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LACUNARY INVARIANT STATISTICAL EQUIVALENCE FOR DOUBLE SET SEQUENCES

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Abstract. In this paper, we introduce the notions of asymptotical strong σ_2-equivalence, asymptotical σ_2-statistical equivalence, asymptotical lacunary strong σ_2-equivalence and asymptotical lacunary σ_2-statistical equivalence in the Wijsman sense for double set sequences. Also, we investigate some relations between these new asymptotical equivalence notions.

1. Introduction

Long after the notion of convergence for double sequences was introduced by Pringsheim [1], this notion was extended to the notion of statistical convergence by Móricz [2] and Mursaleen and Edely [3] in the same year, to the notion of lacunary statistical convergence by Patterson and Savaş [4] and to the notion of double σ-convergent lacunary statistical sequence by Savaş and Patterson [5]. Moreover, for double sequences, the notion of asymptotical equivalence was introduced by Patterson [6].

Over the years, on the various convergence notions for set sequences have been studied by many authors (see, [7–9]). One of them, discussed in this paper, is the notion of convergence in the Wijsman sense [10]. Using the notions of statistical convergence, double lacunary sequence and invariant mean, this notion was extended to the notions of convergence for double set sequences by some authors [11–13]. Furthermore, for double set sequences, the notions of asymptotical equivalence in the Wijsman sense were introduced by Nuray et al. [14] and then these notions were studied by some authors [15–17]. In this paper, using the notion of invariant

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mean, we study on new asymptotical equivalence notions in the Wijsman sense for double set sequences. More information on the notions of asymptotical equivalence for set sequences can be found in [18, 19].

2. Basic Definitions and Notations

In this section, let us remind the basic notions necessary for a better understanding of our paper.

Definition 1. *[1] A double sequence (x_{jk}) is called convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{jk} - L| < \varepsilon$, whenever $j, k > N_\varepsilon$. It is denoted by $P - \lim_{j,k\to\infty} x_{kj} = L$ or $\lim_{j,k\to\infty} x_{jk} = L$.*

Definition 2. *[3] A double sequence (x_{jk}) is called statistically convergent to L if for every $\varepsilon > 0$,*

$$P - \lim_{m,n\to\infty} \frac{1}{mn}\Big|\big\{(j,k) : \ j \le m, k \le n, \ |x_{jk} - L| \ge \varepsilon\big\}\Big| = 0.$$

For a metric space (Y, d), $\mu(y, B)$ denote the distance from y to B where

$$\mu(y, B) = \inf_{b\in B} d(y, b)$$

for any $y \in Y$ and any nonempty $B \subseteq Y$.

Throughout this study, (Y, d) will be considered as a metric space and B, B_{jk}, D_{jk} will be considered as any nonempty closed subsets of Y.

Definition 3. *[13] A double set sequence $\{B_{jk}\}$ is called convergent to the set B in the Wijsman sense if for each $y \in Y$,*

$$P - \lim_{j,k\to\infty} \mu(y, B_{jk}) = \mu(y, B).$$

Let σ be a mapping such that $\sigma : \mathbb{N} \to \mathbb{N}$ (the set of positive integers). A continuous linear functional ψ on ℓ_∞, the space of real bounded sequences, is called an invariant mean (or a σ-mean) if it satisfies the following conditions:

(1) $\psi(x_s) \ge 0$, when the sequence (x_s) has $x_s \ge 0$ for all s,
(2) $\psi(e) = 1$, where $e = (1, 1, 1, \ldots)$ and
(3) $\psi(x_{\sigma(s)}) = \psi(x_s)$ for all $(x_s) \in \ell_\infty$.

The mapping σ is assumed to be one-to-one and such that $\sigma^j(s) \neq s$ for all $j, s \in \mathbb{N}$, where $\sigma^j(s)$ denotes the j th iterate of the mapping σ at s. Thus ψ extends the limit functional on c, the space of convergent sequences, in the sense that $\psi(x_s) = \lim x_s$ for all $(x_s) \in c$.

