,2} = (1+3u_1)\, x^4 - (1-u_1)\, x^3 + (1+u_1)\, x^2 - (1+u_1)\, x + 1 + u_1$. So, $\theta_1(h_1) = u_1 x^4 - u_1 x^3 + (2+3u_1)\, x^2 - (2+3u_1)\, x + 2 + 3u_1$. Since $m_1 = 6$, we consider the generator matrix of C $\begin{bmatrix} E_0 \\ F_0 \\ E_1 \\ F_1 \\ \vdots \\ E_5 \\ F_5 \end{bmatrix}$, where $E_0 = f_1, E_1 = x f_1, E_2 = x^2 f_1, E_3 = x^3 f_1, E_4 = x^4 f_1, E_5 = x^5 f_1, F_0 = \theta_1(h_1), F_1 = x\theta_1(h_1), F_2 = x^2\theta_1(h_1), F_3 = x^3\theta_1(h_1), F_4 = x^4\theta_1(h_1), F_5 = x^5\theta_1(h_1)$. If we take $\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = u_1, \alpha_3 = 0, \alpha_4 = 0, \alpha_5 = 0, \beta_0 = 1, \beta_1 = 0, \beta_2 = 1, \beta_3 = 0, \beta_4 = 0, \beta_5 = 3$, then $\alpha_0 E_0 + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 + \alpha_5 E_5 + \beta_0 F_0 + \beta_1 F_1 + \beta_2 F_2 + \beta_3 F_3 + \beta_4 F_4 + \beta_4 F_4 = 3u_1 x^9 + u_1 x^8 + (2+u_1) x^7 + (2+2u_1)\, x^6 + (3+3u_1)\, x^5 + (1+u_1)\, x^4 + (3+u_1) x^3 + (3+3u_1) x^2 + 3x + 2 + 3u_1$. It corresponds to the codeword

$$\mathbf{d}_1 = (2+3u_1, 3, 3+3u_1, 3+u_1, 1+u_1, 3+3u_1, 2+2u_1, 2+u_1, u_1, 3u_1)$$

Hence, $\xi_1(\mathbf{d}_1) = GTCCCGCATGCGGAGCATAC$. Moreover, $\theta_1(\alpha_0) F_5 + \theta_1(\alpha_1) F_4 + \theta_1(\alpha_2) F_3 + \theta_1(\alpha_3) F_2 + \theta_1(\alpha_4) F_1 + \theta_1(\alpha_5) F_0 + \theta_1(\beta_0) E_5 + \theta_1(\beta_1) E_4 + \theta_1(\beta_2) E_3 + \theta_1(\beta_3) E_2 + \theta_1(\beta_4) E_1 + \theta_1(\beta_5) E_0 = (1+u_1)\, x^9 +$

$3x^8+(2+u_1)x^7+3u_1x^6+(2+3u_1)x^5+(2+u_1)x^4+2u_1x^3+(3+3u_1)x^2+(1+3u_1)x+3+u_1$ corresponds to the codeword

$$\mathbf{d}_2=(3+u_1,1+3u_1,3+3u_1,2u_1,2+u_1,2+3u_1,3u_1,2+u_1,3,1+u_1)$$

Hence, $\xi_1(\mathbf{d}_2)=CATACGAGGCGTACGCCCTG$. So, $(\xi_1(\mathbf{d}_2))^r=\xi_1(\mathbf{d}_1)$.

Example 7.2.

$$\begin{aligned} f_{0,1}(x) &= x+1 \\ f_{0,2}(x) &= x^2+x+1 \\ f_{0,3}(x) &= x^6+x^3+1 \\ f_{0,4}(x) &= x+1 \end{aligned}$$

where all of them divide x^9-1 over Z_4. Hence,

$$f_2=u_2\left(u_1f_{0,1}+(1+u_1)\,f_{0,2}\right)+(1+u_2)$$

over A_2. That is

$$f_2=u_1\,(1+u_2)\,x^6+u_1\,(1+u_2)\,x^3+u_2\,(1+u_1)\,x^2+(1+u_1+2u_2+3u_1u_2)\,x+1+2u_1+2u_2.$$

