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A fixed point theorem without a Picard operator

Erdal Karapınar[a,b]

[a]Faculty of Fundamental Science, Industrial University of Ho Chi Minh City, Ho Chi Minh, Vietnam
[b]Department of Mathematics, Çankaya University, 06790, Etimesgut, Ankara, Turkey.

Abstract

In this short note, we propose a fixed point theorem in the setting of a Banach space without using a Picard operator.

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1. Introduction and Preliminaries

The first paper in which the fixed point concept was discussed separately was by Banach: Every contraction possesses a unique fixed point on a complete metric space. However, the fixed point concept has been discussed and used by many authors in solving differential equations before. Among all, the most famous article belongs to Picard. Roughly speaking, Banach abstracted the fixed point theorem from the result of Picard. For this reason, in some old books and papers, this first fixed point theorem is called as "Picard-Banach theorem." Indeed, the proof of the Picard-Banach theorem is constructive and elegant: "For any given (starting) point, a constructive sequence is obtained recursively. Then, one can observe that the sequence is Cauchy and completeness yields the convergency. The limit of the sequence turns to be a fixed point of the given mapping due to continuity of mapping." In most of the fixed point theorems are proved in this way.

In this article, inspired by Gornicki and Rhoades [1], we propose a new fixed point theorem in the setting of Banach spaces without using a Picard operator.

Let T be a self-mapping on a Banach space $(X, \|\cdot\|)$ and $x_0 \in X$. The constructive sequence $\{x_n\}$ is called Picard sequence if $x_n = Tx_{n-1}$ for $n = 1, 2, \cdots$. Here, the mapping T is called Picard operator.

We recall the main results of Gornicki and Rhoades [1].

Email address: erdal.karapinar@cankaya.edu.tr erdalkarapinar@yahoo.com (Erdal Karapınar)

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Theorem 1.1. *[1] Let T be selfmapping on a nonempty closed convex subset C of a Banach space X. If for some $x_0 \in C$, there exists a constant c, $0 \le c < 1$ such that*

$$\|x_{n+2} - x_{n+1}\| \le c\|x_{n+1} - x_n\| \text{ for all } n = 0, 1, 2, \cdots, \tag{1.1}$$

where

$$x_{n+1} := \frac{1}{2}(x_n + Tx_n). \tag{1.2}$$

Then, $\{x_n\}$ converges to a point u in C. If, in addition, there exists nonnegative constants $\alpha, \beta, \gamma, \delta$, $0 \le \gamma < 1$, such that

$$\|Tx_n - Tu\| \le \alpha\|x_n - u\| + \beta\|x_n - Tx_n\| + \gamma \max\{\|u - Tu\|, \|x_n - Tu\|, \delta\|u - Tx_n\|\} \tag{1.3}$$

for all n sufficiently large, then u is a fixed point of T.

A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a *comparison function* if it is increasing and $\varphi^n(t) \to 0$, $n \to \infty$, for any $t \in [0, \infty)$. We denote by Φ, the class of the corporation function $\varphi : [0, \infty) \to [0, \infty)$. For more details and examples , see e.g. [6, 2]. Among them, we recall the following essential result.

Lemma 1.2. *(Berinde [2], Rus [6]) If $\varphi : [0, \infty) \to [0, \infty)$ is a comparison function, then:*

(1) each iterate φ^k of φ, $k \ge 1$, is also a comparison function;

(2) φ is continuous at 0;

(3) $\varphi(t) < t$, for any $t > 0$.

Later, Berinde [2] introduced the concept of (c)-*comparison function* in the following way.

Definition 1.3. (Berinde [2]) A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a (c)-comparison function if

(c_1) φ is increasing,

(c_2) there exists $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \le a\varphi^k(t) + v_k$, for $k \ge k_0$ and any $t \in [0, \infty)$.

The collection of all (c)-comparison functions will be denote by Ψ.

Theorem 1.4. *Let T be selfmapping on a nonempty closed convex subset C of a Banach space X. If for some $x_0 \in C$, there exists a constant $\varphi \in \Psi$ such that*

$$\|x_{n+2} - x_{n+1}\| \le \varphi(\|x_{n+1} - x_n\|) \text{ for all } n = 0, 1, 2, \cdots, \tag{1.4}$$

where

$$x_{n+1} := \frac{1}{2}(x_n + Tx_n). \tag{1.5}$$

Then, $\{x_n\}$ converges to a point u in C. If, in addition, there exist nonnegative constants $\alpha, \beta, \gamma, \delta$, $0 \le \gamma < 1$, such that

$$\|Tx_n - Tu\| \le \alpha\|x_n - u\| + \beta\|x_n - Tx_n\| + \gamma \max\{\|u - Tu\|, \|x_n - Tu\|, \delta\|u - Tx_n\|\} \tag{1.6}$$

for all n sufficiently large, then u is a fixed point of T.

Proof. Suppose there is $x_0 \in C$ such that the inequality (1.4) is fulfilled. Recursively, we derive from the inequality (1.4) that

$$\|x_{n+1} - x_n\| \leq \varphi^n(\|x_1 - x_0\|) \text{ for all } n = 0, 1, 2, \cdots , \tag{1.7}$$

Since φ is a c-comparison function (hence, comparison function), we conclude that the sequence $\{x_n\}$ is asymptotically regular.

On the other hand,

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \sum_{k=n}^{n+p-1} \|x_{k+1} - x_k\| \leq \sum_{k=n}^{n+p-1} \varphi^k(\|x_1 - x_0\|) \\ &\leq \sum_{k=n}^{\infty} \varphi^k(\|x_1 - x_0\|) \to 0 \text{ as } n \to \infty, \end{aligned} \tag{1.8}$$

since φ is a c-comparison function. Hence, $\{x_n\}$ is Cauchy sequence. Since, C is closed subset of Banach, we deduce that $\{x_n\}$ converges a point in C, say u.

In what follows we prove that u is a fixed point of T. Assume that the inequality (1.6) is satisfied. Note that the expression can be restated as $2x_{n+1} = x_n + Tx_n$ and we find $2x_{n+1} - 2x_n = Tx_n - x_n$. Consequently, we shall observe that

$$\|x_n - Tx_n\| = 2\|x_{n+1} - x_n\| \text{ for each } n = 0, 1, 2, \cdots \tag{1.9}$$

On account of (1.6), the above inequality can be estimated above as follows

$$\|x_n - Tx_n\| = 2\|x_{n+1} - x_n\| \leq 2\varphi^n(\|x_1 - x_0\|) \text{ for each } n = 0, 1, 2, \cdots \tag{1.10}$$

By taking $n \to \infty$ in the inequality above, we find

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{1.11}$$

By uniqueness of the limit and the fact that $\lim_{n\to\infty} x_n = u$, we conclude, from the expression (1.6), that $\lim_{n\to\infty} Tx_n = u$.

Taking the limit of (1.6) as $n \to \infty$, we obtain that

$$\|u - Tu\| \leq \gamma\|u - Tu\| \tag{1.12}$$

that yields $u = Tu$. □

Remark 1.5. Notice that Theorem 1.1 can de derived from Theorem 1.4 easily, by letting $\varphi(t) := ct$, for all $t \in [0, \infty)$ and $c \in (0, 1]$.

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