Hyperbolic Raisa Orbits of the Second Order in an Extended Hyperbolic Plane

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(Dedicated to the memory of **Prof. Dr. Aurel BEJANCU (1946 - 2020)**)

ABSTRACT

In this paper, we study conics, which are invariant under the hyperbolic inversion with respect to the absolute of an extended hyperbolic plane H^2 of curvature radius ρ, $\rho \in \mathbb{R}_+$. They are called the hyperbolic Raisa Orbits of the second order. We prove that each hyperbolic Raisa Orbits of the second order in H^2 belongs to one of four conics types of this plane. These types are as follows: the bihyperbolas of one sheet; the hyperbolas; the hyperbolic parabolas of one sheet and two branches; the elliptic cycles of radius $\pi\rho/4$. The family of all hyperbolic Raisa Orbits from the family of all bihyperbolas of one sheet (or all hyperbolas) defined exactly up to motions, is one-parametric. The family of all hyperbolic Raisa Orbits from the family of all hyperbolic parabolas of one sheet and two branches (or all elliptic cycles) contains a unique conic defined exactly up to motions.

1. Introduction

1.1. The extended hyperbolic plane H^2

In the Cayley – Klein projective interpretation (see, for instance, [4, 10, 9]), an extended hyperbolic plane H^2 is the projective plane P_2 with an infinitely removed oval curve γ, called the *absolute* of the plane H^2. According to Staudt's definition, the oval curve is also named a *non-degenerate conic* or simply a *conic* (see, for instance, [1]). The plane H^2 contains two connected components adjacent to the absolute. A complete Lobachevskii plane Λ_2 is realized on the interior domain of the plane P_2 with respect to the conic γ. A hyperbolic plane $\widehat{H}$ of positive curvature is realized on the exterior domain of P_2 with respect to γ [13, 14]. The group G of projective automorphisms of the oval curve γ is the *fundamental group* of transformations for H^2, Λ_2, and $\widehat{H}$.

The lines of the plane H^2 are as follows:
(1) the elliptic lines, crossing the absolute at a pair of imaginary conjugate points;
(2) the hyperbolic lines, intersecting the absolute at two real points;
(3) the parabolic lines that touch the absolute.

1.2. The conics types of the extended hyperbolic plane H^2

Each conic of the plane H^2, different from the absolute, belongs to one of twenty types. Each proper conic of the Lobachevskii plane Λ_2 belongs to one of five types [9, p. 181], and each proper conic of the plane $\widehat{H}$ belongs to one of eight types [14, 11]. Notice, the refined conics classification in the plane Λ_2 was provided in [9]. One of the conics types, namely the hyperbolic parabolas of one branches in Λ_2, was skipped in [5], [4, p. 229].

The classification of conics in the planes Λ_2 and $\widehat{H}$ (see [4, 10, 14, 11, 5]) is based on the classification of collineations, which are the product of polar transformations of two conics. In this classification, the geometric covariants of two conics of the projective plane P_2 are considered, namely the common points and the common