Definition 4. *[12] A double set sequence $\{B_{jk}\}$ is called invariant convergent to the set B in the Wijsman sense if for each $y \in Y$,*

$$P - \lim_{n,m\to\infty} \frac{1}{nm} \sum_{j,k=1,1}^{n,m} \mu(y, B_{\sigma^j(s)\sigma^k(t)}) = \mu(y, B), \quad \textit{uniformly in } s, t.$$

Definition 5. *[12] A double set sequence $\{B_{jk}\}$ is called strong invariant convergent to the set B in the Wijsman sense if for each $y \in Y$,*

$$P-\lim_{n,m\to\infty} \frac{1}{nm} \sum_{j,k=1,1}^{n,m} \left|\mu(y, B_{\sigma^j(s)\sigma^k(t)}) - \mu(y,B)\right| = 0, \quad \textit{uniformly in } s,t.$$

Definition 6. *[12] A double set sequence $\{B_{jk}\}$ is called invariant statistically convergent to the set B in the Wijsman sense if for every $\varepsilon > 0$ and each $y \in Y$,*

$$P-\lim_{n,m\to\infty} \frac{1}{nm} \left|\left\{(j,k) : j \le n, k \le m, \left|\mu(y, B_{\sigma^j(s)\sigma^k(t)}) - \mu(y,B)\right| \ge \varepsilon\right\}\right| = 0,$$

uniformly in s,t.

A double sequence $\theta_2 = \{(j_r, k_u)\}$ is called a double lacunary sequence if there exist increasing sequences (j_r) and (k_u) of the integers such that

$$j_0 = 0,\ h_r = j_r - j_{r-1} \to \infty \ \text{ and } \ k_0 = 0,\ \bar{h}_u = k_u - k_{u-1} \to \infty \ \text{ as } \ r,u \to \infty.$$

In general, the following notations is used for any double lacunary sequence:

$$h_{ru} = h_r \bar{h}_u, \quad I_{ru} = \{(j,k) : j_{r-1} < j \le j_r \ \text{ and } \ k_{u-1} < k \le k_u\},$$

$$q_r = \frac{j_r}{j_{r-1}} \ \text{ and } \ q_u = \frac{k_u}{k_{u-1}}.$$

Throughout this study, $\theta_2 = \{(j_r, k_u)\}$ will be considered as a double lacunary sequence.

Definition 7. *[12] A double set sequence $\{B_{jk}\}$ is called lacunary invariant convergent to the set B in the Wijsman sense if for each $y \in Y$,*

$$P-\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(j,k)\in I_{ru}} \mu(y, B_{\sigma^j(s)\sigma^k(t)}) = \mu(y,B), \quad \textit{uniformly in } s,t.$$

Definition 8. *[12] A double set sequence $\{B_{jk}\}$ is called lacunary strong invariant convergent to the set B in the Wijsman sense if for each $y \in Y$,*

$$P-\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(j,k)\in I_{ru}} \left|\mu(y, B_{\sigma^j(s)\sigma^k(t)}) - \mu(y,B)\right| = 0, \quad \textit{uniformly in } s,t.$$

Definition 9. *[12] A double set sequence $\{B_{jk}\}$ is called lacunary invariant statistically convergent to the set B in Wijsman sense if for every $\varepsilon > 0$ and each $y \in Y$,*

$$P-\lim_{r,u\to\infty} \frac{1}{h_{ru}} \left|\left\{(j,k) \in I_{ru} : \left|\mu(y, B_{\sigma^j(s)\sigma^k(t)}) - \mu(y,B)\right| \ge \varepsilon\right\}\right| = 0,$$

uniformly in s,t.

