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NONOSCILLATION AND OSCILLATION CRITERIA FOR CLASS OF HIGHER-ORDER DIFFERENCE EQUATIONS INVOLVING GENERALIZED DIFFERENCE OPERATOR

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Abstract. In this paper, sufficient conditions are obtained for nonoscillation/oscillation of all solutions of a class of higher-order difference equations involving the generalized difference operator of the form

$$\Delta_a^k(p_n\Delta_a^2 y_n) = f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, ..., \Delta_a^{k+1} y_n),$$

where Δ_a is generalized difference operator which is defined as $\Delta_a y_n = y_{n+1} - a y_n$, $a \neq 0$.

1. Introduction

In this paper, we study nonoscillation and oscillation of solutions of a class of higher-order difference equations of the form

$$\Delta_a^k(p_n\Delta_a^2 y_n) = f(n, y_n, \Delta_a y_n, ..., \Delta_a^{k+1} y_n), \quad n \in \mathbb{N}, \tag{1}$$

where $\mathbb{N}$ is the set of natural numbers, $a \in \mathbb{R}\backslash\{0\}$, $\mathbb{R}$ is the set of real numbers, $\{p_n\}$ is a real sequence with $p_n \neq 0$ for $n \in \mathbb{N}$ and $f : \mathbb{N} \times \mathbb{R}^{k+2} \longrightarrow \mathbb{R}$. The generalized difference operator Δ_a is defined as $\Delta_a y_n = y_{n+1} - a y_n$. For $a = 1$, we write $\Delta_1 = \Delta$ where Δ is known forward difference operator. We define inductively $\Delta_a^k y_n = \Delta_a(\Delta_a^{k-1} y_n)$ for $k \geq 2$. By a solution of Eq. (1) we mean a sequence $\{y_n\}$ of real numbers which satisfies Eq. (1) identically. We consider only nontrivial solutions, i.e., such for which $\sup\{|y_n| : n \geq i\} > 0$ for every $i \in \mathbb{N}$. A solution of Eq. (1) is called non-oscillatory if it is eventually of constant sign (positive or negative) otherwise it is called oscillatory. For $a \in \mathbb{R}\backslash\{0\}$, Eq. (1) always admits a solution on $\mathbb{N}$. The oscillation and nonoscillation of solutions of difference

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equations are very popular for researchers in the last two decades. For this we refer the monograps [1,2,3]. The oscillation and nonoscillation of solutions of higher order difference equations has been studied by many authors. For example in [9], oscillation criteria are obtained for higher-order neutral-type nonlinear delay difference equations of the form

$$\Delta\left(r_n\left(\Delta^{k-1}\left(y_n+p_n y_{\tau_n}\right)\right)\right)+q_n f\left(y_{\sigma_n}\right)=0, n \geq n_0,$$

where $r_n, p_n, q_n \in [n_0, \infty)$, $r_n > 0$, $q_n > 0; 0 \leq p_n \leq p_0 < \infty$, $\lim\limits_{n\to\infty} \tau_n = \infty$, $\lim\limits_{n\to\infty} \sigma_n = \infty$, $\sigma_n \leq n$, σ_n is nondecreasing, $\Delta\tau_n \geq \tau_0 > 0$, $\tau_\sigma = \sigma_\tau$; $\frac{f(u)}{u} \geq m > 0$ for $u \neq 0$. In [5], Agarval et al. established some new criteria for the oscillation of higher order difference equations of the form

$$\Delta\left(\Delta^{m-1}\left(x_n\right)\right)^{\alpha}+q_n x^{\alpha}[n-\tau]=0,$$

where $m \geq 2$, $\tau \geq 1$ and α is the ratio of positive odd integers. In [4], Agarval et al. established sufficient conditions for the oscillation of all solutions of the even order difference equations of the form

$$\Delta^m x_n+p_n \Delta^{m-1} x_n+F\left(x_{n-g}, \Delta x_{n-h}\right)=0, \ m \text{ is even},$$

by comparing it with certain difference equations of lower order whose oscillatory character is known. In [6], some oscillation criteria for solutions of nonlinear higher-order forced difference equations are established. The investigations are carried out without assuming that the coefficients of the equations are of a definite sign and by showing that the forcing term needs not be the mth difference of an oscillatory function. In [13], Saker et al. established some new oscillation criteria for a certain class of third order nonlinear delay difference equations by employing the generalized Riccati transformation technique. In [7], sufficient conditions are established for the oscillatory and asymptotic behavior of higher–order half–linear delay difference equation of the form

$$\Delta\left(p_n\left(\Delta^{m-1}\left(x_n+q_n x_{\tau_n}\right)\right)^{\alpha}\right)+r_n x_{\sigma_n}^{\beta}=0, \ n \geq n_0,$$

where it is assumed that $\sum\limits_{s=n_0}^{\infty} \frac{1}{p_s{\frac{1}{\alpha}}} < \infty$. In [8] Bolat et al. investigated the oscillatory behavior of solutions of the th order half-linear functional difference equations with damping term of the form

$$\Delta\left(p_n Q\left(\Delta^{m-1} y_n\right)\right)+q_n Q\left(\Delta^{m-1} y_n\right)+r_n Q\left(y_{\tau_n}\right)=0, n \geq n_0,$$

where m is even and $Q(s)=|s|^{\alpha-2}s$, $\alpha > 1$ is a fixed real number.

The generalized difference operator Δ_a is a generalization of the difference operator Δ. Due to the relation between the ordinary difference operator Δ and generalized difference operator Δ_a, most difference equation can be considered more effectively by using generalized difference operator Δ_a. In the literature there are number of papers on the behavior of the difference equations involving operator

Δ_a. In [12], Popenda obtained sufficient conditions for nonoscillation/oscillation of solutions of a class of nonlinear nonhomogeneous second order difference equations involving generalized difference of the form

$$\Delta_a^2 x_n = F(n, x_n, \Delta_b x_n). \tag{2}$$

For some results of this type we refer the reader to the recent papers [11,14,15]. In [16], Tan and Yang generalized and improved the result of Popenda by considering the equation

$$\Delta_a(p_n \Delta_a x_n) + q_n \Delta_a x_n = F(n, x_n, \Delta_b x_n). \tag{3}$$

In [10], Parhi and Panda obtained sufficient conditions for nonoscillation /oscillation of all solutions of a class of nonlinear third order difference equations of the form

$$\Delta_a(p_n \Delta_a^2 y_n) + q_n \Delta_a^2 y_n = f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n). \tag{4}$$

Our purpose is to establish oscillation and nonoscillation criteria for a class of higher-order difference equations involving generalized difference operator of the form Eq. (1).

