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Stochastic sub-diffusion equation with conformable derivative driven by standard Brownian motion

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Abstract

This article is concerned with a forward problem for the following sub-diffusion equation driven by standard Brownian motion

$$\left({}^{\mathcal{C}}\partial_t^\gamma + A\right) u(t) = f(t) + B(t)\dot{W}(t), \quad t \in J := (0,T),$$

where ${}^{\mathcal{C}}\partial_t^\gamma$ is the conformable derivative, $\gamma \in (\frac{1}{2}, 1]$. Under some flexible assumptions on f, B and the initial data, we investigate the existence, regularity, continuity of the solution on two spaces $L^r(J; L^2(\Omega, \dot{H}^\sigma))$ and $C^\alpha(\overline{J}; L^2(\Omega, H))$ separately.

Keywords: diffusion equation, standard Brownian motion, fractional Brownian motion, conformable derivative, existence and regularity.
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1. Introduction

The normal diffusion processes are characterized by a linear growth in a time of the mean squared displacement (variance increases linearly in time). Besides, there is also a concept of anomalous diffusion

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regimes. They are characterized by a variance growing slower (sub-diffusion) or faster than normal (super-diffusion). Many different underlying processes can lead to anomalous diffusion, with qualitative differences between mechanisms producing sub-diffusion and mechanisms resulting in super-diffusion.

Recent decades have witnessed the escalating popularity of fractional calculus, which replaces the classical time derivative in a partial differential equation with a fractional derivative. Because of containing information of the considered function at the previous time, fractional derivatives in time (for instance, Caputo, Riemann–Liouville, and Conformable) reflect memory, history, and non-local spatial effects, which are paramount crucial in better modeling and understanding of the complex and dynamic system behavior. We can list here some successful applications of fractional calculus to specific problems of dynamics [18, 38, 43]. Some applications of Conformable derivative in modeling neuronal dynamics, dynamic cobweb, dynamics of a Particle in a viscoelastic medium, and fractional-order chaotic system can be found in [15, 16, 24, 32, 55]. One of the efficient and widely used approaches to modeling sub-diffusion is based on the theory of fractional derivatives. Such equations were widely used to model many phenomena in nature. For example: Population dynamics of water confined in soft environments [33]; Transport phenomena of proteins in cellular environments [10, 27, 35, 45]; Ion movement in dendritic spines [28, 50], and RNA molecules within cells [31]. Some interesting works can be found in [2, 3, 4, 5, 6, 7, 8, 9, 63, 64, 65].

Let T be a positive constant, $\gamma \in (\frac{1}{2}, 1]$ and $J := (0, T)$. Let D be a bounded subset of $\mathbb{R}^d$, with $d \geq 1$, and possess a smooth boundary when $d > 1$. Let $H := L^2(D)$ and $A = -\Delta$ be the negative Laplacian. Without loss of generality, the operator A is assumed to possess eigenpairs (λ_k, e_k) satisfying $0 < \lambda_k \nearrow \infty$ and $Ae_k = \lambda_k e_k$, for every $k \geq 1$.

Let $\{W(t)\}_{t\in\overline{J}}$ be a standard Brownian motion (sBm) taking value in H (see Section 2 for more details). This article is concerned with the following problem for a stochastic conformable diffusion equation (SCDE) perturbed by sBm

$$\begin{cases} \left({}^{\mathcal{C}}\partial_t^{\gamma} + A\right) u(t) = f(t) + B(t)\dot{W}(t), & t \in J, \\ u(t)\big|_{\partial D} = 0, \quad u(0) = \varphi, & t \in J, \end{cases} \tag{1}$$

where $\dot{W}(t) = \partial W(t)/\partial t$ describes a white noise, $\varphi : \Omega \to H$ is known as the initial value, $f : \overline{J} \times \Omega \to H$ is called the source function, and B is an operator coming from $\overline{J}$ to L_0^2 defined in Section 2. The notation ${}^{\mathcal{C}}\partial_t^{\gamma}$, with $\gamma \in (\frac{1}{2}, 1]$, stands for the conformable derivative [36] defined as

$${}^{\mathcal{C}}\partial_t^{\gamma} g(t) := \lim_{\varepsilon\to 0} \frac{g(t + \varepsilon t^{1-\gamma}) - g(t)}{\varepsilon}, \quad t > 0,$$

which was invented by Khalil in [36] (refer to [1, 12, 29, 59] for some other works). Noting that if $\gamma = 1$ and there is no the appearance of the stochastic term $B(t)\dot{W}(t)$, then the SCDE in (1) becomes the normal deterministic diffusion equation, which was considered in [20, 21].

With the appearance of the conformable derivative, an alternative expression of the diffusion equation (called conformable diffusion equation) is proposed to improve the modeling of anomalous diffusion (see [59]). This new fractional derivative has been found to possess many successful applications of in many fields of science [11, 23, 32, 42, 44, 53, 54]. In recent time, the number of articles concerning with the deterministic conformable diffusion equation has increased significantly [1, 13, 14, 29, 39, 54, 56]. For some recent studies on diffusion equations with random noise, the readers can refer to [51, 52, 62].

Stochastic partial differential equations (SPDEs) are crucial issues modeling the phenomena in a lot of fields of science [19, 34, 46, 48]. Additionally, it is effective to use fractional differential equations (FDEs) to model some anomalous diffusion phenomena in physics, chemistry, engineering, etc. [22, 37, 47, 49, 57, 58]. The area of SFDEs is interesting to mathematicians since it contains various hard open problems [30, 40, 41, 61]. It is a fact that our considered equation in this paper is included in the topic of SFDEs. Notwithstanding the importance, as we know, there is no result concerning the initial value problem (or called forward problem) for SCDE (1). This motivated us to contribute the existence, regularity, and continuity results for Problem (1).