tangents of a conic and the absolute. These geometric covariants are called the *basic covariants* of a conic [11]. Each a basic covariant can be real or imaginary. In the conics classification, the number of basic covariants and their nature are taken into account. Other ways for the conics classification are known. For example, focal-director properties of conics were used in the classification of conics of the plane Λ_2 [7, 8]. Interestingly, the relative position of two projective conics is also investigated regardless of hyperbolic geometry, without taking into account the classic results from the works [4, 10, 5]. For example, incomplete lists of the possible relative positions of conics are presented in [6, 3]. Such investigations, despite their omissions, confirm the timeliness of the study of the conics in the plane H^2. In the given paper, we adhere to the conics classification from [10, 14, 11] and use terms from the works [14, 11]. All types of conics of the plane H^2 are shown in Figure 1.
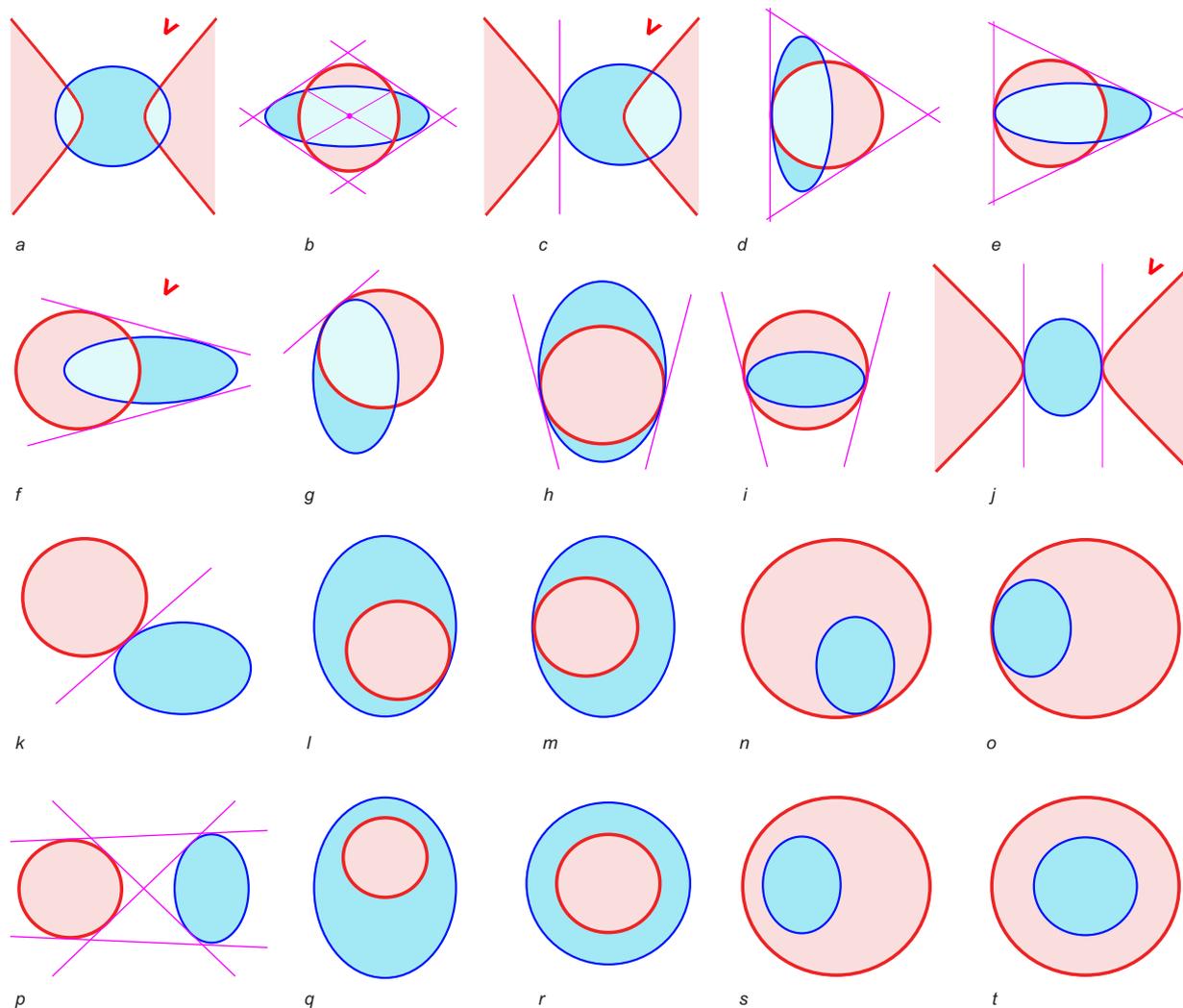
Figure 1. The conics of the plane H^2: a bihyperbola of one sheet in $\widehat{H}$ (*a*); a bihyperbola of two sheets in $\widehat{H}$ (*b*); a hyperbolic parabola of one sheet and two branches in $\widehat{H}$ (*c*); a hyperbolic parabola of two sheets and two branches in $\widehat{H}$ (*d*); a hyperbolic parabola of one sheet and one branch in $\widehat{H}$ (*e*); a hyperbola (*f*); a parabola (*g*); a hyperbolic cycle of $\widehat{H}$ (*h*); an equidistance of Λ_2 (*i*); an elliptic cycle of $\widehat{H}$ (*j*); a convex elliptic parabole (*k*); a nonconvex elliptic parabole of $\widehat{H}$ (*l*); a horocycle of $\widehat{H}$ (*m*); an elliptic parabole of Λ_2 (*n*); a horocycle of Λ_2 (*o*); a convex ellipce of $\widehat{H}$ (*p*); a nonconvex ellipce of $\widehat{H}$ (*q*); a hypercycle of $\widehat{H}$ (*r*); an ellipce of Λ_2 (*s*); a circle of Λ_2 (*t*).

On the plane H^2 there are seven types of cycles. These types are as follows: the hyperbolic and elliptic cycles of the plane $\widehat{H}$ (see Figures 1*h* and 1*j*, respectively), the equidistantes of the plane Λ_2 (see Figure 1*i*), the horocycles of the planes $\widehat{H}$ and Λ_2 (see Figures 1*m* and 1*o*, respectively), the hypercycles of the plane $\widehat{H}$ (see Figure 1*r*), the circles of the plane Λ_2 (see Figure 1*t*) (see [4, 9, 14, 11, 15]). These conics are paths of an arbitrary point in the plane H^2 under the transformations of the groop G.

1.3. Problem statement

Inversions with respect to the cycles of the plane H^2 were studied in [16, 18, 21]. Since the absolute of the plane H^2 can be considered as the limit position of a hypercycle, a horocycle, and a hyperbolic cycle of the plane $\widehat{H}$, as well as of a circle, an equidistant and a horocycle of the plane Λ_2, we enter three types of inversions with respect to the absolute of H^2 [19, 20]. An inversion with respect to the absolute of H^2 we call *elliptic*, *hyperbolic* or *parabolic* if its center is respectively an inner, outer or eigen point with respect to the absolute.

According to the definitions of inverse points, each line containing the inversion center, is invariant under this inversion. We study higher order curves, which are invariant under inversions with respect to the absolute. Such curves we called the Raisa Orbits of the respective order (or R-orbits, for short). Depending on the type of the inversion we distinguish *elliptic*, *hyperbolic* and *parabolic* R-orbits. Elliptic second-order Raisa Orbits were studied in [20]. In this work we present the hyperbolic second-order R-orbits of the plane H^2. We find canonical equations of these conics and prove that a hyperbolic R-orbit of the second order in the plane H^2 of curvature radius ρ, $\rho \in \mathbb{R}_+$, can be a conic of one of four following types:

- the bihyperbolas of one sheet in $\widehat{H}$;
- the hyperbolas;
- the hyperbolic parabolas of one sheet and two branches in $\widehat{H}$;
- the elliptic cycles of radius $\pi\rho/4$.

In addition, we define the range of each family of all R-orbits of the same type.