2. Auxiliary Lemmas

Lemma 1. *[10] Let $\{y_n\}$be a real sequence. If $\{\Delta_b y_n\}$, $b > 0$, is eventually of one sign, then $\{y_n\}$ is non-oscillatory.*

Lemma 2. *[10] For $b > 0$, a real sequence $\{y_n\}$ is oscillatory if and only if $\{\Delta_b^l y_n\}$ is oscillatory for all integers $l \geq 0$, where $\Delta_b^0 y_n \equiv y_n$*

Lemma 3. *For $n \in \mathbb{Z}$, $\Delta_a y_{n+1} = \Delta_a^2 y_n + a\Delta_a y_n$.*

Proof. By the definition of generalized difference operator, we write $\Delta_a y_n = y_{n+1} - ay_n$. Thus, If we apply the generalized difference to the both sides of this equality, we obtain that $\Delta_a^2 y_n = \Delta_a y_{n+1} - a\Delta_a y_n$. $\square$

Lemma 4. *[10] Let $b < 0$ and $k \in \mathbb{N}$. Then $\Delta_b^k y_l = b^{l+k}\Delta^k(\frac{y_l}{b^l})$, $l \in \mathbb{N}$, for any sequence $\{y_n\}$ of real numbers.*

Lemma 5. *For $m \geq 1$, $\Delta_a^m(p_n \Delta_a^2 y_n) = \sum\limits_{i=0}^{m} (-1)^i \binom{m}{i} a^i p_{n+m-i} \Delta_a^2 y_{n+m-i}$.*

Proof. One can easily show it using the definition of the generalized difference operator. $\square$

Lemma 6. *For $k \geq 1$, $\Delta_a y_l = \sum\limits_{i=0}^{k} a^i \binom{k}{i} \Delta_a^{k+1-i} y_{l-k}$.*

Proof. From Lemma 3, we can write $\Delta_a y_{l-1} = \Delta_a^2 y_{l-2} + a\Delta_a y_{l-2}$. If we apply the generalized difference operator to the both sides of this equality, we obtain $\Delta_a^2 y_{l-1} = \Delta_a^3 y_{l-2} + a\Delta_a^2 y_{l-2}$. Also from Lemma 3, we can write $\Delta_a y_l = \Delta_a^2 y_{l-1} + a\Delta_a y_{l-1}$. Then we have

$$\Delta_a y_l = \Delta_a^3 y_{l-2} + 2a\Delta_a^2 y_{l-2} + a^2 \Delta_a y_{l-2}.$$

Thus we obtain

$$\Delta_a y_{l-1} = \Delta_a^3 y_{l-3} + 2a\Delta_a^2 y_{l-3} + a^2 \Delta_a y_{l-3}.$$

Similarly, by applying the generalized difference operator to the both sides of last equality , we obtain that

$$\Delta_a^2 y_{l-1} = \Delta_a^4 y_{l-3} + 2a\Delta_a^3 y_{l-3} + a^2 \Delta_a^2 y_{l-3}.$$

By writing $\Delta_a y_{l-1}$ and $\Delta_a^2 y_{l-1}$ in the last equation, we obtain

$$\Delta_a y_l = \Delta_a^4 y_{l-3} + 3a\Delta_a^3 y_{l-3} + 3a^2 \Delta_a^2 y_{l-3} + a^3 \Delta_a y_{l-3},$$

and so on , we reach

$$\Delta_a y_l = \sum_{i=0}^{k} a^i \binom{k}{i} \Delta_a^{k+1-i} y_{l-k}.$$

□

Lemma 7. $\Delta_a^2 y_{n+k-1} = \sum_{i=0}^{k-1} \binom{k-1}{i} a^i \Delta_a^{k+1-i} y_n$, *for* $k \geq 1$, $n \in \mathbb{N}$.

Proof. By the Lemma 3, we have

$$\Delta_a^2 y_{n+1} = \Delta_a^3 y_n + a\Delta_a^2 y_n. \tag{5}$$

From (5) we can write

$$\Delta_a^2 y_{n+2} = \Delta_a^3 y_{n+1} + a\Delta_a^2 y_{n+1}. \tag{6}$$

Applying generalized difference operator to the Equation (5), we obtain $\Delta_a^3 y_{n+1} = \Delta_a^4 y_n + a\Delta_a^3 y_n$. Hence from (5) and (6) we have

$$\Delta_a^2 y_{n+2} = \Delta_a^4 y_n + 2a\Delta_a^3 y_n + a^2 \Delta_a^2 y_n.$$

Similarly, we obtain

$$\Delta_a^2 y_{n+3} = \Delta_a^5 y_n + 3a\Delta_a^4 y_n + 3a^2 \Delta_a^3 y_n + a^3 \Delta_a^2 y_n$$

and so on we reach

$$\Delta_a^2 y_{n+k-1} = \sum_{i=0}^{k-1} \binom{k-1}{i} a^i \Delta_a^{k+1-i} y_n, \text{ for } k \geq 1, \; n \in \mathbb{N}. \tag{7}$$

From (7) we can write

$$\Delta_a^2 y_n = \sum_{j=0}^{k-1} \binom{k-1}{j} a^j \Delta_a^{k+1-j} y_{n-k+1}, \quad \text{for } k \geq 1, \; n \in \mathbb{N}. \tag{8}$$

The proof is completed. □

3. Nonoscillation of Solutions

In this section non-oscillatory behaviour of solutions of Eq. (1) is studied.

Theorem 1. *Let $a > 0$. Assume that*

$$\frac{\sum_{i=0}^{k} a^i \binom{k}{i} \Delta_a^{k+1-i} y_n}{p_{n+k}} \Big[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{k+1} y_n) + \sum_{j=1}^{k} \binom{k}{j} a^j \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} p_{n+k-m} \right) \Delta_a^{k+2-m} y_n \Big] \geq 0, \tag{9}$$

is satisfied. Then all solutions of Eq. (1) are non-oscillatory.