We now mention the organization of this paper. Section (2) introduces notations, functional spaces, and the definition of the mild solution. Section 3 is divided into two subsections. Subsection 3.1 gives some prior estimates for the terms appearing in the representation of the solution. In Subsection 3.2, the existence, regularity, and continuity results are stated.

2. Preliminaries

For $\varsigma > 0$, we define by $\dot{H}^\varsigma$ the following space

$$\dot{H}^\varsigma := \Big\{ h \in H : \|h\|_{\dot{H}^\varsigma} := \Big(\sum_{k\geq 1} \lambda_k^{2\varsigma} |(h, e_k)|^2 \Big)^{1/2} < \infty \Big\},$$

and denote by $\dot{H}^{-\varsigma}$ its dual space. We define by $A^\sigma : \dot{H}^{\varsigma/2} \to \dot{H}^{-\varsigma/2}$ [17, 26] the following operator

$$A^\varsigma v = \sum_{k\geq 1} \lambda_k^\varsigma (v, e_k) e_k, \quad \text{for } v \in \dot{H}^{\varsigma/2}.$$

Let Q be a covariance operator [25, 60] on H with finite trace, i.e. $Tr(Q) = \sum_{k\geq 1} \chi_k < \infty$, and satisfy $Qe_k = \chi_k e_k$, where $\{\chi_k\}_{k\geq 1}$ is the spectrum of Q. Assume that $\{\dot{W}(t)\}_{t\in \overline{J}}$ be an H-valued Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, with covariance operator Q and posses the following representation (see [25, 60])

$$W(t) = \sum_{k\geq 1} Q^{\frac{1}{2}} e_k \xi_k(t) = \sum_{k\geq 1} \chi_k^{\frac{1}{2}} e_k \xi_k(t),$$

where $\xi_k(t)$ are independent one-dimensional Brownian motions. Let $L^2(\Omega, \dot{H}^\varsigma)$ be the Bochner space defined by

$$\mathbb{E}\, \|\varrho\|^2_{H^\varsigma} := \int_\Omega \|\varrho(\omega)\|^2_{H^\varsigma}\, d\mathbb{P}(\omega) < \infty, \quad \omega \in \Omega.$$

Let $\mathcal{L}(H, \dot{H}^\varsigma)$ be the space of bounded linear operators $\mathbf{T}$ coming from H to $\dot{H}^\varsigma$. Let $\mathcal{HS}(H, \dot{H}^\varsigma)$ be the space of all operators $\mathbf{T} \in \mathcal{L}(H, \dot{H}^\varsigma)$ such that $\|\mathbf{T}\|_{\mathcal{HS}(H,\dot{H}^\varsigma)} := \Big(\sum_{k\geq 1} \|\mathbf{T} e_k\|^2_{\dot{H}^\varsigma} \Big)^{1/2} < \infty$. By $L_0^2(H, \dot{H}^\varsigma)$ we denote the space of $\mathbf{T} \in \mathcal{HS}(H, \dot{H}^\varsigma)$ such that

$$\|\mathbf{T}\|_{L_0^2(H,\dot{H}^\varsigma)} := \|\mathbf{T} Q^{\frac{1}{2}}\|_{\mathcal{HS}(H,\dot{H}^\varsigma)} < \infty.$$

In the case $\varsigma = 0$, it is obvious that $\dot{H}^0 = H$. For short, we denote $\mathcal{L}(H) := \mathcal{L}(H, H)$, $L_0^2 := L_0^2(H, H)$.

Given a Hilbert space K. For $p \geq 1$, we define the following space

$$L^p(J; K) := \Big\{ \psi : J \to K : \|\psi\|_{L^p(J;K)} := \Big(\int_0^T \|\psi(t)\|_K^p\, dt \Big)^{\frac{1}{p}} < \infty \Big\}.$$

Let $C(\overline{J}; K)$ be space of continuous functions $\psi : \overline{J} \to K$ such that

$$\|\psi\|_{C(\overline{J};K)} := \sup_{t\in\overline{J}} \|\psi(t)\|_K < \infty.$$

For $\alpha > 0$, we recall the following subspace of $C(\overline{J}; K)$

$$C^\alpha(\overline{J}; K) := \Big\{ \psi \in C(\overline{J}; K) : \|v\|_{C^\alpha(\overline{J};K)} := \sup_{0\leq s<t\leq T} \frac{\|\psi(t) - \psi(s)\|_K}{|t-s|^\alpha} < \infty \Big\}.$$

Now, we aim find an expression for $u(t)$ in the form $u(t) = \sum_{k\geq 1}(u(t), e_k)e_k$. From the SCDE in (1), we immediately obtain

$$^{\mathcal{C}}\partial_t^\gamma (u(t), e_k) - \lambda_k(u(t), e_k) = (f(t), e_k) + B(t)\chi_k^{\frac{1}{2}}\xi_k(t).$$

By using the method in Theorem 5 of [29] and Theorem 3.3 of [39], we arrive at

$$(u(t), e_k) = e^{-\frac{t^\gamma \lambda_k}{\gamma}}(u(0), e_k) + \int_0^t z^{\gamma-1}e^{\frac{(z^\gamma - t^\gamma)\lambda_k}{\gamma}}(f(z), e_k)dz + \chi_k^{\frac{1}{2}}\int_0^t z^{\gamma-1}e^{\frac{(z^\gamma - t^\gamma)\lambda_k}{\gamma}}B(z)d\xi_k(z).$$

Using $u(0) = \varphi$, we obtain the following equation

$$u(t) = e^{-\frac{t^\gamma A}{\gamma}}\varphi + \int_0^t z^{\gamma-1}e^{\frac{(z^\gamma - t^\gamma)A}{\gamma}}f(z)dz + \int_0^t z^{\gamma-1}e^{\frac{(z^\gamma - t^\gamma)A}{\gamma}}B(z)dW(z). \tag{2}$$

Basing the above expression for $u(t)$, we define mild solution.