The term $\mu_y\left(\frac{B_{jk}}{D_{jk}}\right)$ is defined as follows:

$$\mu_y\left(\frac{B_{jk}}{D_{jk}}\right)=\begin{cases}\dfrac{\mu(y,B_{jk})}{\mu(y,D_{jk})} & , \quad y\notin B_{jk}\cup D_{jk}\\ \lambda & , \quad y\in B_{jk}\cup D_{jk}.\end{cases}$$

Definition 10. *[14] Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are called asymptotically equivalent of multiplicity λ in the Wijsman sense if for each $y\in Y$,*

$$P-\lim_{j,k\to\infty}\mu_y\left(\frac{B_{jk}}{D_{jk}}\right)=\lambda.$$

It is denoted by $B_{jk}\overset{W_2^\lambda}{\sim}D_{jk}$ and simply called asymptotically equivalent in the Wijsman sense if $\lambda=1$.

As an example to asymptotically equivalent double set sequences, the following sequences can be considered:

$$B_{jk}=\{(a,b)\in\mathbb{R}^2: a^2+b^2-2jkb=0\}$$

and

$$D_{jk}=\{(a,b)\in\mathbb{R}^2: a^2+b^2+2jkb=0\}.$$

Since

$$P-\lim_{j,k\to\infty}\mu_y\left(\frac{B_{jk}}{D_{jk}}\right)=1$$

for every $y\in\mathbb{R}^2$, the double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are asymptotically equivalent in the Wijsman sense, i.e., $B_{jk}\overset{W_2}{\sim}D_{jk}$.

3. Main results

In this section, for double set sequences, we introduce the notions of asymptotical σ_2-equivalence, asymptotical strong σ_2-equivalence, asymptotical σ_2-statistical equivalence, asymptotical lacunary σ_2-equivalence, asymptotical strong lacunary σ_2-equivalence and asymptotical lacunary σ_2-statistical equivalence in the Wijsman sense. Also, we investigate some relations between some of these new asymptotical equivalence notions.

Definition 11. *Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are said to be asymptotically σ_2-equivalent of multiplicity λ in the Wijsman sense if for each $y\in Y$,*

$$P-\lim_{n,m\to\infty}\frac{1}{nm}\sum_{j,k=1,1}^{n,m}\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)=\lambda,\quad \textit{uniformly in } s,t.$$

This type of equivalence is denoted by $B_{jk}\overset{W_{\sigma_2}^\lambda}{\sim}D_{jk}$ and simply called asymptotically σ_2-equivalent in the Wijsman sense if $\lambda=1$.

Definition 12. *Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are said to be asymptotically strong σ_2-equivalent of multiplicity λ in the Wijsman sense if for each $y \in Y$,*

$$P - \lim_{n,m\to\infty} \frac{1}{nm} \sum_{j,k=1,1}^{n,m} \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| = 0, \quad \textit{uniformly in } s,t.$$

This type of equivalence is denoted by $B_{jk} \overset{[W_{\sigma_2}^{\lambda}]}{\sim} D_{jk}$ and simply called asymptotically strong σ_2-equivalent in the Wijsman sense if $\lambda = 1$.

The set of all asymptotically strong σ_2-equivalent double set sequences of multiplicity λ in the Wijsman sense is denoted by $\{[W_{\sigma_2}^{\lambda}]\}$.

Definition 13. *Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are said to be asymptotically σ_2-statistical equivalent of multiplicity λ in the Wijsman sense if for every $\varepsilon > 0$ and each $y \in Y$,*

$$P - \lim_{n,m\to\infty} \frac{1}{nm} \left| \left\{ (j,k) : j \le n, k \le m, \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \ge \varepsilon \right\} \right| = 0,$$

uniformly in s,t. This type of equivalence is denoted by $B_{jk} \overset{WS_{\sigma_2}^{\lambda}}{\sim} D_{jk}$ and simply called asymptotically σ_2-statistical equivalent in the Wijsman sense if $\lambda = 1$.