1.4. Main notions

We study the objects of the plane H^2 in canonical frames of two types. The equations of conics in the frames of the first type allow us to obtain the most informative images by means the interactive mathematics software GeoGebra. The frames of the second type are the most convenient in analytical proofs. Here we briefly provide main notions (see [13, p. 87]). The main definitions and facts on conics in the projective plane are described, for example, in books [1, 2].

A *canonical frame of the first type* of the plane H^2 is a projective frame $R^* = \{A_1, A_2, A_3, E\}$ whose vertices A_1, A_2, A_3 form an autopolar trilateral of the first order with respect to the absolute γ, and the unit point E lies on the tangents to the curve γ drawn from the vertices A_1 and A_2.

A *canonical frame of the second type* of the plane H^2 is a projective frame $R^* = \{A_1, A_2, A_3, E\}$ whose vertices A_1, A_2, A_3 form an autopolar trilateral of the second order with respect to the absolute conic γ, and points A_1, A_2, and E lie on γ.

The absolute curve γ in any canonical frame R^* of the first type (or in any canonical frame R of the second type) of the plane H^2 is given by the equation

$$x_1^2 + x_2^2 - x_3^2 = 0 \qquad \left(x_1 x_2 - x_3^2 = 0\right). \tag{1.1}$$

The tangential equation of the absolute γ in any canonical frame R of the second type has the form

$$4X_1 X_2 - X_3^2 = 0. \tag{1.2}$$

Denote the quadric form $x_1 x_2 - x_3^2$ from the second equation in (1.1) by $\varphi(x_1, x_2, x_3)$. In the frame R, the real coordinates $(a_1 : a_2 : a_3)$ of an exterior or interior point with respect to the absolute satisfy respectively the inequality

$$\varphi(a_1, a_2, a_3) < 0 \qquad \text{or} \qquad \varphi(a_1, a_2, a_3) > 0. \tag{1.3}$$

Any point of the plane $\widehat{H}$ defines four components in the plane H^2. Let us determine them by analogy with components of the plane $\widehat{H}$ (see, for instance, [13, p. 104], [14, p. 11], [12, p. 9]).

Let X be an arbitrary point of the plane $\widehat{H}$, and let l_1, l_2 be the parabolic lines containing the point X.

By the *valiana* of a point X in H^2 we mean the set of all interior points of the angle between lines l_1, l_2 that does not contain the absolute γ. The polar line l of the point X with respect to γ divides the valiana of X into two connected components. We call them the *semivalianas* of the point X.

The complement of the valiana of a point X in H^2 is called the *covaliana* of X. The covaliana of a point X on H^2 consists of two components bounded by the lines l_1, l_2, and l. We call each of these components the *semicovaliana* of a point X.

2. Hyperbolic inversion in the plane H^2

Let $\omega(S; h)$ be a cycle with centre S in the plane H^2, and let M be an arbitrary point in the plane H^2. Denote the polar line of the point M with respect to the cycle ω by p_M, and the common point of lines SM and p_M by M'.

The points M and M' are called *inverse points* with respect to the cycle ω.

The transformation of the plane H^2 under which each point M passes to its inverse point M' with respect to a cycle ω, is called the inversion with respect to this cycle and is denoted by I. The cycle ω is called *the base cycle* (or *base*), the point S is the *centre* of inversion I (see [16, 18, 21]).

When the base cycle ω is not an elliptic cycle of the plane $\widehat{H}$, the absolute γ can be considered as a limit position of ω. When ω is a hyperbolic cycle of the plane $\widehat{H}$, that is, S is an inner point of the plane $\widehat{H}$, the inversion I is called a *hyperbolic inversion with respect to the absolute*, or a *hyperbolic inversion* for short.

Let us find the analytical expression for the hyperbolic inversion I with centre S.

To this end we consider a canonical frame $R = \{A_1, A_2, S, E\}$ of the second types. For an arbitrary point M of the plane H^2 with coordinates $(m_1 : m_2 : m_3)$ the line SM has the coordinates $(-m_2 : m_1 : 0)$ in R. According to the second equation from (1.1) the polar line p_M of the point M with respect to the absolute can be set by the coordinates $(m_2 : m_1 : -2m_3)$. Consequently, the point M', where $M' = I(M) = SM \cap p_M$, can be given in R by the coordinates $(m_1 m_3 : m_2 m_3 : m_1 m_2)$. Thus, for the inversion I with centre S the following analytical expressions in the frame R hold:

$$\lambda x_1' = x_1 x_3, \ \lambda x_2' = x_2 x_3, \ \lambda x_3' = x_1 x_2, \ \lambda \in \mathbb{R}. \tag{2.1}$$

We formulate and prove several almost obvious assertions in the following theorem.

Theorem 2.1. *The hyperbolic inversion I of the plane H^2 with centre S possesses the following properties.*

1. *Inversion I is not a bijective transformation of the plane H^2.*

2. *Under I a line of the plane H^2 corresponds to itself if and only if it passes through the point S.*

3. *Let M, M' be inverse points under I, and let l be the polar line of the point S with respect to the absolute γ. Then the distance from M to S is equal to the distance from M' to l.*

4. *If an arbitrary point M of the plane H^2 belongs to the covaliana of the point S, then the inverse points M, M' under I belong to the same semicovaliana of S and to different components Λ_2, $\widehat{H}$ of the plane H^2.*

5. *If an arbitrary point M of the plane H^2 belongs to the valiana of the point S, then the inverse points M, M' under I belong to different semivalianas of S.*

Proof. 1. Accoding to the definition of the hyperbolic inversion each point of a parabolic line containing the point S, corresponds under I to the common point of this line and the absolute γ. Moreover, each point of the polar line of S with respect to γ corresponds under I to S. This violates the condition of injectiveness of I and thereby proves the first assertion of the theorem.