Definition 2.1. An H-valued process $\{u(t)\}_{t\in\overline{J}}$ is said to be a mild solution of (1) if for almost all $t \in \overline{J}$, it satisfies the equation (2) almost surely.

Let σ be a non-negative constant satisfying $\sigma < \frac{1}{2}$. The following assumptions are needed to establish our results.

(H1) $\varphi \in L^2(\Omega, H)$,

(H2) $f \in L^p(J; L^2(\Omega, H))$, for some $p > 2\gamma^{-1}$,

(H3) $B \in L^q(J; L^2(\Omega, L_0^2))$, for some $q > \max(\frac{2}{2\gamma-1}, \frac{2}{1-2\sigma})$,

(H1') $\varphi \in L^2(\Omega, \dot{H}^\mu)$, for some $\mu \in (0, 1]$,

(H2') $f \in L^p(J; L^2(\Omega, \dot{H}^\nu))$, for some $\nu \in (\frac{1}{2}, 1]$, $p > 2\gamma^{-1}$,

(H3') $B \in L^q(J; L^2(\Omega, L_0^2(H, \dot{H}^s)))$, for some $s \in (\frac{1}{2}, 1]$, $q > \max(\frac{2}{2\gamma-1}, \frac{2}{2s-1})$.

3. Main results

3.1. Some prior estimates

This subsection is aimed to give some prior estimates which will be used throughout this paper. From now on, we employ the notation $a_1 \lesssim a_2$ to describe $a_1 \leq Ca_2$, where C is a positive constant.

Lemma 3.1. *i) If f satisfies (H2), then there holds*

$$\mathbb{E}\left\|\int_0^t z^{\gamma-1}e^{\frac{(z^\gamma - t^\gamma)A}{\gamma}}f(z)dz\right\|_{\dot{H}^\sigma}^2 \lesssim \|f\|_{L^p(J;L^2(\Omega,H))}^2. \tag{3}$$

ii) If f satisfies (H2'), then there holds

$$\mathbb{E}\left\|\int_0^{t+\delta} z^{\gamma-1}e^{\frac{(z^\gamma - (t+\delta)^\gamma)A}{\gamma}}f(z)dz - \int_0^t z^{\gamma-1}e^{\frac{(z^\gamma - t^\gamma)A}{\gamma}}f(z)dz\right\|_H^2 \lesssim \delta^{2(\gamma-\frac{1}{p})}\|f\|_{L^p(J;L^2(\Omega,H^\nu))}^2, \tag{4}$$

where $\delta > 0$ is small enough.

Proof. i) By applying the inequality $e^{-y} \lesssim y^{-2\sigma}$ with $y = \frac{(z^\gamma - t^\gamma)\lambda_k}{\gamma}$, we have

$$e^{\frac{(z^\gamma - t^\gamma)\lambda_k}{\gamma}} \lesssim \left[\frac{(t^\gamma - z^\gamma)\lambda_k}{\gamma}\right]^{-2\sigma} \lesssim \lambda_k^{-2\sigma}(t^\gamma - z^\gamma)^{-2\sigma}. \tag{5}$$

Now, we aim to show the following estimate holds for every $k \geq 1$

$$\mathcal{E}_k(t) := \mathbb{E}\left|\int_0^t z^{\gamma-1} e^{\frac{(z^\gamma - t^\gamma)\lambda_k}{\gamma}} (f(z), e_k) dz\right|^2 \lesssim \lambda_k^{-2\sigma} \int_0^t z^{\gamma-1}\mathbb{E}|(f(z), e_k)|^2 dz. \tag{6}$$

The Hölder inequality associated with the property (5) allow that

$$\begin{aligned}
\mathcal{E}_k(t) &\leq \left(\int_0^t z^{\gamma-1} e^{\frac{(z^\gamma - t^\gamma)\lambda_k}{\gamma}} dz\right)\left(\int_0^t z^{\gamma-1} e^{\frac{(z^\gamma - t^\gamma)\lambda_k}{\gamma}} \mathbb{E}|(f(z), e_k)|^2 dz\right) \\
&\lesssim \lambda_k^{-2\sigma}\left(\int_0^t z^{\gamma-1}(t^\gamma - z^\gamma)^{-2\sigma} dz\right)\left(\int_0^t z^{\gamma-1} e^{\frac{(z^\gamma - t^\gamma)\lambda_k}{\gamma}} \mathbb{E}|(f(z), e_k)|^2 dz\right) \\
&= \gamma^{-1}\lambda_k^{-2\sigma}\left(\int_0^{t^\gamma} (t^\gamma - z)^{-2\sigma} dz\right)\left(\int_0^t z^{\gamma-1} e^{\frac{(z^\gamma - t^\gamma)\lambda_k}{\gamma}} \mathbb{E}|(f(z), e_k)|^2 dz\right) \\
&\lesssim \gamma^{-1}\lambda_k^{-2\sigma}(1-2\sigma)^{-1} T^{(1-2\sigma)\gamma} \int_0^t z^{\gamma-1}\mathbb{E}|(f(z), e_k)|^2 dz,
\end{aligned}$$

where it should be noted that $1 - 2\sigma > 0$ since $\sigma \in (0, \frac{1}{2})$.