The set of all asymptotically σ_2-statistical equivalent double set sequences of multiplicity λ in the Wijsman sense is denoted by $\{WS_{\sigma_2}^{\lambda}\}$.

Definition 14. *Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are said to be asymptotically lacunary σ_2-equivalent of multiplicity λ in the Wijsman sense if for each $y \in Y$,*

$$P - \lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(j,k)\in I_{ru}} \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) = \lambda, \quad \textit{uniformly in } s,t.$$

This type of equivalence is denoted by $B_{jk} \overset{W_{\theta\sigma_2}^{\lambda}}{\sim} D_{jk}$ and simply called asymptotically lacunary σ_2-equivalent in the Wijsman sense if $\lambda = 1$.

Definition 15. *Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are said to be asymptotically lacunary strong σ_2-equivalent of multiplicity λ in the Wijsman sense if for each $y \in Y$,*

$$P - \lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(j,k)\in I_{ru}} \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| = 0, \quad \textit{uniformly in } s,t.$$

This type of equivalence is denoted by $B_{jk} \overset{[W_{\theta\sigma_2}^{\lambda}]}{\sim} D_{jk}$ and simply called asymptotically lacunary strong σ_2-equivalent in the Wijsman sense if $\lambda = 1$.

Example 1. *Let $Y = \mathbb{R}^2$ and double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ be defined as following:*

$$B_{jk} := \begin{cases} \left\{(a,b) \in \mathbb{R}^2 : a^2 + (b+1)^2 = \frac{1}{jk}\right\} & ; \text{ if } (j,k) \in I_{ru},\ j \text{ and } k \text{ are square integers}, \\ \{(2,0)\} & ; \text{ otherwise.} \end{cases}$$

and

$$D_{jk} := \begin{cases} \left\{(a,b) \in \mathbb{R}^2 : a^2 + (b-1)^2 = \frac{1}{jk}\right\} & ; \text{ if } (j,k) \in I_{ru},\ j \text{ and } k \text{ are square integers}, \\ \{(2,0)\} & ; \text{ otherwise.} \end{cases}$$

In this case, the double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are asymptotically lacunary strong σ_2-equivalent in the Wijsman sense.

The set of all asymptotically lacunary strong σ_2-equivalent double set sequences of multiplicity λ in the Wijsman sense is denoted by $\{[W^{\lambda}_{\theta\sigma_2}]\}$.

Definition 16. *Two double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are said to be asymptotically lacunary σ_2-statistical equivalent of multiplicity λ in the Wijsman sense if for every $\varepsilon > 0$ and each $y \in Y$,*

$$P - \lim_{r,u\to\infty} \frac{1}{h_{ru}} \left| \left\{ (j,k) \in I_{ru} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| = 0,$$

uniformly in s,t. This type of equivalence is denoted by $B_{jk} \overset{WS^{\lambda}_{\theta\sigma_2}}{\sim} D_{jk}$ and simply called asymptotically lacunary σ_2-statistical equivalent in the Wijsman sense if $\lambda = 1$.

Example 2. *Let $Y = \mathbb{R}^2$ and double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ be defined as following:*

$$B_{jk} := \begin{cases} \left\{(a,b) \in \mathbb{R}^2 : (a-j)^2 + (b+k)^2 = 4\right\} & ; \text{ if } (j,k) \in I_{ru},\ j \text{ and } k \text{ are square integers}, \\ \{(-2,1)\} & ; \text{ otherwise.} \end{cases}$$

and

$$D_{jk} := \begin{cases} \left\{(a,b) \in \mathbb{R}^2 : (a+j)^2 + (b-k)^2 = 4\right\} & ; \text{ if } (j,k) \in I_{ru},\ j \text{ and } k \text{ are square integers}, \\ \{(-2,1)\} & ; \text{ otherwise.} \end{cases}$$

In this case, the double set sequences $\{B_{jk}\}$ and $\{D_{jk}\}$ are asymptotically lacunary σ_2-statistical equivalent in the Wijsman sense.