Proof. Assume that $\{y_n\}$ is a possible oscillatory solution of Eq. (1). Hence, for every $s \in \mathbb{N}$, there exists $l > s$ such that $y_l \geq 0$ and $y_{l+1} < 0$ or $y_l > 0$ and $y_{l+1} \leq 0$. Therefore, $\Delta_a y_l = y_{l+1} - a y_l < 0$. By the Lemma 5 and Lemma 7, for $n \geq l$, Eq. (1) can be written as

$$\Delta_a y_{n+1} = a \Delta_a y_n + \frac{1}{p_n} \Big[f(n-k, y_{n-k}, ..., \Delta_a^{k+1} y_{n-k}) + \sum_{j=1}^{k} \binom{k}{j} a^j \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} p_{n-m} \right) \Delta_a^{k+2-m} y_{n-k} \Big]. \tag{10}$$

Multiplying (10) by $\Delta_a y_l$ and considering (9) we have

$$\Delta_a y_l \Delta_a y_{l+1} = a (\Delta_a y_l)^2 + \frac{\Delta_a y_l}{p_l} \Big[f(l-k, y_{l-k}, ..., \Delta_a^{k+1} y_{l-k}) + \sum_{j=1}^{k} \binom{k}{j} a^j \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} p_{l-m} \right) \Delta_a^{k+2-m} y_{l-k} \Big] > 0$$

Hence $\Delta_a y_{l+1} < 0$, since $\Delta_a y_l = \sum_{i=0}^{k} a^i \binom{k}{i} \Delta_a^{k+1-i} y_{l-k} < 0$. Putting $n = l+1$ in (10) and proceeding as above, we obtain $\Delta_a y_{l+1} \Delta_a y_{l+2} > 0$. Hence $\Delta_a y_{l+2} < 0$. Generally, we see that $\Delta_a y_{l+t} < 0$ for $t \in \mathbb{N}$. That is, $\Delta_a y_{l+t}$ is eventually of one sign. From Lemma 1 it follows that $\{y_n\}$ is eventually non-oscillatory. This is a contradiction to our assumption. Thus the theorem is proved. □

Theorem 2. *Let $a > 0$. Assume that*

$$\frac{1}{p_{n+k}} \Big[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{k+1} y_n)$$

$$+\sum_{j=1}^{k}\binom{k}{j}a^{j}\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n\Bigg] \le 0, \tag{11}$$

is satisfied. Then all solutions of Eq. (1) are non-oscillatory.

Proof. Let $\{y_n\}$ be a solution of Eq. (1). We claim that it is non-oscillatory. If not, then $\{y_n\}$ is oscillatory. Hence, for every $s \in \mathbb{N}$, there exists $l > s$ such that $y_l \ge 0$ and $y_{l+1} < 0$ or $y_l > 0$ and $y_{l+1} \le 0$. Therefore, $\Delta_a y_l = y_{l+1} - a y_l < 0$. For $n \ge l$, we can write Eq. (1) as in (10). Considering (10) and $\Delta_a y_l < 0$, we have

$$\begin{aligned}\Delta_a y_{l+1} &= a\Delta_a y_l + \frac{1}{p_l}\Big[f(l-k, y_{l-k}, ..., \Delta_a^{k+1}y_{l-k}) \\ &\quad + \sum_{j=1}^{k}\binom{k}{j}a^{j}\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{l-m}\right)\Delta_a^{k+2-m}y_{l-k}\Bigg] \\ &< 0.\end{aligned}$$

Putting $n = l+1$ in (10) and by (11) we obtain $\Delta_a y_{l+2} < 0$. By similar processes, we reach that $\Delta_a y_{l+s} < 0$ for $s \in \mathbb{N}$. Hence $\{y_n\}$ is eventually non-oscillatory by Lemma 1. This contradiction completes the proof. □

Theorem 3. *Let $a > 0$. Assume that*

$$\begin{aligned}&\frac{1}{p_{n+k}}\Big[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{k+1}y_n) \\ &+ \sum_{j=1}^{k}\binom{k}{j}a^{j}\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n \\ &\ge 0,\end{aligned}$$

is satisfied. Then all solutions of Eq. (1) are non-oscillatory.

Proof. Assume that $\{y_n\}$ is an oscillatory solution of Eq. (1). So we choose $n > n_0$, where $n_0 \in \mathbb{N}$, such that $y_n \le 0$ and $y_{n+1} > 0$ or $y_n < 0$ and $y_{n+1} \ge 0$. Thus $\Delta_a y_n = y_{n+1} - a y_n > 0$. The rest of proof can be made. □

Theorem 4. *Let $a > 0$. Assume that*

$$\begin{cases} f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, ..., \Delta_a^{k+1}y_n) = 0, & \text{if } \Delta_a^2 y_n = 0 \\ \frac{\sum_{j=0}^{k-1}\binom{k-1}{j}a^j\Delta_a^{k+1-j}y_n}{p_{n+k}}\Big[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, ..., \Delta_a^{k+1}y_n) + \sum_{j=1}^{k}\binom{k}{j}a^j & \\ \quad\times\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n\Big] > 0, & \text{if } \Delta_a^2 y_n \ne 0 \end{cases} \tag{12}$$

is satisfied. Then all solutions of Eq. (1) are non-oscillatory.

Proof. Let X is the set of all solutions $y = \{y_n\}$ of Eq. (1). Assume that $X_1 = \{y \in X : \Delta_a^2 y_n = 0 \text{ for some } n \in \mathbb{N}\}$ and $X_2 = X - X_1$. Suppose that $y = \{y_n\}$ is a solution of Eq. (1). If $y \in X_1$, then there exists $t \in \mathbb{N}$ such that $\Delta_a^2 y_t = 0$. From the first part of assumption (12) it follows that $f(t, y_t, \Delta_a y_t, \Delta_a^2 y_t, ..., \Delta_a^{k+1} y_t) = 0$. Thus from Eq. (1) we obtain

$$\Delta_a^k (p_t \Delta_a^2 y_t) = 0.$$

Hence, by from Lemma 5, we have

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} a^i p_{t+k-i} \Delta_a^2 y_{t+k-i} = 0.$$