Next, we will use the property (6) to prove that (3) holds. Indeed, we have

$$\mathbb{E}\left\|\int_0^t z^{\gamma-1} e^{\frac{(z^\gamma - t^\gamma)A}{\gamma}} f(z) dz\right\|_{\dot{H}^\sigma}^2 = \sum_{k\geq 1} \lambda_k^{2\sigma}\mathcal{E}_k(t) \lesssim \int_0^t z^{\gamma-1}\mathbb{E}\left\|f(z)\right\|_H^2 dz. \tag{7}$$

In addition, by using the Hölder inequality again, we obtain

$$\begin{aligned}
\int_0^t z^{\gamma-1}\mathbb{E}\left\|f(z)\right\|_H^2 dz &\lesssim \left(\int_0^t z^{\frac{p}{p-2}(\gamma-1)} dz\right)^{\frac{p-2}{p}} \left(\int_0^t \left(\mathbb{E}\left\|f(z)\right\|_H^2\right)^{\frac{p}{2}} dz\right)^{\frac{2}{p}} \\
&\lesssim T^{\gamma - \frac{2}{p}} \left(\int_0^t \left\|f(z)\right\|_{L^2(\Omega,H)}^p dz\right)^{\frac{2}{p}},
\end{aligned} \tag{8}$$

Combining (7), (8), we now obtain the desired estimate (3).
ii) Firstly, for $\delta > 0$, we can see that

$$\begin{aligned}
&\int_0^{t+\delta} z^{\gamma-1} e^{\frac{(z^\gamma - (t+\delta)^\gamma)A}{\gamma}} f(z) dz - \int_0^t z^{\gamma-1} e^{\frac{(z^\gamma - t^\gamma)A}{\gamma}} f(z) dz \\
&= \int_0^t z^{\gamma-1}\Big(e^{\frac{(z^\gamma - (t+\delta)^\gamma)A}{\gamma}} - e^{\frac{(z^\gamma - t^\gamma)A}{\gamma}}\Big) f(z) dz + \int_t^{t+\delta} z^{\gamma-1} e^{\frac{(z^\gamma - (t+\delta)^\gamma)A}{\gamma}} f(z) dz \\
&=: \mathcal{Z}_1(t,\delta) + \mathcal{Z}_2(t,\delta).
\end{aligned} \tag{9}$$

The first term can be estimated by using the inequality $\left|e^{-a} - e^{-b}\right| \leq (a-b)e^{-b}$, for $0 \leq b \leq a$, as follows

$$\begin{aligned}
\mathbb{E}\left\|\mathcal{Z}_1(t,\delta)\right\|_H^2 &= \sum_{k\geq 1}\mathbb{E}\left|\int_0^t z^{\gamma-1}\Big(e^{-\frac{(t+\delta)^\gamma\lambda_k}{\gamma}} - e^{-\frac{t^\gamma\lambda_k}{\gamma}}\Big) e^{\frac{z^\gamma\lambda_k}{\gamma}} (f(z), e_k) dz\right|^2 \\
&\leq \sum_{k\geq 1} \lambda_k^2 \left|\frac{(t+\delta)^\gamma - t^\gamma}{\gamma}\right|^2 \mathbb{E}\left|\int_0^t z^{\gamma-1} e^{\frac{(z^\gamma - t^\gamma)\lambda_k}{\gamma}} (f(z), e_k) dz\right|^2 \\
&\lesssim \delta^{2\gamma}\sum_{k\geq 1}\lambda_k^2\mathbb{E}\left|\int_0^t z^{\gamma-1} e^{\frac{(z^\gamma - t^\gamma)\lambda_k}{\gamma}} (f(z), e_k) dz\right|^2.
\end{aligned} \tag{10}$$

By the Hölder inequality and noting that $e^{\frac{(z^\gamma-t^\gamma)\lambda_k}{\gamma}} \lesssim (t^\gamma-z^\gamma)^{-(1-\nu)}\lambda_k^{-(1-\nu)}$, we have

$$\begin{aligned}
&\lambda_k^2\mathbb{E}\left|\int_0^t z^{\gamma-1}e^{\frac{(z^\gamma-t^\gamma)\lambda_k}{\gamma}}(f(z),e_k)dz\right|^2\\
\le&\lambda_k^2\left(\int_0^t z^{\gamma-1}e^{\frac{2(z^\gamma-t^\gamma)\lambda_k}{\gamma}}dz\right)\left(\int_0^t z^{\gamma-1}\mathbb{E}|(f(z),e_k)|^2dz\right)\\
\lesssim&\left(\int_0^t z^{\gamma-1}(t^\gamma-z^\gamma)^{-2(1-\nu)}dz\right)\left(\int_0^t z^{\gamma-1}\lambda_k^{2\nu}\mathbb{E}|(f(z),e_k)|^2dz\right). \qquad (11)
\end{aligned}$$

Combining (10), (11), we obtain

$$\begin{aligned}
\mathbb{E}\left\|\mathcal{Z}_1(t,\delta)\right\|_H^2 &\lesssim \delta^{2\gamma}\left(\int_0^t z^{\gamma-1}(t^\gamma-z^\gamma)^{-2(1-\nu)}dz\right)\left(\int_0^t z^{\gamma-1}\mathbb{E}\left\|f(z)\right\|_{\dot H^\nu}^2 dz\right)\\
&\lesssim \delta^{2\gamma}\left(\int_0^{t^\gamma}(t^\gamma-z)^{-2(1-\nu)}dz\right)\left(\int_0^t z^{\frac{p}{p-2}(\gamma-1)}dz\right)^{\frac{p-2}{p}}\left(\int_0^t\left(\mathbb{E}\left\|f(z)\right\|_{\dot H^\nu}^2\right)^{\frac{p}{2}}dz\right)^{\frac{2}{p}}\\
&\lesssim \delta^{2\gamma}t^{\gamma(1-2(1-\nu))}t^{\gamma-\frac{2}{p}}\left(\int_0^t\left\|f(z)\right\|_{L^2(\Omega,\dot H^\nu)}^p dz\right)^{\frac{2}{p}}\\
&\lesssim \delta^{2\gamma}T^{\gamma(2\nu-\frac{2}{p})}\left\|f\right\|_{L^p(J;L^2(\Omega,\dot H^\nu))}^2. \qquad (12)
\end{aligned}$$