The set of all asymptotically lacunary σ_2-statistical equivalent double set sequences of multiplicity λ in the Wijsman sense is denoted by $\{WS^{\lambda}_{\theta\sigma_2}\}$.

Theorem 1.
(i) *If* $B_{jk} \overset{[W^{\lambda}_{\theta\sigma_2}]}{\sim} D_{jk}$*, then* $B_{jk} \overset{WS^{\lambda}_{\theta\sigma_2}}{\sim} D_{jk}$.
(ii) *If for each* $y \in Y$ $\sup_{j,k,s,t} \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)\right| < \infty$ *and* $B_{jk} \overset{WS^{\lambda}_{\theta\sigma_2}}{\sim} D_{jk}$*, then* $B_{jk} \overset{[W^{\lambda}_{\theta\sigma_2}]}{\sim} D_{jk}$.

Proof. (i) Let $B_{jk} \overset{[W^{\lambda}_{\theta\sigma_2}]}{\sim} D_{jk}$. For every $\varepsilon > 0$ and each $y \in Y$, we have

$$\begin{aligned}
\sum_{(j,k)\in I_{ru}} \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| &\geq \sum_{\substack{(j,k)\in I_{ru} \\ \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)-\lambda\right|\geq\varepsilon}} \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \\
&\geq \varepsilon \left|\left\{(j,k) \in I_{ru} : \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

for all s, t, which gives the result.

(ii) Let $B_{jk} \overset{WS^{\lambda}_{\theta\sigma_2}}{\sim} D_{jk}$. Also, suppose that $\sup_{j,k,s,t} \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)\right| < \infty$ for each $y \in Y$. Then, there exists an $M > 0$ such that for each $y \in Y$

$$\left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \leq M$$

for all j, k and s, t. Thus, for every $\varepsilon > 0$ and each $y \in Y$ we have

$$\begin{aligned}
&\frac{1}{h_{ru}} \sum_{(j,k)\in I_{ru}} \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \\
&= \frac{1}{h_{ru}} \sum_{\substack{(j,k)\in I_{ru} \\ \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)-\lambda\right|\geq\varepsilon}} \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \\
&\quad + \frac{1}{h_{ru}} \sum_{\substack{(j,k)\in I_{ru} \\ \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)-\lambda\right|<\varepsilon}} \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \\
&\leq \frac{M}{h_{ru}} \left|\left\{(j,k) \in I_{ru} : \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \geq \varepsilon\right\}\right| + \varepsilon
\end{aligned}$$

for all s, t, which gives the result. □

With a technique similar to that of Theorem 1, the following theorem can be proved.

Theorem 2.
(i) *If* $B_{jk} \overset{[W^{\lambda}_{\sigma_2}]}{\sim} D_{jk}$, *then* $B_{jk} \overset{WS^{\lambda}_{\sigma_2}}{\sim} D_{jk}$.
(ii) *If for each* $y \in Y$ $\sup_{j,k,s,t} \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)\right| < \infty$ *and* $B_{jk} \overset{WS^{\lambda}_{\sigma_2}}{\sim} D_{jk}$, *then* $B_{jk} \overset{[W^{\lambda}_{\sigma_2}]}{\sim} D_{jk}$.

Theorem 3. *If* $\liminf_r q_r > 1$ *and* $\liminf_u q_u > 1$ *for any* $\theta_2 = \{(j_r, k_u)\}$, *then*

$$B_{jk} \overset{WS^{\lambda}_{\sigma_2}}{\sim} D_{jk} \quad \textit{implies} \quad B_{jk} \overset{WS^{\lambda}_{\theta\sigma_2}}{\sim} D_{jk}.$$