2. By definition, inverse points under I are collinear with the centre S of I. From this follows the second assertion of the theorem.

3. By the condition of the third theorem assertion, the point S is the pole of the line l with resprct to the absolute. Hence, the line MS is orthogonal to the line l. Let $M_0 = MS \cap l$. Then the distance from M' to l is equal to $|M'M_0|$. Let us prove that $|MS| = |M'M_0|$.

Denote the common points of the line MM' and the absolute γ by H_1, H_2. These points can be real or imaginary conjugate. When the line MS is parabolic, these points coincide. By construct of inverse points, the point M' lies on the polar line of M with respect to γ. Thus, for the cross-ratios $(SM_0H_1H_2)$, $(MM'H_1H_2)$ of the quadruples of collinear points the following equalities hold

$$(SM_0H_1H_2) = -1, \qquad (MM'H_1H_2) = -1. \tag{2.2}$$

Let us choose a frame $R_0 = \{H_1, H_2, S\}$ on the line MS. Accoding to the first equality from (2.2) the point M_0 has the coordinates $(1 : -1)$ in R_0. Denote coordinates of the points M, M' in this frame by $(a : 1)$,

$(x : 1)$, where $a, x \in \mathbb{R}$. Using the second equality from (2.2), we find that $x = -a$. Expressing the numbers (MSH_1H_2), $(M'M_0H_1H_2)$ via coordinates of points in the frame R_0, we obtain

$$(MSH_1H_2) = \frac{\begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} a & 1 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{1}{a}, \qquad (M'M_0H_1H_2) = \frac{\begin{vmatrix} -a & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} -a & 1 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}} = \frac{1}{a}.$$

Thus, the equality $(MSH_1H_2) = (M'M_0H_1H_2)$ holds. Based on this equality and the definition of the length of an elliptic or hyperbolic segment (see, for instance, [13, pp. 118, 122]), we provide support for the equality $|MS| = |M'M_0|$, from which the third assertion of the theorem follows.

4. To prove the following assertions, we use the canonical frame $R = \{A_1, A_2, S, E\}$ of the second type. By definition, the points A_1, A_2, E from R lie in γ, the point S is the pole of the line A_1A_2 with respect γ.

   Let M be an arbitrary point from the covaliana of the point S, and let $M_0 = SM \cap l$, where l is the polar line of S with respect to γ. Without losing of generality in reasonings we place the unit point E of R on the line MM' in the same semicovaliana of S that the point M. After that, the points M, M_0 can be given in the frame R by the coordinates $(1 : 1 : m)$, $m \in \mathbb{R}$, and $(1 : 1 : 0)$, respectively. According to formulae (2.1) the point M' has the coordinates $(m : m : 1)$. Expressing the cross-ratio $(MM'M_0S)$ of the quadruple of collinear points via their coordinates in the frame R, we find

$$(MM'M_0S) = \frac{\begin{vmatrix} 1 & m \\ 1 & 0 \end{vmatrix} \begin{vmatrix} m & 1 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & m \\ 0 & 1 \end{vmatrix} \begin{vmatrix} m & 1 \\ 1 & 0 \end{vmatrix}} = m^2.$$

Since $(MM'M_0S) = m^2 > 0$ for any m, the points M, M' are not divided by the points M_0, S. Consequently, M and M' lie in the same semicovaliana of the point S.

The cross-ratio $(MM'ES)$ has the following expression in coordinates

$$(MM'ES) = \frac{\begin{vmatrix} 1 & m \\ 1 & 1 \end{vmatrix} \begin{vmatrix} m & 1 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & m \\ 0 & 1 \end{vmatrix} \begin{vmatrix} m & 1 \\ 1 & 1 \end{vmatrix}} = -m. \tag{2.3}$$

Since the points M, E lie in the same semicovaliana of S, the inequality $(MEM_0S) > 0$ holds. Writing down this inequality in coordinates, we obtain

$$(MEM_0S) = \frac{\begin{vmatrix} 1 & m \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & m \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}} = m > 0. \tag{2.4}$$

By the conditions (2.3), (2.4), the inequality $(MM'ES) < 0$ holds. Consequently, the points M, M' are divided by the points E, S, that is, the points M, M' belong to different components Λ_2, $\widehat{H}$ of the plane H^2. So, the fourth assertion of the theorem is proved.

5. Now we assume that an arbitrary point M lies in the valiana of the point S. Keeping the previous notations, we place the unit point E of the frame R on the line, containing the point S and orthogonal to the line SM. The points M, M_0 have the coordinates $(1 : -1 : m)$, $m \in \mathbb{R}$, and $(1 : -1 : 0)$, respectively. By formulae (2.1), the point M' has the coordinates $(-m : m : 1)$. Hence, for the cross-ratio $(MM'M_0S)$ we obtain

$$(MM'M_0S) = \frac{\begin{vmatrix} 1 & m \\ 1 & 0 \end{vmatrix} \begin{vmatrix} -m & 1 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & m \\ 0 & 1 \end{vmatrix} \begin{vmatrix} -m & 1 \\ 1 & 0 \end{vmatrix}} = -m^2.$$

Since $(MM'M_0S) = -m^2 < 0$ for any m, the points M, M' are divided by the points M_0, S. Consequently, the points M, M' belong to different semivaliana of the point S. This completes the theorem proof.