Next, we continue to estimate $\mathcal{Z}_2(t,\delta)$. We first see that

$$\mathbb{E}\left\|\mathcal{Z}_2(t,\delta)\right\|_H^2=\sum_{k\ge1}\mathbb{E}\left|\int_t^{t+\delta}z^{\gamma-1}e^{\frac{(z^\gamma-(t+\delta)^\gamma)\lambda_k}{\gamma}}(f(z),e_k)dz\right|^2. \qquad (13)$$

In addition, the Hölder inequality and $e^{\frac{(z^\gamma-(t+\delta)^\gamma)\lambda_k}{\gamma}}\le 1$ yield

$$\begin{aligned}
&\mathbb{E}\left|\int_t^{t+\delta}z^{\gamma-1}e^{\frac{(z^\gamma-(t+\delta)^\gamma)\lambda_k}{\gamma}}(f(z),e_k)dz\right|^2\\
\le&\left(\int_t^{t+\delta}z^{\gamma-1}e^{\frac{z^\gamma\lambda_k-(t+\delta)^\gamma\lambda_k}{\gamma}}dz\right)\left(\int_t^{t+\delta}z^{\gamma-1}e^{\frac{z^\gamma\lambda_k-(t+\delta)^\gamma\lambda_k}{\gamma}}\mathbb{E}|(f(z),e_k)|^2dz\right)\\
\le&\left(\int_t^{t+\delta}z^{\gamma-1}dz\right)\left(\int_t^{t+\delta}z^{\gamma-1}\mathbb{E}|(f(z),e_k)|^2dz\right), \qquad (14)
\end{aligned}$$

From (13) and (14) and noting that $\sum_{k\ge1}\mathbb{E}|(f(z),e_k)|^2\le\lambda_1^{-2\nu}\sum_{k\ge1}\lambda_k^{2\nu}\mathbb{E}|(f(z),e_k)|^2\lesssim\mathbb{E}\left\|f(z)\right\|_{\dot H^\nu}^2$, we deduce that

$$\begin{aligned}
\mathbb{E}\left\|\mathcal{Z}_2(t,\delta)\right\|_H^2&\le\left(\int_t^{t+\delta}z^{\gamma-1}dz\right)\left(\int_t^{t+\delta}z^{\gamma-1}\mathbb{E}\left\|f(z)\right\|_{\dot H^\nu}^2dz\right)\\
&\lesssim((t+\delta)^\gamma-t^\gamma)\left(\int_t^{t+\delta}z^{\gamma-1}\mathbb{E}\left\|f(z)\right\|_{\dot H^\nu}^2dz\right).
\end{aligned}$$

Using $(t+\delta)^\gamma-t^\gamma\le\delta^\gamma$ and a similar estimate as in (8), we arrive at

$$\begin{aligned}
\mathbb{E}\left\|\mathcal{Z}_2(t,\delta)\right\|_H^2&\lesssim\delta^\gamma\left(\int_t^{t+\delta}z^{\frac{p}{p-2}(\gamma-1)}dz\right)^{\frac{p-2}{p}}\left(\int_0^t\left(\mathbb{E}\left\|f(z)\right\|_{\dot H^\nu}^2\right)^{\frac{p}{2}}dz\right)^{\frac{2}{p}}\\
&\lesssim\delta^\gamma\left((t+\delta)^{\gamma-\frac{2}{p}}-t^{\gamma-\frac{2}{p}}\right)\left(\int_t^{t+\delta}\left\|f(z)\right\|_{L^2(\Omega,\dot H^\nu)}^pdz\right)^{\frac{2}{p}}\\
&\lesssim\delta^{2(\gamma-\frac{1}{p})}\left\|f\right\|_{L^p(J;L^2(\Omega,\dot H^\nu)))}^2. \qquad (15)
\end{aligned}$$

Now, combining (9), (12), (15), we conclude that the estimate (4) holds. □

Lemma 3.2. *i) If B satisfies Assumption (H3), then there holds*

$$\mathbb{E}\left\|\int_0^t z^{\gamma-1}e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}B(z)dW(z)\right\|^2_{\dot{H}^\sigma} \lesssim t^{\gamma(c_1+c_2-1)\frac{q-2}{q}}\left\|B\right\|^2_{L^q(J;L^2(\Omega,L_0^2))}, \tag{16}$$

where $c_1 := c_1(q,\gamma) = 1-\frac{(1-\gamma)(q+2)}{\gamma(q-2)}$ and $c_2 := c_2(q,\sigma) = 1-2\sigma\frac{q}{q-2}$.
ii) If B satisfies Assumption (H3'), then there holds

$$\begin{aligned}\mathbb{E}\bigg\|\int_0^{t+\delta} z^{\gamma-1}e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}}B(z)dW(z) - \int_0^t z^{\gamma-1}e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}B(z)dW(z)\bigg\|^2_H \\ \lesssim \delta^{2\gamma-\frac{q+2}{q}}\left\|B\right\|^2_{L^q(J;L^2(\Omega,L_0^2(H,\dot{H}^s)))},\end{aligned} \tag{17}$$

where $\delta > 0$ is small enough.