In here we know that $p_{t+k}\Delta_a^2 y_{t+k} = \binom{k}{1} a p_{t+k-1} \Delta_a^2 y_{t+k-1} - \binom{k}{2} a^2 p_{t+k-2} \Delta_a^2 y_{t+k-2} + ... - (-1)^k a^k p_t \Delta_a^2 y_t$. If $\Delta_a^2 y_t = 0$, $\Delta_a y_{t+1} = a\Delta_a y_t$. Thus, If we apply the generalized difference operator to both sides of the last equality, we obtain that $\Delta_a^2 y_{t+1} = a\Delta_a^2 y_t$. Since $\Delta_a^2 y_t = 0$, $\Delta_a^2 y_{t+1} = 0$. Since $\Delta_a^2 y_{t+1} = 0$, $\Delta_a y_{t+2} = a\Delta_a y_{t+1}$. Likewise if we apply the generalized difference operator to both sides of the last equality , we obtain that $\Delta_a^2 y_{t+2} = a\Delta_a^2 y_{t+1}$. Since $\Delta_a^2 y_{t+1} = 0$, $\Delta_a^2 y_{t+2} = 0$. Continuing the progress in the same way yields $p_{t+k}\Delta_a^2 y_{t+k} = 0$, that is, $\Delta_a^2 y_{t+k} = 0$. Writing $t + k$ instead of n in Eq. (1) and using the first part of (15), we obtain $\Delta_a^k(p_{t+k}\Delta_a^2 y_{t+k}) = 0$, that is, $\sum_{i=0}^{k} (-1)^i \binom{k}{i} a^i p_{t+2k-i} \Delta_a^2 y_{t+2k-i} = 0$. If $\Delta_a^2 y_{t+k} = 0$, $\Delta_a y_{t+k+1} = a\Delta_a y_{t+k}$. If we apply the generalized difference operator to both sides of the last equality, we obtain that $\Delta_a^2 y_{t+k+1} = a\Delta_a^2 y_{t+k}$. Thus $\Delta_a^2 y_{t+k+1} = 0$. Continuing the progress in the same way for the first part of (12) yields $\Delta_a^2 y_{t+s} = 0$ for $s \in \mathbb{N}$. We may observe that $\Delta_a^2 y_{t+1} = 0$ implies $\Delta_a y_{t+2} = a\Delta_a y_{t+1}$ and $\Delta_a^2 y_{t+2} = 0$ implies $\Delta_a y_{t+3} = a\Delta_a y_{t+2} = a^2 \Delta_a y_{t+1}$. In general case, we obtain

$$\Delta_a y_{t+l} = a^{l-1} \Delta_a y_{t+1},\ l \in \mathbb{N}. \tag{13}$$

If $\Delta_a y_{t+1} = 0$, then $\Delta_a y_{t+l} = 0$ for $l \in \mathbb{N}$. Hence

$$y_{t+l+1} = a y_{t+l},\ l \in \mathbb{N}. \tag{14}$$

Since the solution $\{y_n\}$ of Eq. (1) is nontrivial, we can find $n_0 \in \mathbb{N}$, $n_0 \geq t + 1$, such that $y_{n_0} \neq 0$. Putting $l = n_0 - t, n_0 - t + 1, ...$ in (13) we get $y_{n_0+1} = a y_{n_0}$, $y_{n_0+2} = a y_{n_0+1} = a^2 y_{n_0}$, etc. In general, $y_{n_0+s} = a^s y_{n_0}$, $s \in \mathbb{N}$. Hence $\{y_n\}$ is eventually of one sign, that is, $\{y_n\}$ is non-oscillatory. From (13) it follows that since $\Delta_a y_{t+1} > 0$ or < 0, $\Delta_a y_{t+l} > 0$ or < 0 for $l \in \mathbb{N}$. Hence $\{\Delta_a y_n\}$ is eventually of one sign. Thus $\{y_n\}$ is eventually of one sign by Lemma 1. Consequently, $\{y_n\}$ is non-oscillatory.

Now let $y \in X_2$. Then $\Delta_a^2 y_n \neq 0$ for all $n \in \mathbb{N}$. Eq. (1) can be written in the form

$$\Delta_a^2 y_{n+k} = \frac{1}{p_{n+k}} \left[f(n, y_n, ..., \Delta_a^{k+1} y_n) \right.$$

$$+\sum_{j=1}^{k}\binom{k}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n\Bigg]. \qquad (15)$$

Putting $n = l-k+1$ in (15) for a fixed l and multiplying (15) by $\Delta_a^2 y_l$, we obtain

$$\begin{aligned}\Delta_a^2 y_l \Delta_a^2 y_{l+1} &= \frac{\Delta_a^2 y_l}{p_{l+1}}\big[f(l-k+1, y_{l-k+1}, ..., \Delta_a^{k+1}y_{l-k+1})\\ &\quad +\sum_{j=1}^{k}\binom{k}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{l-m+1}\right)\Delta_a^{k+2-m}y_{l-k+1}\Bigg]\\ &> 0\end{aligned}$$

by the second part of the assumption (12). Since $\Delta_a^2 y_l \neq 0$, $\Delta_a^2 y_l > 0$ or < 0, also $\Delta_a^2 y_{l+1} > 0$ or < 0. Putting $n = l-k+2$ in (15) and considering the second part of (12), we have $\Delta_a^2 y_{l+2}\Delta_a^2 y_{l+1} > 0$. Therefore since $\Delta_a^2 y_l > 0$ or < 0, $\Delta_a^2 y_{l+2} > 0$ or < 0. The repeated considering of the second part of (12), we yield $\Delta_a^2 y_{l+k} > 0$ or < 0 for $k \in \mathbb{N}$. Hence from (8) we have $\Delta_a^2 y_l = \sum_{j=0}^{k-1}\binom{k-1}{j}a^j\Delta_a^{k+1-j}y_{l-k+1} > 0$ or < 0.Thus $\{\Delta_a^2 y_n\}$ is non-oscillatory. From Lemma 2 it follows that $\{y_n\}$ is non-oscillatory. Thus the theorem is proved. □

4. Oscillation of Solutions

In this section, we study oscillatory behavior of all solutions of Eq. (1).

Theorem 5. *Let $a < 0$. Assume that*

$$\frac{\sum_{i=0}^{k}a^i\binom{k}{i}\Delta_a^{k+1-i}y_n}{p_{n+k}}\big[f(n, y_n, ..., \Delta_a^{k+1}y_n)$$
$$+\sum_{j=1}^{k}\binom{k}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n\Bigg] \leq 0, \qquad (16)$$

is satisfied. Then all solutions of Eq. (1) are oscillatory.