□

3. Main Theorem on hyperbolic Raisa Orbits of the second order

Theorem 3.1. *The family Θ of all hyperbolic Raisa Orbits of the second order in the plane H^2 of curvature radius ρ, $\rho \in \mathbb{R}_+$, possesses the following properties.*

1. Each conic from Θ belongs to one of four conics types of the plane H^2. These types are as follows: the bihyperbolas of one sheet in $\widehat{H}$; the hyperbolas; the hyperbolic parabolas of one sheet and two branches in $\widehat{H}$; the elliptic cycles of radius $\pi\rho/4$.

2. The family of all bihyperbolas of one sheet in $\widehat{H}$ (or all hyperbolas) defined exactly up to motions and belonging to the family Θ, is one-parametric. Each conic from this family can be set in some canonical frame of the second type by the equation

$$x_3^2 + x_1 x_2 + a x_1 x_3 - (a+2) x_2 x_3 = 0, \ a \in \mathbb{R}, \ a \neq -1 \quad \left(x_3^2 + x_1 x_2 + a(x_1 x_3 + x_2 x_3) = 0, \ a \in \mathbb{R}, \ |a| > 1 \right). \quad (3.1)$$

All hyperbolic R-orbits from the hyperbolas family belong to the same class of hyperbolas.

3. The family Θ contains a unique hyperbolic parabola of one sheet and two branches in $\widehat{H}$ and a unique elliptic cycle, which are defined exactly up to motions. A hyperbolic parabola of one sheet and two branches in $\widehat{H}$ (or elliptic cycle) from Θ can be set in some canonical frame of the second type by the equation

$$x_3^2 + x_1 x_2 - 2 x_1 x_3 = 0, \quad \left(x_1 x_2 + x_3^2 = 0 \right). \quad (3.2)$$

Proof. Let non-degenerate conic σ of the plane H^2 be invariant under hyperbolic inversion I with centre S. To find all posible types of σ as an oval curve of the plane H^2, we choose a canonical frame $R = \{A_1, A_2, S, E\}$ of the second type. The family U of all such frames depends on one parameter. Indeed, the point S is fixed. Hence, its polar line $A_1 A_2$ with respect to the absolute γ is fixed too. Moreover, by the frame definition, $A_1 \in \gamma$, $A_2 \in \gamma$. Thus, the vertices A_1 and A_2 of the frame R are defined with exactly up the points order. The unit point E can move along γ, that is, its position depends on one parameter. This parameter defines a frame in the family U.

If S lies in σ, then each line containing the point S, except for the tangent of the conic σ, has another point from σ. Assume that a line t contains S and $t \cap \sigma = \{S, V\}$, where $V \neq S$. The point V', where $V' = I(V)$, lies in both the line t and the conic σ. Hence t contains three points from the conic σ. This is not possible because the conic σ is non-degenerate. Thus, σ does not contain the point $S(0:0:1)$. In that case, the conic σ can be given in the frame R by the equation

$$\sum_{i,j=1}^{3} a_{ij} x_i x_i = 0, \ a_{ij} \in \mathbb{R}, \ a_{33} = 1. \quad (3.3)$$

Using the analytical expression (2.1) of the inversion I, we get the equation of a curve σ', containing image of the conic σ under I

$$a_{11} x_1^2 x_3^2 + a_{22} x_2^2 x_3^2 + x_1^2 x_2^2 + 2 a_{12} x_1 x_2 x_3^2 + 2 a_{13} x_1^2 x_2 x_3 + 2 a_{23} x_1 x_2^2 x_3 = 0. \quad (3.4)$$

Since σ is invariant under I, the curve σ' breaks down into a pair of conics, one of which is σ. Let $\sigma' = \sigma \cup \beta$, where the conic β is given in R by the equation

$$\sum_{i,j=1}^{3} b_{ij} x_i x_i = 0, \ b_{ij} \in \mathbb{R}. \quad (3.5)$$

Comparing the product of quadric forms from Eqs. (3.3), (3.5) with the quadric form from Eq. (3.4) under the condition $a_{33} = 1$, we obtain the following conditions for coefficients a_{ij} and b_{ij}:

$$a_{11} = a_{22} = 0, \ 2a_{12} = 1, \ b_{11} = b_{22} = b_{33} = b_{13} = b_{23} = 0, \ b_{12} \neq 0.$$

Under these conditions the conic β has the equation $x_1 x_2 = 0$ and is the pair of parabolic lines containing S. The conic σ equation can be written in the form

$$x_3^2 + x_1 x_2 + a x_1 x_3 + b x_2 x_3 = 0, \quad a, b \in \mathbb{R}, \quad ab \neq 1. \quad (3.6)$$

Denote the quadric form $x_3^2 + x_1x_2 + ax_1x_3 + bx_2x_3$ from Eq. (3.6) by $\varphi_0(x_1, x_2, x_3)$. The determinant of a coordinate matrix of the form $\varphi_0(x_1, x_2, x_3)$ is equal to $(ab - 1)/4$. Hence the condition $ab \neq 1$ guarantees the nondegeneracy of the conic σ.

We investigate the hyperbolic R-orbit σ, considering all fundamentally different possibilities for coefficients a, b from Eq. (3.6).

1. At first, we assume that the conic σ is given by Eq. (3.6), where $ab \neq 0$. Under this condition from Eqs. (1.1), (3.6) we obtain coordinates of common points of the R-orbit σ and the absolute γ in R:

$$A_1(1:0:0), \ Q_1\left(b\left(1 + \sqrt{1 - ab}\right) : a\left(1 - \sqrt{1 - ab}\right) : -ab\right),$$

$$A_2(0:1:0), \ Q_2\left(b\left(1 - \sqrt{1 - ab}\right) : a\left(1 + \sqrt{1 - ab}\right) : -ab\right).$$

Note that the points A_1, A_2 are real, and under the conditions $ab \neq 1$, $ab \neq 0$ there are no matching points in the set A_1, A_2, Q_1, Q_2. According to the nature of the points Q_1, Q_2 we distinguish the following cases.