Proof. i) By a similar way as in (5), we also have $e^{\frac{(z^\gamma-t^\gamma)\lambda_k}{\gamma}} \lesssim \lambda_k^{-\sigma}(t^\gamma-z^\gamma)^{-\sigma}$, which follows that

$$\left\|A^\sigma e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}\right\|_{\mathcal{L}(H)} \lesssim (t^\gamma-z^\gamma)^{-\sigma}.$$

This together with the definition of the norm in L_0^2 yields

$$\mathbb{E}\left\|A^\sigma e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}B(z)\right\|^2_{L_0^2} = \sum_{k\geq 1}\mathbb{E}\left\|A^\sigma e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}B(z)Q^{\frac{1}{2}}e_k\right\|^2_H \lesssim (t^\gamma-z^\gamma)^{-2\sigma}\mathbb{E}\left\|B(z)\right\|^2_{L_0^2}.$$

By the Itô isometry and the above estimate, it is obvious that

$$\begin{aligned}\mathbb{E}\left\|\int_0^t z^{\gamma-1}e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}B(z)dW(z)\right\|^2_{\dot{H}^\sigma} &= \int_0^t z^{2(\gamma-1)}\mathbb{E}\left\|A^\sigma e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}B(z)\right\|^2_{L_0^2}dz \\ &\lesssim \int_0^t z^{2(\gamma-1)}(t^\gamma-z^\gamma)^{-2\sigma}\mathbb{E}\left\|B(z)\right\|^2_{L_0^2}dz.\end{aligned}$$

The Hölder inequality allows that

$$\begin{aligned}&\int_0^t z^{2(\gamma-1)}(t^\gamma-z^\gamma)^{-2\sigma}\mathbb{E}\left\|B(z)\right\|^2_{L_0^2}dz \\ &\quad\leq \left(\int_0^t z^{2(\gamma-1)\frac{q}{q-2}}(t^\gamma-z^\gamma)^{-2\sigma\frac{q}{q-2}}dz\right)^{\frac{q-2}{q}}\left(\int_0^t\left(\mathbb{E}\left\|B(z)\right\|^2_{L_0^2}\right)^{\frac{q}{2}}dz\right)^{\frac{2}{q}} \\ &\quad= \left(\int_0^t z^{\gamma-1}z^{(\gamma-1)\frac{q+2}{q-2}}(t^\gamma-z^\gamma)^{-2\sigma\frac{q}{q-2}}dz\right)^{\frac{q-2}{q}}\left(\int_0^t\left\|B(z)\right\|^q_{L^2(\Omega,L_0^2)}dz\right)^{\frac{2}{q}}.\end{aligned} \tag{18}$$

By using the substituting method to calculate the first integral, we arrive at

$$\begin{aligned}&\int_0^t z^{2(\gamma-1)}(t^\gamma-z^\gamma)^{-2\sigma}\mathbb{E}\left\|B(z)\right\|^2_{L_0^2}dz \\ &\quad\lesssim \gamma^{-\frac{q-2}{q}}\left(\int_0^{t^\gamma} z^{-\frac{(1-\gamma)(q+2)}{\gamma(q-2)}}(t^\gamma-z)^{-2\sigma\frac{q}{q-2}}dz\right)^{\frac{q-2}{q}}\left\|B\right\|^2_{L^q(J;L^2(\Omega,L_0^2))} \\ &\quad= \gamma^{-\frac{q-2}{q}}t^{\gamma(c_1+c_2-1)\frac{q-2}{q}}\beta(c_1,c_2)\left\|B\right\|^2_{L^q(J;L^2(\Omega,L_0^2))},\end{aligned} \tag{19}$$

where $\beta(.,.)$ is the beta function. It should be noted that $c_1, c_2 > 0$, since $q > \max(\frac{2}{2\gamma-1}, \frac{2}{1-2\sigma})$. Now, from three later estimates, we obtain (16) as desired.

ii) Let us turn our attention to estimate two terms defined as follows

$$\begin{aligned}&\int_0^{t+\delta} z^{\gamma-1} e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}} B(z)dW(z) - \int_0^t z^{\gamma-1} e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}} B(z)dW(z)\\ &= \int_0^t z^{\gamma-1}\Big(e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}} - e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}\Big)B(z)dW(z) + \int_t^{t+\delta} z^{\gamma-1} e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}} B(z)dW(z)\\ &=: \mathcal{I}_1(t,\delta) + \mathcal{I}_2(t,\delta). \end{aligned} \tag{20}$$

We begin to estimate the first term $\mathcal{I}_1(t,\delta)$ by using the Itô isometry as

$$\mathbb{E}\left\|\mathcal{I}_1(t,\delta)\right\|_H^2 = \int_0^t z^{2(\gamma-1)}\mathbb{E}\Big\|\Big(e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}} - e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}\Big)B(z)\Big\|_{L_0^2}^2 dz. \tag{21}$$

Consider the expectation under the integral sign, we can see

$$\mathbb{E}\Big\|\Big(e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}} - e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}\Big)B(z)\Big\|_{L_0^2}^2 = \sum_{k\geq 1}\mathbb{E}\Big\|\Big(e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}} - e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}\Big)B(z)Q^{\frac{1}{2}}e_k\Big\|_H^2. \tag{22}$$

The inequality $\left|e^{-a}-e^{-b}\right| \leq (a-b)e^{-b}$, for $0\leq b\leq a$, allows that

$$\Big|e^{\frac{(z^\gamma-(t+\delta)^\gamma)\lambda_k}{\gamma}} - e^{\frac{(z^\gamma-t^\gamma)\lambda_k}{\gamma}}\Big| = e^{\frac{z^\gamma\lambda_k}{\gamma}}\Big|e^{\frac{-(t+\delta)^\gamma\lambda_k}{\gamma}} - e^{\frac{-t^\gamma\lambda_k}{\gamma}}\Big| \leq \gamma^{-1}e^{\frac{(z^\gamma-t^\gamma)\lambda_k}{\gamma}}((t+\delta)^\gamma-t^\gamma)\lambda_k.$$