Proof. Let $\{y_n\}$ be a solution of Eq. (1). If $\Delta_a y_n = 0$, then $y_{n+1} = ay_n$. Hence $\{y_n\}$ is oscillatory because of $a < 0$. Suppose that $\Delta_a y_n \neq 0$. If we write Eq. (1) as in (10) and multiply both of this equality $\Delta_a y_n = \sum_{i=0}^{k}a^i\binom{k}{i}\Delta_a^{k+1-i}y_{n-k}$ for $\Delta_a y_n \neq 0$, we have

$$\begin{aligned}\Delta_a y_n \Delta_a y_{n+1} &= a(\Delta_a y_n)^2 + \frac{\Delta_a y_n}{p_n}\big[f(n-k, y_{n-k}, ..., \Delta_a^{k+1}y_{n-k})\\ &\quad +\sum_{j=1}^{k}\binom{k}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n-m}\right)\Delta_a^{k+2-m}y_{n-k}\Bigg]\\ &< 0.\end{aligned}$$

Hence (16) holds. By the Lemma 4 we have $a^{n+1}\Delta\left(\frac{y_n}{a^n}\right)a^{n+2}\Delta\left(\frac{y_{n+1}}{a^{n+1}}\right)<0$, that is, $a^{2n+3}\Delta\left(\frac{y_n}{a^n}\right)\Delta\left(\frac{y_{n+1}}{a^{n+1}}\right)<0$. Since $a<0$, then

$$\Delta\left(\frac{y_n}{a^n}\right)\Delta\left(\frac{y_{n+1}}{a^{n+1}}\right)>0, n\in\mathbb{N}. \tag{17}$$

If $\Delta\left(\frac{y_n}{a^n}\right)>0$, then $\Delta\left(\frac{y_{n+1}}{a^{n+1}}\right)>0$. As (17) holds for every $n\in\mathbb{N}$, then $\Delta\left(\frac{y_{n+1}}{a^{n+1}}\right)>0$ implies that $\Delta\left(\frac{y_{n+2}}{a^{n+2}}\right)>0$ and so on. Hence $\{\Delta\left(\frac{y_n}{a^n}\right)\}$ is eventually of one sign. Consequently, $\{\frac{y_n}{a^n}\}$ is eventually of one sign by Lemma 1 for $b=1$. This implies that $\{y_n\}$ is oscillatory because $a<0$. Similarly, if $\Delta\left(\frac{y_n}{a^n}\right)<0$, then $\{y_n\}$ is oscillatory. Thus the theorem is proved. □

Remark 1. *If*

$$\frac{1}{p_{n+k}}\left[f(n,y_n,\Delta_a y_n,...,\Delta_a^{k+1}y_n)\right.$$
$$\left.+\sum_{j=1}^{k}\binom{k}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n\right]=0, \tag{18}$$

then all solutions of Eq. (1) are oscillatory. Indeed, if $\{y_n\}$ is a non-oscillatory solution of Eq. (1), then there exists $k_o\in\mathbb{N}$ such that $y_n>0$ or <0 for $n\geq k_0$. Eq. (1) can be written in the form

$$\begin{aligned}\Delta_a^2 y_{n+k} &= \frac{1}{p_{n+k}}\left[f(n,y_n,...,\Delta_a^{k+1}y_n)\right.\\ &\quad\left.+\sum_{j=1}^{k}\binom{k}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+k-m}\right)\Delta_a^{k+2-m}y_n\right].\end{aligned}$$

Then considering (18), we have $\Delta_a^2 y_{n+k}=0$, $n\in\mathbb{N}$. Then for $k\geq 1$, $\Delta_a^2 y_{n+1}=0$ implies that $\Delta_a y_{n+2}=a\Delta_a y_{n+1}$. Similarly, $\Delta_a^2 y_{n+2}=0$ implies that $\Delta_a y_{n+3}=a\Delta_a y_{n+2}=a^2\Delta_a y_{n+1}$. In general case, $\Delta_a^2 y_{n+k}=0$ implies that $\Delta_a y_{n+k+1}=a^k\Delta_a y_{n+1}$, $k\in\mathbb{N}$. In particular, $\Delta_a y_{k_0+k+1}=a^k\Delta_a y_{k_0+1}$ for $n\geq k_0$. Let $y_n>0$ for $n\geq k_0$. We consider three possibilities for $\Delta_a y_{k_0+1}$, viz., $\Delta_a y_{k_0+1}=0$, >0 and <0 and obtain a contradiction in each case. If $\Delta_a y_{k_0+1}=0$, then $\Delta_a y_{k_0+k+1}=0$, that is, $y_{k_0+k+2}=ay_{k_0+k+1}<0$ for $k\in\mathbb{N}$, a contradiction to the fact that $y_n>0$ for $n\geq k_0$. Let $\Delta_a y_{k_0+1}>0$. Then $\Delta_a y_{k_0+2k+2}=a^{2k+1}\Delta_a y_{k_0+1}<0$ implies that $y_{k_0+2k+3}=ay_{k_0+2k+2}<0$, a contradiction. If $\Delta_a y_{k_0+1}<0$, then $\Delta_a y_{k_0+2k+1}=a^{2k}\ \Delta_a y_{k_0+1}<0$ implies that $y_{k_0+2k+2}<ay_{k_0+2k+1}<0$, a contradiction. Thus $y_n>0$ for $n\geq k_0$ is not possible. Let $y_n<0$ for $n\geq k_0$. Proceeding as above we arrive at a contradiction in each of the three cases, viz., $\Delta_a y_{k_0+1}=0$, >0 and <0. Hence $y_n<0$ for $n\geq k_0$ is not possible. Thus $\{y_n\}$ is oscillatory.

Theorem 6. *Let $a < 0$. Assume that*

$$\begin{cases} f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, ..., \Delta_a^{k+1} y_n) = 0, & \text{if } \Delta_a^2 y_n = 0 \\ \frac{\sum_{j=0}^{k-1} \binom{k-1}{j} a^j \Delta_a^{k+1-j} y_n}{p_{n+k}} \Big[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, ..., \Delta_a^{k+1} y_n) + \sum_{j=1}^{k} \binom{k}{j} a^j \\ \quad \times \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} p_{n+k-m} \right) \Delta_a^{k+2-m} y_n \Big] < 0, \ \text{if } \Delta_a^2 y_n \neq 0 \end{cases} \tag{19}$$

is satisfied. Then all solutions of Eq. (1) are oscillatory.