(a) Assume that $ab < 1$. In this case the points Q_1, Q_2 are real. Consequently, σ has four real points from the absolute. By the classification of oval curves of the plane $\widehat{H}$ (see [14, 11]), the conic σ can be a bihyperbola of one sheet or a bihyperbola of two sheets in $\widehat{H}$ (see Figures 1a and 1b, respectively). Among the diagonal points of a full quadrilateral with vertices at the common points of the absolute and a bihyperbola of two sheets, there is a point which is interior both with respect to the absolute and with respect to the bihyperbola. We use this fact to clarify the conic σ type. Let us consider the full quadrilateral $A_1A_2Q_1Q_2$. Let $A_1A_2 \cap Q_1Q_2 = A$, $A_1Q_1 \cap A_2Q_2 = B$, and $A_1Q_2 \cap A_2Q_1 = C$. The diagonal points A, B, C of the full quadrilateral $A_1A_2Q_1Q_2$ have the following coordinates in R:

$$A(b : -a : 0), \quad B\left(b\left(1 - \sqrt{1 - ab}\right) : a\left(1 - \sqrt{1 - ab}\right) : -ab\right),$$

$$C\left(b\left(1 + \sqrt{1 - ab}\right) : a\left(1 + \sqrt{1 - ab}\right) : -ab\right).$$

Let K be a pole of the line A_1A_2 with respect σ. Since the points A_1, A_2 are real, the point K is an exterior point with respect σ. Using coordinates $(b : a : -1)$ of the point K, we find the following inequalities for real coordinates $(a_1 : a_2 : a_3)$ of an arbitrary exterior (interior) point with respect to the conic σ in the frame R under the condition $ab < 1$:

$$\varphi_0(a_1, a_2, a_3) > 0 \quad (\varphi_0(a_1, a_2, a_3) < 0). \tag{3.7}$$

The quadric forms φ and φ_0 on the coordinates of points A, B, C have the following expressions:

$$\varphi(A) = -ab, \quad \varphi_0(A) = -ab,$$

$$\varphi(B) = -2ab\sqrt{1 - ab}\left(1 - \sqrt{1 - ab}\right), \quad \varphi_0(B) = 2ab(1 - ab)\left(1 - \sqrt{1 - ab}\right),$$

$$\varphi(C) = 2ab\sqrt{1 - ab}\left(1 + \sqrt{1 - ab}\right), \quad \varphi_0(C) = 2ab(1 - ab)\left(1 + \sqrt{1 - ab}\right).$$

Using these expressions under the condition $ab < 1$, we proved the following inequalities:

$$\varphi(A)\varphi_0(A) > 0, \quad \varphi(B) < 0, \quad \varphi_0(B) > 0, \quad \varphi(C)\varphi_0(C) > 0. \tag{3.8}$$

By virtue of the inequalities (1.3), (3.7), and (3.8), the following assertions hold:
1) the point A (or C) is interior with respect to the conic σ if and only if it is exterior with respect to the absolute;
2) the point B is exterior with respect to the conic σ and with respect to the absolute.

Thus, none of the points A, B, C lie in the intersection of the interior domains of the conics γ and σ. From this we conclude that under the conditions $ab \neq 0$, $ab < 1$ the conic σ is a bihyperbola of one sheet in the plane $\widehat{H}$.

Note that Eq. (3.6) contains two parameters a, b, while the point E position depends on one parameter. By fixing the point E, for example, at the point Q_1, we obtain the condition $b = -a - 2$, that is, we will reduce the number of parameters in Eq. (3.6) to one. After that, the frame R is determined uniquely, and the conic σ can be set in R by the first equation from (3.1).

(b) Assume that $ab > 1$. In this case the points Q_1, Q_2 are imaginary conjugate. Consequently, the conic σ has two real and two imaginary points from the absolute. By the classification of oval curves of the plane $\widehat{H}$, this means that the conic σ is a hyperbole (see Figure 1f). The class of a hyperbola in the plane $\widehat{H}$ is defined by the position of poles of the line connecting the common real points of the hyperbola and the absolute with respect to the absolute and with respect to the hyperbola. There are nine classes of hyperbolas of the plane $\widehat{H}$ (see [11, p. 44]).

Under the condition $ab > 1$ real coordinates $(a_1 : a_2 : a_3)$ of an arbitrary interior point with respect to the conic σ satisfy the inequality $\varphi_0(a_1, a_2, a_3) > 0$. Hence the point $S(0 : 0 : 1)$ lies in the interior domain with respect σ. By virtue of the inequalities (1.3) and $ab > 1$, the point $K(b : a : -1)$ lies in the interior domain with respect γ. Therefore, all hyperbolic R-orbits from the hyperbolas family belong to the same class of hyperbolas.

Note that Eq. (3.6) contains two parameters a, b under such condition that the point E position depends on one parameter. By fixing the point E, for example, in the line SK, we obtain the condition $a = b$, that is, we will reduce the number of parameters in Eq. (3.6) to one. After that, the frame R is determined uniquely, and the conic σ can be set in R by the second equation from (3.1).

2. Now we assume that the conic σ is given by Eq. (3.6), where $a \neq 0$, $b = 0$. The case, when $a = 0$, $b \neq 0$, gives a similar result because it can be obtained by permuting vertices A_1, A_2 of the frame R.