Since $e^{\frac{(z^\gamma-t^\gamma)\lambda_k}{\gamma}} \lesssim (t^\gamma-z^\gamma)^{-(1-s)}\lambda_k^{-(1-s)}$ and $(t+\delta)^\gamma - t^\gamma \leq \delta^\gamma$, we obtain

$$e^{\frac{(z^\gamma-(t+\delta)^\gamma)\lambda_k}{\gamma}} - e^{\frac{(z^\gamma-t^\gamma)\lambda_k}{\gamma}} \lesssim \delta^\gamma(t^\gamma-z^\gamma)^{-(1-s)}\lambda_k^s.$$

By using the above property, we can deduce that

$$\Big\|e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}} - e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}\Big\|_{\mathcal{L}(\dot{H}^s,H)} \lesssim \delta^\gamma(t^\gamma-z^\gamma)^{-(1-s)},$$

which leads to

$$\mathbb{E}\Big\|\Big(e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}} - e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}\Big)B(z)Q^{\frac{1}{2}}e_k\Big\|_H^2 \lesssim \delta^{2\gamma}(t^\gamma-z^\gamma)^{-2(1-s)}\mathbb{E}\Big\|B(z)Q^{\frac{1}{2}}e_k\Big\|_{\dot{H}^s}^2. \tag{23}$$

Combining (22) and (23), we obtain

$$\begin{aligned}\mathbb{E}\Big\|\Big(e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}} - e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}\Big)B(z)\Big\|_{L_0^2}^2 &\lesssim \delta^{2\gamma}(t^\gamma-z^\gamma)^{-2(1-s)}\sum_{k\geq 1}\mathbb{E}\Big\|B(z)Q^{\frac{1}{2}}e_k\Big\|_{\dot{H}^s}^2\\ &\lesssim \delta^{2\gamma}(t^\gamma-z^\gamma)^{-2(1-s)}\mathbb{E}\left\|B(z)\right\|_{L_0^2(H,\dot{H}^s)}^2. \end{aligned} \tag{24}$$

From (21) and (24), we deduce that

$$\mathbb{E}\left\|\mathcal{I}_1(t,\delta)\right\|_H^2 \lesssim \delta^{2\gamma}\int_0^t z^{2(\gamma-1)}(t^\gamma-z^\gamma)^{-2(1-s)}\mathbb{E}\left\|B(z)\right\|_{L_0^2(H,\dot{H}^s)}^2 dz.$$

By the same way as in (18)-(19), one can check that

$$\begin{aligned}\mathbb{E}\left\|\mathcal{I}_1(t,\delta)\right\|_H^2 &\lesssim \delta^{2\gamma}\gamma^{-\frac{q-2}{q}}t^{\gamma(c_1+c_3-1)\frac{q-2}{q}}\beta(c_1,c_3)\left\|B\right\|_{L^q(J;L^2(\Omega,L_0^2(H,\dot{H}^s)))}^2\\ &\lesssim \delta^{2\gamma}T^{\gamma(c_1+c_3-1)\frac{q-2}{q}}\left\|B\right\|_{L^q(J;L^2(\Omega,L_0^2(H,\dot{H}^s)))}^2, \end{aligned} \tag{25}$$

where $c_1 = 1 - \frac{(1-\gamma)(q+2)}{\gamma(q-2)}$, $c_3 = 1 - 2(1-s)\frac{q}{q-2}$. In the above estimate, it should be noted that $c_1, c_3 > 0$ and $c_1 + c_3 - 1 > 0$, since $q > \max(\frac{2}{2\gamma-1}, \frac{2}{2s-1})$.

Next, we continue to estimate the second term $\mathcal{I}_2(t,\delta)$ by using the Itô isometry as

$$\mathbb{E}\left\|\mathcal{I}_2(t,\delta)\right\|_H^2 = \int_t^{t+\delta} z^{2(\gamma-1)}\mathbb{E}\Big\|e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}}B(z)\Big\|_{L_0^2}^2 dz.$$

Since $e^{\frac{(z^\gamma-(t+\delta)^\gamma)\lambda_k}{\gamma}} \le 1$, one can check that

$$\mathbb{E}\Big\|e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}}B(z)\Big\|_{L_0^2}^2 \le \mathbb{E}\big\|B(z)\big\|_{L_0^2}^2 \le \lambda_1^{-2}\mathbb{E}\big\|B(z)\big\|_{L_0^2(H,\dot{H}^s)}^2.$$

This together with the Hölder inequality gives us

$$\begin{aligned}
\mathbb{E}\left\|\mathcal{I}_2(t,\delta)\right\|_H^2 &\lesssim \int_t^{t+\delta} z^{2(\gamma-1)}\mathbb{E}\big\|B(z)\big\|_{L_0^2(H,\dot{H}^s)}^2 dz \\
&\lesssim \left(\int_t^{t+\delta} z^{\frac{q(2\gamma-2)}{q-2}}dz\right)^{\frac{q-2}{q}} \left(\int_t^{t+\delta}\left(\mathbb{E}\big\|B(z)\big\|_{L_0^2(H,\dot{H}^s)}^2\right)^{\frac{q}{2}}dz\right)^{\frac{2}{q}} \\
&\lesssim \left((t+\delta)^{1+\frac{q(2\gamma-2)}{q-2}} - t^{1+\frac{q(2\gamma-2)}{q-2}}\right)^{\frac{q-2}{q}} \left(\int_t^{t+\delta}\big\|B(z)\big\|_{L^2(\Omega,L_0^2(H,\dot{H}^s))}^q dz\right)^{\frac{2}{q}} \\
&\lesssim \delta^{2\gamma-\frac{q+2}{q}}\big\|B\big\|_{L^q(J;L^2(\Omega,L_0^2(H,\dot{H}^s)))}^2. \qquad (26)
\end{aligned}$$

Now, from (20), (25) and (26), we obtain (17) as desired. □

Now, we are ready to state our main results.