Proof. Let X be the set of all solutions $y = \{y_n\}$ of Eq. (1). Assume that $X_1 = \{y \in X : \Delta_a^2 y_n = 0 \text{ for some } n \in \mathbb{N}\}$ and $X_2 = X - X_1$. Suppose that $y = \{y_n\}$ be a non-oscillatory solution of Eq. (1). Hence $\{y_n\}$ is eventually of one sign. If $y \in X_1$, then there exists $t \in \mathbb{N}$ such that $\Delta_a^2 y_t = 0$. Thus from Eq. (1) and (19) it follows that $\Delta_a^k (p_t \Delta_a^2 y_t) = 0$, that is, $\sum_{i=0}^{k} (-1)^i \binom{k}{i} a^i p_{t+k-i} \Delta_a^2 y_{t+k-i} = 0$. Then

$$p_{t+k} \Delta_a^2 y_{t+k} = \binom{k}{1} a p_{t+k-1} \Delta_a^2 y_{t+k-1} - \binom{k}{2} a^2 p_{t+k-2} \Delta_a^2 y_{t+k-2} + ... - (-1)^k a^k p_t \Delta_a^2 y_t.$$

If $\Delta_a^2 y_t = 0$, then $\Delta_a y_{t+1} = a \Delta_a y_t$. If we apply the generalized difference operator to both sides of the last equality, we obtain that $\Delta_a^2 y_{t+1} = a \Delta_a^2 y_t$. Since $\Delta_a^2 y_t = 0$, $\Delta_a^2 y_{t+1} = 0$. Since $\Delta_a^2 y_{t+1} = 0$, $\Delta_a y_{t+2} = a \Delta_a y_{t+1}$. Likewise , if we apply the generalized difference operator to both sides of the last equality, we obtain that $\Delta_a^2 y_{t+2} = a \Delta_a^2 y_{t+1}$. Since $\Delta_a^2 y_{t+1} = 0$, $\Delta_a^2 y_{t+2} = 0$. By recurrence of the processes ,we obtain that $p_{t+k} \Delta_a^2 y_{t+k} = 0$, that is, $\Delta_a^2 y_{t+k} = 0$. If $\Delta_a^2 y_{t+1} = 0$, for $k \geq 1$, $\Delta_a y_{t+2} = a \Delta_a y_{t+1}$. Since $\Delta_a^2 y_{t+2} = 0$, $\Delta_a y_{t+3} = a \Delta_a y_{t+2} = a^2 \Delta_a y_{t+1}$ and so on. Generally, we have $\Delta_a y_{t+k} = a^{k-1} \Delta_a y_{t+1}$. We can choose $k_0 \in \mathbb{N}$ such that $y_k > 0$ or < 0 for $k \geq k_0$. Let $y_k > 0$ for $k \geq k_0$. If $\Delta_a y_{t+1} = 0$, then $\Delta_a y_{t+k_0} = 0$ and hence $y_{t+k_0+1} = a y_{t+k_0} < 0$, a contradiction. If $\Delta_a y_{t+1} > 0$, then $\Delta_a y_{t+2k_0} = a^{2k_0-1} \Delta_a y_{t+1} < 0$ and hence $y_{t+2k_0+1} = a y_{t+2k_0} < 0$, a contradiction. If $\Delta_a y_{t+1} < 0$, then $\Delta_a y_{t+2k_0+1} = a^{2k_0} \Delta_a y_{t+1} < 0$ implies that $y_{t+2k_0+2} = a y_{t+2k_0+1} < 0$, a contradiction. Similar contradiction is obtained if $y_k < 0$ for $k \geq k_0$. Thus $y \notin X_1$. Now let $y \in X_2$. Hence $\Delta_a^2 y_n \neq 0$ for all $n \in \mathbb{N}$. Writing Eq. (1) as we obtain

$$\begin{aligned} \Delta_a^2 y_n \Delta_a^2 y_{n+1} &= \frac{\Delta_a^2 y_n}{p_{n+1}} \Big[f(n-k+1, y_{n-k+1}, \Delta_a y_{n-k+1}, ..., \Delta_a^{k+1} y_{n-k+1}) \\ &\quad + \sum_{j=1}^{k} \binom{k}{j} a^j \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} p_{n-m+1} \right) \Delta_a^{k+2-m} y_{n-k+1} \Big] \\ &< 0, \end{aligned}$$

by the second of (19). In here $\Delta_a^2 y_n = \sum_{j=0}^{k-1} \binom{k-1}{j} a^j \Delta_a^{k+1-j} y_{n-k+1}$. Applying Lemma 4 we get $a^{2n+5} \Delta^2 \left(\frac{y_n}{a^n}\right) \Delta^2 \left(\frac{y_{n+1}}{a^{n+1}}\right) < 0$. Hence $\Delta^2 \left(\frac{y_n}{a^n}\right) \Delta^2 \left(\frac{y_{n+1}}{a^{n+1}}\right) > 0$, $n \in \mathbb{N}$, since $a < 0$. If $\Delta^2 \left(\frac{y_n}{a^n}\right) > 0$, then $\Delta^2 \left(\frac{y_{n+1}}{a^{n+1}}\right) > 0$. This in turn implies that $\Delta^2\left(\frac{y_{n+2}}{a^{n+2}}\right) > 0$ and so on. If $\Delta^2\left(\frac{y_n}{a^n}\right) < 0$, then $\Delta^2\left(\frac{y_{n+1}}{a^{n+1}}\right) < 0$ which in turn implies that $\Delta^2\left(\frac{y_{n+2}}{a^{n+2}}\right) < 0$ and so on. Therefore $\{\Delta^2\left(\frac{y_n}{a^n}\right)\}$ is of one sign. By Lemma 1, $\{\Delta\left(\frac{y_n}{a^n}\right)\}$ is eventually of one sign and hence $\{\frac{y_n}{a^n}\}$ is eventually of one sign . Consequently $\{y_n\}$ is oscillatory. This contradicts our assumption $y = \{y_n\}$ be a non-oscillatory solution of Eq. (1) . Thus $y \notin X_2$. Consequently, all solutions of Eq. (1) are oscillatory and this completes the proof of the theorem. □