Proof. Let $B_{jk} \overset{WS^{\lambda}_{\sigma_2}}{\sim} D_{jk}$. Also, suppose that $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$. Then, there exist $\eta, \rho > 0$ such that $q_r \geq \eta + 1, q_u \geq \rho + 1$ for all $r, u > 1$, which implies that

$$\frac{h_{ru}}{j_r k_u} \geq \frac{\eta\rho}{(\eta+1)(\rho+1)}.$$

Thus, for every $\varepsilon > 0$ and each $y \in Y$ we have

$$\begin{aligned}
&\frac{1}{j_r k_u}\left|\left\{(j,k) : j \leq j_r, k \leq k_u, \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \geq \varepsilon\right\}\right| \\
&\qquad \geq \frac{1}{j_r k_u}\left|\left\{(j,k) \in I_{ru} : \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \geq \varepsilon\right\}\right| \\
&\qquad = \frac{h_{ru}}{j_r k_u}\frac{1}{h_{ru}}\left|\left\{(j,k) \in I_{ru} : \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \geq \varepsilon\right\}\right| \\
&\qquad \geq \frac{\eta\rho}{(\eta+1)(\rho+1)}\frac{1}{h_{ru}}\left|\left\{(j,k) \in I_{ru} : \left|\mu_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

for all s, t, which gives the result. □

Theorem 4. *If* $\limsup_r q_r < \infty$ *and* $\limsup_u q_u < \infty$ *for any* $\theta_2 = \{(j_r, k_u)\}$, *then*

$$B_{jk} \overset{WS^{\lambda}_{\theta\sigma_2}}{\sim} D_{jk} \quad \textit{implies} \quad B_{jk} \overset{WS^{\lambda}_{\sigma_2}}{\sim} D_{jk}.$$

Proof. Let $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$. Then, there exist $\alpha, \beta > 0$ such that $q_r < \alpha, q_u < \beta$ for all $r, u > 1$. Also, suppose that $B_{jk} \overset{WS^{\lambda}_{\theta\sigma_2}}{\sim} D_{jk}$ and

$\delta > 0$. Then, there exist $n_0, m_0 \in \mathbb{N}$ such that for every $\varepsilon > 0$, each $y \in Y$ and all $j \geq n_0, k \geq m_0$

$$\mathcal{S}_{jk} := \frac{1}{h_{jk}} \left| \left\{ (j,k) \in I_{jk} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| < \delta$$

for all s, t. We can also find an $M > 0$ such that $\mathcal{S}_{jk} < M$ for all $j, k = 1, 2, \ldots$.

Now, let n and m be any integers satisfying $j_{r-1} < n \leq j_r, k_{u-1} < m \leq k_u$ where $r > n_0, u > m_0$. Then, for every $y \in Y$ we have

$$\begin{aligned}
&\frac{1}{nm} \left| \left\{ (j,k) : j \leq n, k \leq m, \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&\leq \frac{1}{j_{r-1}k_{u-1}} \left| \left\{ (j,k) : j \leq j_r, k \leq k_u, \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&= \frac{1}{j_{r-1}k_{u-1}} \left| \left\{ (j,k) \in I_{11} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&\quad + \frac{1}{j_{r-1}k_{u-1}} \left| \left\{ (j,k) \in I_{12} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&\quad + \frac{1}{j_{r-1}k_{u-1}} \left| \left\{ (j,k) \in I_{21} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&\quad + \frac{1}{j_{r-1}k_{u-1}} \left| \left\{ (j,k) \in I_{22} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&\quad \vdots \\
&\quad + \frac{1}{j_{r-1}k_{u-1}} \left| \left\{ (j,k) \in I_{ru} : \left| \mu_y \left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| \\
&= \frac{j_1 k_1}{j_{r-1}k_{u-1}} \mathcal{S}_{11} + \frac{j_1(k_2 - k_1)}{j_{r-1}k_{u-1}} \mathcal{S}_{12} + \frac{(j_2 - j_1)k_1}{j_{r-1}k_{u-1}} \mathcal{S}_{21} + \frac{(j_2 - j_1)(k_2 - k_1)}{j_{r-1}k_{u-1}} \mathcal{S}_{22} \\
&\quad \vdots \\
&\quad + \frac{(j_{n_0} - j_{n_0 - 1})(k_{m_0} - k_{m_0 - 1})}{j_{r-1}k_{u-1}} \mathcal{S}_{n_0 m_0} \\
&\quad \vdots
\end{aligned}$$