Since $h_{2,1,1}=u_1xf_{0,1}+(1+u_1)f_{0,2}$ and $h_{2,1,2}=u_1f_{0,3}+x^5(1+u_1)f_{0,4}$, we get $h_2=u_2x^4h_{2,1,1}+(1+u_2)h_{2,1,2}=(1+2u_1+2u_2)\,x^6+(1+u_1+2u_2+3u_1u_2)\,x^5+(1+u_1)u_2x^4+(1+u_2)u_1x^3+u_1(1+u_2)$. So, $\theta_2\,(h_2)=(1+2u_1+2u_2)\,x^6+(3+3u_2+3u_1u_2)\,x^5+(2+3u_1+2u_2+u_1u_2)\,x^4+(2+2u_1+3u_2+u_1u_2)\,x^3+(2+2u_1+3u_2+u_1u_2)$.

Since $m_2=3$, we consider the generator matrix of C $\begin{bmatrix} E_0\\ F_0\\ E_1\\ F_1\\ E_2\\ F_2 \end{bmatrix}$, where $E_0=f_2, E_1=xf_2, E_2=x^2f_2, F_0=\theta_2\,(h_2)\,, F_1=x\theta_2\,(h_2)\,, F_2=x^2\theta_2\,(h_2)$. If we take $\alpha_0=0,\alpha_1=0,\alpha_2=3,\beta_0=0,\beta_1=2,\beta_2=0$, then $\alpha_0E_0+\alpha_1E_1+\alpha_2E_2+\beta_0F_0+\beta_1F_1+\beta_2F_2=3u_1(1+u_2)x^8+2x^7+(2+2u_2+2u_1u_2)\,x^6+(u_1+u_1u_2)x^5+(u_2+u_1u_2)x^4+(3+3u_1+2u_2+u_1u_2)x^3+(3+2u_1+2u_2)\,x^2+(2u_2+2u_1u_2)\,x$. It corresponds to the codeword

$$\mathbf{d}_1=\begin{pmatrix} 0,2u_2+2u_1u_2,3+2u_1+2u_2,3+3u_1+2u_2+u_1u_2, \\ u_2+u_1u_2,u_1+u_1u_2,2+2u_2+2u_1u_2,2,3u_1+3u_1u_2 \end{pmatrix}$$

Hence, $\xi_2(\mathbf{d}_1)=AAAAAAGACTTCCGTTAATGATAGGGAGGGGGACAG$. Moreover, $\theta_2\,(\alpha_0)\,F_2+\theta_2\,(\alpha_1)\,F_1+\theta_2\,(\alpha_2)\,F_0+\theta_2\,(\beta_0)\,E_2+\theta_2\,(\beta_1)\,E_1+\theta_2\,(\beta_2)\,E_0=2u_1(1+u_2)x^7+(3+2u_1+2u_2)\,x^6+(1+u_2+u_1u_2)x^5+(2+3u_1+2u_2+u_1u_2)x^4+(2+2u_1+3u_2+u_1u_2)x^3+(2+2u_1+2u_1u_2)\,x^2+2x+2+2u_1+u_2+3u_1u_2$ corresponds to the codeword

$$\mathbf{d}_2=\begin{pmatrix} 2+2u_1+u_2+3u_1u_2,2,2+2u_1+2u_1u_2,2+2u_1+3u_2+u_1u_2, \\ 2+3u_1+2u_2+u_1u_2,1+u_2+u_1u_2,3+2u_1+2u_2,2u_1+2u_1u_2,0 \end{pmatrix}$$

Hence, $\xi_2(\mathbf{d}_2)=GACAGGGGGAGGGATAGTAATTGCCTTCAGAAAAAA$. So, $(\xi_2(\mathbf{d}_2))^r=\xi_2(\mathbf{d}_1)$.

8. Conclusion

The DNA codes are obtained with three different methods by using cyclic, skew cyclic codes and θ_i-set over a family of the rings A_t. A one to one correspondence between A_t and $\{A,T,C,G\}^{2^t}$ is constructed by using a map.The sufficient and necessary conditions of cyclic codes over A_t satisfying the reverse and reverse complement constraints are given, respectively. By defining a non trivial automorphism θ_i on A_t, the skew cyclic codes are introduced. By using the skew cyclic codes over A_t and the θ_i-set, the DNA codes are obtained. In a future work, it can be identified the new ring family and its associated Gray map reversible and reversible complement codes to search for optimal DNA codes that meet all or some of the constraints.

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