5. Examples

Example 1. *Consider*

$$4\Delta_a^4 y_n = (1-8a)\Delta_a^3 y_n + 2a(1-2a)\Delta_a^2 y_n + a^2\Delta_a y_n, \tag{20}$$

where $a > 0$*,* $p_n = 4$*,* $k = 2$ *and* $f(n, y_n, \Delta_a y_n, \Delta^2{}_a y_n, \Delta_a^3 y_n) = (1-8a)\Delta_a^3 y_n + 2a(1-2a)\Delta_a^2 y_n + a^2\Delta_a y_n$*. Since*

$$\begin{aligned}
&\frac{\sum\limits_{i=0}^{2} a^i \binom{2}{i} \Delta_a^{2+1-i} y_n}{p_{n+2}} \Big[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{2+1} y_n) \\
&+ \sum_{j=1}^{2} \binom{2}{j} a^j \left(\sum_{m=1}^{j} \binom{j}{m} (-1)^{m+1} p_{n+2-m} \right) \Delta_a^{2+2-m} y_n \Big] \\
= \;& \frac{\Delta_a^3 y_n + 2a\Delta_a^2 y_n + a^2 \Delta_a y_n}{p_{n+2}} \Big[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n) \\
& + 2a p_{n+1} \Delta_a^3 y_n + a^2(2p_{n+1} - p_n)\Delta_a^2 y_n \Big] \\
= \;& \frac{\Delta_a^3 y_n + 2a\Delta_a^2 y_n + a^2 \Delta_a y_n}{4} \Big[(1-8a)\Delta_a^3 y_n + 2a(1-2a)\Delta_a^2 y_n + a^2 \Delta_a y_n \\
& + 4a\Delta_a^3 y_n + 2a^2 \Delta_a^2 y_n \Big] \\
= \;& \frac{(\Delta_a^3 y_n + 2a\Delta_a^2 y_n + a^2\Delta_a y_n)^2}{4} \\
\geq \;& 0,
\end{aligned}$$

all solutions of (20) are non-oscillatory by Theorem 1. In other way, Equation (20) can be written as

$$4y_{n+4} + (-1-8a)y_{n+3} + (4a^2 + a)y_{n+2} = 0.$$

The characteristic equation concerning with this equation is given by

$$4\lambda^4 + (-1-8a)\lambda^3 + (4a^2 + a)\lambda^2 = 0,$$

that is,

$$(\lambda - a)(4\lambda^3 + (-1-4a)\lambda^2) = 0.$$

A fundamental set of all solutions of (20) equation is $\left\{\{a^n\}, \{(\frac{1+4a}{4})^n\}\right\}$. Thus we again see that all solutions of (20) are non-oscillatory.

Example 2. *Consider the equation*

$$-2\Delta^5 y_n = 6\Delta^4 y_n + 6\Delta^3 y_n + 2\Delta^2 y_n + (\Delta y_n)^2, \tag{21}$$

where $a = 1$*,* $p_n = -2$*,* $k = 3$ *and* $f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n, \Delta_a^4 y_n) = 6\Delta^4 y_n + 6\Delta^3 y_n + 2\Delta^2 y_n + (\Delta y_n)^2$*. Thus*

$$\begin{aligned}
&\frac{1}{p_{n+3}}\Big[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{3+1} y_n) \\
&+\sum_{j=1}^{3}\binom{3}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+3-m}\right)\Delta_a^{3+2-m}y_n\Bigg] \\
= \quad &\frac{1}{p_{n+3}}\big[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n, \Delta_a^4 y_n) \\
&+3ap_{n+2}\Delta_a^4 y_n + 3a^2(2p_{n+2} - p_{n+1})\Delta_a^3 y_n + a^3(3p_{n+2} - 3p_{n+1} + p_n)\Delta_a^2 y_n\big] \\
= \quad &\frac{1}{-2}\big[6\Delta^4 y_n + 6\Delta^3 y_n + 2\Delta^2 y_n + (\Delta y_n)^2 \\
&-6\Delta^4 y_n - 6\Delta^3 y_n - 2\Delta^2 y_n\big] \\
= \quad &-\frac{(\Delta y_n)^2}{2} \le 0
\end{aligned}$$

and the condition of Theorem 2 is satisfied. Hence it follows that all solutions of (21) are non-oscillatory. In particular, $y_n \equiv c$*, where* $c \neq 0$ *is a constant, is a non-oscillatory solution of the equation.*

Example 3. *Consider*

$$-2\Delta^5 y_n = 6\Delta^4 y_n + 6\Delta^3 y_n + 2\Delta^2 y_n - (\Delta y_n)^2, \tag{22}$$

where $a = 1$*,* $p_n = -2$*,* $k = 3$ *and* $f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n, \Delta_a^4 y_n) = 6\Delta^4 y_n + 6\Delta^3 y_n + 2\Delta^2 y_n - (\Delta y_n)^2$*. Thus*

$$\begin{aligned}
&\frac{1}{p_{n+3}}\Big[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{3+1} y_n) \\
&+\sum_{j=1}^{3}\binom{3}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+3-m}\right)\Delta_a^{3+2-m}y_n\Bigg] \\
= \quad &\frac{1}{p_{n+3}}\big[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n, \Delta_a^4 y_n) \\
&+3ap_{n+2}\Delta_a^4 y_n + 3a^2(2p_{n+2} - p_{n+1})\Delta_a^3 y_n + a^3(3p_{n+2} - 3p_{n+1} + p_n)\Delta_a^2 y_n\big] \\
= \quad &\frac{1}{-2}\big[6\Delta^4 y_n + 6\Delta^3 y_n + 2\Delta^2 y_n - (\Delta y_n)^2 \\
&-6\Delta^4 y_n + -6\Delta^3 y_n + -2\Delta^2 y_n\big] \\
= \quad &\frac{(\Delta y_n)^2}{2} \ge 0.
\end{aligned}$$

Then all solutions of the equation (22) are non-oscillatory due to Theorem 3.

Example 4. *Consider*

$$3\Delta_a^3 y_n = 2\Delta_a^2 y_n, \tag{23}$$

where $a>0$, $p_n=3, k=1$ *and* $f(n,y_n,\Delta_a y_n,\Delta_a^2 y_n)=2\Delta_a^2 y_n$. $f(n,y_n,\Delta_a y_n,\Delta_a^2 y_n)=0$ *if* $\Delta_a^2 y_n = 0$, *and if* $\Delta_a^2 y_n \neq 0$,

$$\begin{aligned}
&\frac{\sum_{j=0}^{1-1}\binom{1-1}{j}a^j\Delta_a^{1+1-j}y_n}{p_{n+1}}\left[f(n,y_n,\Delta_a y_n,\Delta_a^2 y_n,...,\Delta_a^{1+1}y_n)+\sum_{j=1}^{1}\binom{1}{j}a^j\right.\\
&\left.\times\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+1-m}\right)\Delta_a^{1+2-m}y_n\right]\\
=\;&\frac{\Delta_a^2 y_n}{p_{n+1}}\left[f(n,y_n,\Delta_a y_n,\Delta_a^2 y_n)+ap_n\Delta_a^2 y_n\right]\\
=\;&\frac{\Delta_a^2 y_n}{3}\left[2\Delta_a^2 y_n+3a\Delta_a^2 y_n\right]\\
=\;&\frac{(\Delta_a^2 y_n)^2(2+3a)}{3}>0,
\end{aligned}$$