From Eqs. (1.1), (3.6) we find coordinates of common points of the R-orbit σ and the absolute:

$$A_1(1 : 0 : 0), \quad A_2(0 : 1 : 0), \quad Q\left(4 : a^2 : -2a\right), \, a \neq 0.$$

Since there are no matching points in the set A_1, A_2, Q, the conic σ is a hyperbolic parabola (see [14, p. 29]). Each hyperbolic parabola of the plane H^2 belongs to one of three types (see Figures 1c, 1d, and 1e). We refine the conic σ type, taking into account the number and nature of its common tangents with the absolute.

In the case on consideration the tangential equation of the conic σ can be written in the form

$$a^2 X_2^2 + X_3^2 + 4X_1X_2 - 2aX_2X_3 = 0, \quad a \neq 0. \tag{3.9}$$

From the system of Eqs. (1.2), (3.9) we find common tangents of the conics σ and γ:

$$A_2A_3(1 : 0 : 0), \quad l_1^*\left(a^2 : 8i : 4a(1 + i)\right), \quad l_2^*\left(a^2 : -8i : 4a(1 - i)\right).$$

Two of three common tangents of the conics σ and γ are imaginary. Hence the conic σ is a hyperbolic parabola of one sheet and two branches in $\widehat{H}$.

By fixing the point E at the point Q, we obtain the condition $a = -2$, that is, we fix the last free parameter a in Eq. (3.6) under $b = 0$. In this step, the frame R is fixed, and the conic σ can be set in R by the first equation from (3.2).

3. Finally, we assume that the conic σ is given by Eq. (3.6), where $a = b = 0$. In this case, the conic σ is not depend on the point E position and can be set by the second equation from (3.2). It is an elliptic cycle of the plane H^2 with centre S and radius $\pi\rho/4$ (see [14, p. 39]).

We considered all possible cases and uncovered four types of hyperbolic second-order R-orbits in the plane H^2. This yields the first theorem assertion. In addition, for each conic under study we presented the canonical frame in which the conic can be given by one of the equations from (3.1), (3.2). In each case, having regard to the number of free parameters of the equations from (3.1), (3.2), we get the second and third assertions of the theorem. Thus, the theorem is proved. $\square$

4. Visualization of stadied objects

In order to display the studied conics, we represent them in the plane E_2 (see, for instance, [4, 17, 22]).

Let the coordinate line $A_1 A_2$ of the frame R be the absolute line of the plane E_2. The absolute circle-points J_1 and J_2 of E_2 we give by coordinates $(\pm i : 1 : 0)$. Under this agreement the following relations between the Cartesian coordinates (x, y) and the projective coordinates $(x_1 : x_2 : x_3)$ of the same point hold

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3}. \tag{4.1}$$

Using the relations (4.1), we move from Eqs. (1.1), (3.1), and (3.2) to the following equations of the absolute γ and hyperbolic R-orbits in the Cartesian coordinates (x, y) corresponding to the frame R:

$$\gamma : \ xy - 1 = 0, \tag{4.2}$$

$$xy + ax - (a + 2)y + 1 = 0, \ a \in \mathbb{R}, \ a \neq -1, \tag{4.3}$$

$$xy + a(x + y) + 1 = 0, \ a \in \mathbb{R}, \ |a| > 1, \tag{4.4}$$

$$xy - 2x + 1 = 0, \tag{4.5}$$

$$xy + 1 = 0. \tag{4.6}$$

We construct images of the absolute and hyperbolic second-order R-orbits in the plane E_2 through the equations (4.2), (4.3), (4.4), (4.5), (4.6) in the Cartesian system corresponding to the frame R. The obtained conics are shown in Figures 1a, 2a, 3a, and 4a. The interior domains with respect to the conics γ and σ are highlighted.
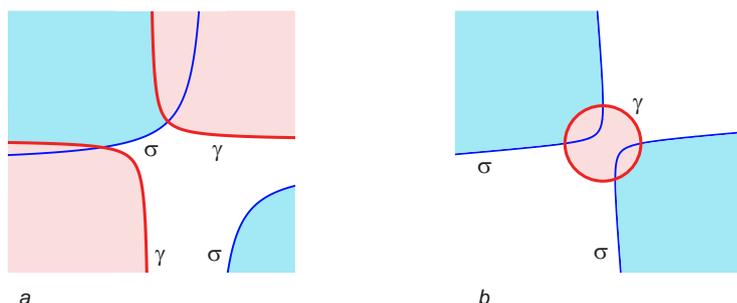
Figure 2. Images of the absolute γ of the plane H^2 and the hyperbolic second-order R-orbit σ with Eqs. (4.3) (a), (4.14) (b), where $a = 5$, in E_2. The conic σ is a bihyperbola of one sheet.
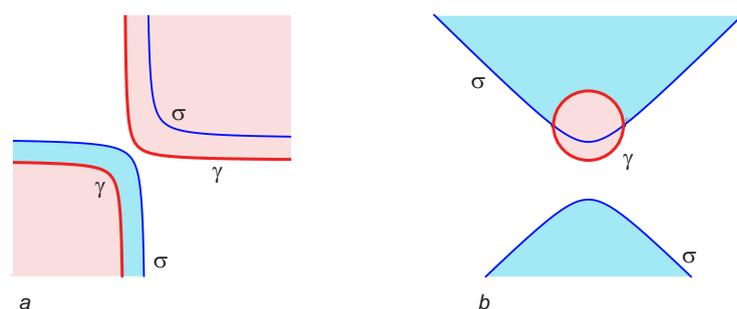
Figure 3. Images of the absolute γ of the plane H^2 and the hyperbolic second-order R-orbit σ with Eqs. (4.4) (a), (4.15) (b), where $a = 1.3$, in E_2. The conic σ is a hyperbola.