$$+\frac{(j_r-j_{r-1})(k_u-k_{u-1})}{j_{r-1}k_{u-1}}\mathcal{S}_{ru}$$

$$\leq \left\{\sup_{1\leq j,\,1\leq k}\mathcal{S}_{jk}\right\}\frac{j_{n_0}k_{m_0}}{j_{r-1}k_{u-1}}+\left\{\sup_{j\geq n_0,\,k\geq m_0}\mathcal{S}_{jk}\right\}\frac{(j_r-j_{n_0})(k_u-k_{m_0})}{j_{r-1}k_{u-1}}$$

$$\leq M\frac{j_{n_0}k_{m_0}}{j_{r-1}k_{u-1}}+\delta\,\alpha\,\beta$$

for all s,t, which gives the result. □

Theorem 5. *If*

$$1<\liminf_r q_r\leq\limsup_r q_r<\infty \quad \textit{and} \quad 1<\liminf_u q_u\leq\limsup_u q_u<\infty$$

for any $\theta_2=\{(j_r,k_u)\}$, *then*

$$B_{jk}\overset{WS^{\lambda}_{\theta\sigma_2}}{\sim}D_{jk} \quad \textit{if and only if} \quad B_{jk}\overset{WS^{\lambda}_{\sigma_2}}{\sim}D_{jk}.$$

Proof. The proof is obvious from Theorem 3 and Theorem 4. □

With techniques similar to that of Theorem 3, Theorem 4 and Theorem 5, the following theorems can be respectively proved.

Theorem 6. *If* $\liminf_r q_r>1$ *and* $\liminf_u q_u>1$ *for any* $\theta_2=\{(j_r,k_u)\}$, *then*

$$B_{jk}\overset{[W^{\lambda}_{\sigma_2}]}{\sim}D_{jk} \quad \textit{implies} \quad B_{jk}\overset{[W^{\lambda}_{\theta\sigma_2}]}{\sim}D_{jk}.$$

Theorem 7. *If* $\limsup_r q_r<\infty$ *and* $\limsup_u q_u<\infty$ *for any* $\theta_2=\{(j_r,k_u)\}$, *then*

$$B_{jk}\overset{[W^{\lambda}_{\theta\sigma_2}]}{\sim}D_{jk} \quad \textit{implies} \quad B_{jk}\overset{[W^{\lambda}_{\sigma_2}]}{\sim}D_{jk}.$$

Theorem 8. *If*

$$1<\liminf_r q_r\leq\limsup_r q_r<\infty \quad \textit{and} \quad 1<\liminf_u q_u\leq\limsup_u q_u<\infty$$

for any $\theta_2=\{(j_r,k_u)\}$, *then*

$$B_{jk}\overset{[W^{\lambda}_{\theta\sigma_2}]}{\sim}D_{jk} \quad \textit{if and only if} \quad B_{jk}\overset{[W^{\lambda}_{\sigma_2}]}{\sim}D_{jk}.$$

4. Conclusion

When $(\sigma(s),\sigma(t))=(s+1,t+1)$, from Definitions 11-16 we get the definitions of asymptotical almost equivalence, asymptotical strong almost equivalence, asymptotical almost statistical equivalence, asymptotical lacunary almost equivalence, asymptotical lacunary strong almost equivalence and asymptotical lacunary almost statistical equivalence in the Wijsman sense for double set sequences. So, the analogues of Theorem 1-8 can also be obtained between these definitions, which have not been appeared anywhere by this time.

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