Therefore all solution of (23) are non-oscillatory by Theorem 4. We can make the proof by the another way. For this, we can write the Eq. (23) as in the form

$$3y_{n+3}-(9a+2)y_{n+2}+(9a^2+4a)y_{n+1}-(2a^2+3a^3)y_n=0.$$

The characteristic equation concerning with this equation is

$$3\lambda^3-(9a+2)\lambda^2+(9a^2+4a)\lambda-(2a^2+3a^3)=0,$$

that is,

$$(\lambda-a)(3\lambda^2-(6a+2)\lambda+3a^2+2a)=0.$$

Hence a fundamental set of all solutions of Eq. (23) is $\left\{\{a^n\},\{na^n\},\{\left(\frac{3a+2}{3}\right)^n\right\}$. *Thus all solutions of (23) are non-oscillatory.*

Example 5. *Consider*

$$\Delta_a^3 y_n = -(1+a)\Delta_a^2 y_n - a\Delta_a y_n, \tag{24}$$

where $a<0$, $p_n=1$, $k=1$ *and* $f(n,y_n,\Delta_a y_n,\Delta_a^2 y_n) = -(1+a)\Delta_a^2 y_n - a\Delta_a y_n$. *Since*

$$\begin{aligned}
&\frac{\sum_{i=0}^{1}a^i\binom{1}{i}\Delta_a^{1+1-i}y_n}{p_{n+1}}\left[f(n,y_n,...,\Delta_a^{1+1}y_n)\right.\\
&\left.+\sum_{j=1}^{1}\binom{1}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+1-m}\right)\Delta_a^{1+2-m}y_n\right]\\
=\;&\frac{\Delta_a^2 y_n + a\Delta_a y_n}{p_{n+1}}\left[f(n,y_n,\Delta_a y_n,\Delta_a^2 y_n)+ap_n\Delta_a^2 y_n\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Delta_a^2 y_n + a\Delta_a y_n}{1}\left[-(1+a)\Delta_a^2 y_n - a\Delta_a y_n + a\Delta_a^2 y_n\right] \\
&= -(\Delta_a^2 y_n + a\Delta_a y_n)^2 \\
&\leq 0,
\end{aligned}
$$

all solutions of the equation are oscillatory by Theorem 5. In particular, a fundamental set of all solutions of Eq. (24) is $\{\{(a^n)\},\{(a-1)^n\}\}$*. Thus all of solutions of (24) are oscillatory.*

Example 6. *Consider*

$$2\Delta_a^4 y_n = -(4a\Delta_a^3 y_n + 2a^2\Delta_a^2 y_n), \tag{25}$$

where $a < 0$*,* $p_n = 2$*,* $k = 2$ *and* $f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n) = -(4a\Delta_a^3 y_n + 2a^2\Delta_a^2 y_n)$*. Since*

$$
\begin{aligned}
&\frac{1}{p_{n+2}}\Bigg[f(n, y_n, \Delta_a y_n, ..., \Delta_a^{2+1} y_n) \\
&\quad + \sum_{j=1}^{2}\binom{2}{j}a^j\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+2-m}\right)\Delta_a^{2+2-m}y_n\Bigg] \\
&= \frac{1}{p_{n+2}}\left[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, \Delta_a^3 y_n) + 2ap_{n+1}\Delta_a^3 y_n + a^2(2p_{n+1} - p_n)\Delta_a^2 y_n\right] \\
&= \frac{1}{2}\left[-(4a\Delta_a^3 y_n + 2a^2\Delta_a^2 y_n) + 4a\Delta_a^3 y_n + 2a^2\Delta_a^2 y_n\right] \\
&= 0,
\end{aligned}
$$

all solutions of the equation (25) are oscillatory in view of Remark 1. In particular, $\{a^n\}$ *and* $\{na^n\}$ *are two oscillatory solutions of the equation.*

Example 7. *Consider*

$$3\Delta_a^3 y_n = -2\Delta_a^2 y_n, \tag{26}$$

where $a < 0$*,* $k = 1$*,* $p_n = 3$ *and* $f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n) = -2\Delta_a^2 y_n$*. Hence* $\Delta_a^2 y_n = 0$ *implies that* $f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n) = 0$*. If* $\Delta_a^2 y_n \neq 0$*, then*

$$
\begin{aligned}
&\frac{\sum_{j=0}^{1-1}\binom{1-1}{j}a^j\Delta_a^{1+1-j}y_n}{p_{n+k}}\Bigg[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n, ..., \Delta_a^{1+1} y_n) + \sum_{j=1}^{1}\binom{1}{j}a^j \\
&\times\left(\sum_{m=1}^{j}\binom{j}{m}(-1)^{m+1}p_{n+1-m}\right)\Delta_a^{1+2-m}y_n\Bigg] \\
&= \frac{\Delta_a^2 y_n}{p_{n+1}}\left[f(n, y_n, \Delta_a y_n, \Delta_a^2 y_n) + ap_n\Delta_a^2 y_n\right] \\
&= \frac{\Delta_a^2 y_n}{3}\left[-2\Delta_a^2 y_n + 3a\Delta_a^2 y_n\right]
\end{aligned}
$$

$$\begin{aligned} &= (\Delta_a^2 y_n)^2 \left(\frac{-2+3a}{3}\right) \\ &< 0. \end{aligned}$$

Hence by Theorem 6 all solution of (26) are oscillatory . On the other hand, the characteristic equation of (26) is

$$(\lambda - a)^2(3\lambda^2 + (2-6a)\lambda + 3a^2 - 2a) = 0.$$

Hence a fundamental set of all solutions of Eq. (26) is $\{\{a^n\}, \{na^n\}, \{\left(\frac{3a-2}{3}\right)^n\}\}$ *which consists of all oscillatory solutions.*

6. Conclusion

In this paper we investigated the sufficient conditions of the oscillation and non-oscillation of higher -order difference equations (1). In this study, we used definitions of generalized difference operator and oscillation/non-oscillation for the proof of the results. Also, we have considered both cases of $a < 0$ and $a > 0$. We have obtained non-oscillatory behaviour of solution of Eq. (1) in Section 3, we have studied oscillatory behaviour of solution of Eq. (1) in Section 4, respectively. Finally, we have discussed some examples related to our main results.

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