To get the most informative images of the studied objects on the Euclidean plane E_2, we will go to the canonical frame R^* of the first type. Let us obtain the formulae for transforming the coordinates of an arbitrary point in the plane H^2 which correspond to this transition.

Assume that the canonical frame $R^* = \{A_1^*, A_2^*, A_3^*, E^*\}$ of the first type associated with the canonical frame $R = \{A_1, A_2, A_3, E\}$ of the second type by the following conditions:

$$A_1^* = E_{21} = A_1 - A_2, \quad A_2^* = A_3, \quad A_3^* = E_{12} = A_1 + A_2, \quad E^* = A_2 A_3 \cap E E_{21}.$$
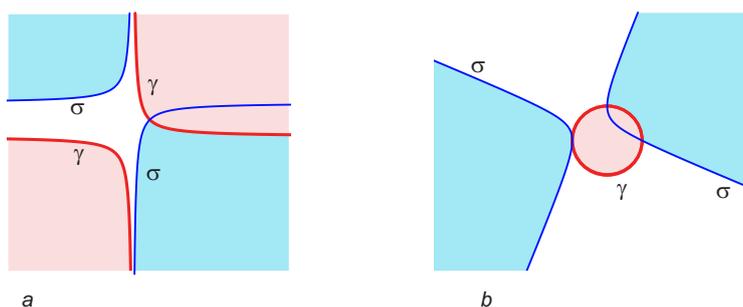
Figure 4. Images of the absolute γ of the plane H^2 and the hyperbolic second-order R-orbit σ with Eqs. (4.5) (*a*), (4.16) (*b*) in E_2. The conic σ is a hyperbolic parabola of one sheet and two branches.
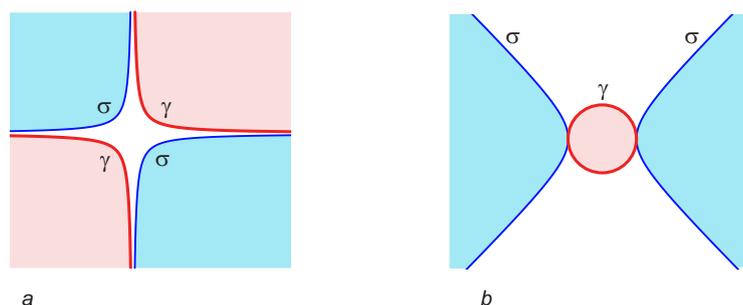
Figure 5. Images of the absolute γ of the plane H^2 and the hyperbolic second-order R-orbit σ with Eqs. (4.6) (*a*), (4.17) (*b*) in E_2. The conic σ is an elliptic cycle.

Under these conditions the vertices and unit point of the frame R^* have the following coordinates in the frame R:

$$A_1^*(1:-1:0), \quad A_2^*(0:0:1), \quad A_3^*(1:1:0), \quad E^*(2:0:1).$$

Consequently, the compatible matrix of the transition from R to R^* can be written in the form:

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \tag{4.7}$$

Denote the coordinates of an arbitrary point of the plane H^2 in the frame R by $(x_1:x_2:x_3)$ and denote the coordinates of the same point in the frame R^* by $(x_1^*:x_2^*:x_3^*)$. Using the matrix (4.7), we get the desired formulae of the coordinates transformation:

$$\lambda x_1 = x_1^* + x_3^*, \quad \lambda x_2 = -x_1^* + x_3^*, \quad \lambda x_3 = x_2^*, \quad \lambda \in \mathbb{R}. \tag{4.8}$$

Rewriting the equations of investigated conics from (3.1), (3.2) using the formulae (4.8), we obtain the equations of these conics in the frame R^*:

$$-x_1^2 + x_2^2 + x_3^2 + 2(a+1)x_1x_2 - 2x_2x_3 = 0, \ a \in \mathbb{R}, \ a \neq -1, \tag{4.9}$$

$$-x_1^2 + x_2^2 + x_3^2 + 2ax_2x_3 = 0, \ a \in \mathbb{R}, \ |a| > 1, \tag{4.10}$$

$$-x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3 = 0, \tag{4.11}$$

$$-x_1^2 + x_2^2 + x_3^2 = 0. \tag{4.12}$$

Using the relations (4.1), we move from Eqs. (1.1), (4.9)–(4.12) to the following equations of the absolute γ and hyperbolic R-orbits in the Cartesian coordinates (x, y) corresponding to the frame R^*:

$$\gamma: \ x^2 + y^2 = 1. \tag{4.13}$$

$$x^2 - y^2 - 2(a+1)xy + 2y = 1, \ a \in \mathbb{R}, \ a \neq -1, \tag{4.14}$$

$$x^2 - y^2 - 2ay = 1, \ a \in \mathbb{R}, \ |a| > 1, \tag{4.15}$$

$$x^2 - y^2 + 2xy + 2y = 1, \tag{4.16}$$

$$x^2 - y^2 = 1. \tag{4.17}$$

By applying the equations (4.13), (4.14), (4.15), (4.16), (4.17), we construct images of the absolute and hyperbolic second-order R-orbits in the Cartesian system corresponding to the frame R^*. The obtained conics are shown in Figures 1*b*, 2*b*, 3*b*, and 4*b*.

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