3.2. The existence, regularity and continuity

Firstly, let us state the following theorem which shows an existence, uniqueness, regularity result on $L^r(J;L^2(\Omega,\dot{H}^\sigma))$.

Theorem 3.1. *Let $\sigma \ge 0$ and φ, f, B satisfy Assumptions (H1), (H2), (H3) respectively. Let $r \ge 1$ and satisfy the following conditions*

$$\sigma\gamma r < 1 \quad \textit{and} \quad \gamma r(c_1 + c_2 - 1) > -\frac{2q}{q-2}, \qquad (27)$$

where c_1, c_2 are defined in Lemma 3.2. Then, Problem (1) *has a unique solution in $L^r(J;L^2(\Omega,\dot{H}^\sigma))$ satisfying*

$$\|u\|_{L^r(J;L^2(\Omega,\dot{H}^\sigma))} \lesssim \|\varphi\|_{L^2(\Omega,H)} + \|f\|_{L^p(J;L^2(\Omega,H))} + \|B\|_{L^q(J;L^2(\Omega,L_0^2))}. \qquad (28)$$

Corollary 3.1 (The case of $\sigma = 0$)**.** *Let $\varphi \in L^2(\Omega,H)$, $f \in L^p(J;L^2(\Omega,H))$, for some $p > 2\gamma^{-1}$, and $B \in L^q(J;L^2(\Omega,L_0^2))$, for some $q > \frac{2}{2\gamma-1}$. Then, for any $r \ge 1$, $u \in L^r(J;L^2(\Omega,H))$ and satisfies*

$$\|u\|_{L^r(J;L^2(\Omega,H))} \lesssim \|\varphi\|_{L^2(\Omega,H)} + \|f\|_{L^p(J;L^2(\Omega,H))} + \|B\|_{L^q(J;L^2(\Omega,L_0^2))}.$$

Proof of Theorem 3.1. Since $e^{-\frac{t^\gamma\lambda_k}{\gamma}} \lesssim \lambda_k^{-\sigma}t^{-\gamma\sigma}$, for every $k \ge 1$, we can see $\big\|e^{-\frac{t^\gamma A}{\gamma}}\big\|_{\mathcal{L}(H,\dot{H}^\sigma)} \lesssim t^{-\gamma\sigma}$. It follows that $\big\|e^{-\frac{t^\gamma A}{\gamma}}\varphi\big\|_{L^2(\Omega,\dot{H}^\sigma)}^2 \lesssim t^{-2\gamma\sigma}\|\varphi\|_{L^2(\Omega,H)}^2$. This allows us to obtain

$$\Big\|e^{-\frac{t^\gamma A}{\gamma}}\varphi\Big\|_{L^r(J;L^2(\Omega,\dot{H}^\sigma))}^r \lesssim \|\varphi\|_{L^2(\Omega,H)}\int_0^T t^{-\gamma\sigma r}dt \lesssim T^{1-\gamma\sigma r}\|\varphi\|_{L^2(\Omega,H)}. \qquad (29)$$

Applying part i) of Lemma 3.1, we can estimate the second term as

$$\begin{aligned}\left\|\int_0^t z^{\gamma-1}e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}f(z)dz\right\|^r_{L^r(J;L^2(\Omega,\dot H^\sigma))} &= \int_0^T\left\|\int_0^t z^{\gamma-1}e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}f(z)dz\right\|^r_{L^2(\Omega,\dot H^\sigma)}dt\\ &\lesssim T\,\|f\|^r_{L^p(J;L^2(\Omega,H))}. \end{aligned}\tag{30}$$

The last term can be estimated by applying part i) of Lemma 3.2 as

$$\begin{aligned}\left\|\int_0^t z^{\gamma-1}e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}B(z)dW(z)\right\|^r_{L^r(J;L^2(\Omega,\dot H^\sigma))} &= \int_0^T\left\|\int_0^t z^{\gamma-1}e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}B(z)dW(z)\right\|^r_{L^2(\Omega,\dot H^\sigma)}dt\\ &\lesssim \|B\|^r_{L^q(J;L^2(\Omega,L_0^2))}\int_0^T t^{\gamma r(c_1+c_2-1)\frac{q-2}{2q}}dt\\ &\lesssim T^{\gamma r(c_1+c_2-1)\frac{q-2}{2q}+1}\,\|B\|^r_{L^q(J;L^2(\Omega,L_0^2))}. \end{aligned}\tag{31}$$

From (2) and (29)-(31), we deduce that

$$\|u\|_{L^r(J;L^2(\Omega,\dot H^\sigma))}\lesssim \|\varphi\|_{L^2(\Omega,H)}+\|f\|_{L^p(J;L^2(\Omega,H))}+\|B\|_{L^q(J;L^2(\Omega,L_0^2))}.$$

We now complete the proof. □

Proof of Corollary 3.1. This result can be proved easily by applying Theorem 3.1. Here, it should be noted that, in the case of $\sigma=0$, the conditions in (27) always hold. □

Next, we state another main result considered in the space $C^\alpha(\overline{J};L^2(\Omega,H))$.

Theorem 3.2. *Let φ,f,B satisfy Assumptions (H1'), (H2'), (H3') respectively. Then, Problem* (1) *has a unique solution in $C^\alpha(\overline{J};L^2(\Omega,H))$, with $\alpha=\min\left(\gamma\mu,\gamma-\frac{q+2}{2q}\right)$. Furthermore, the following regularity property holds*

$$\|u\|_{C^\alpha(\overline{J};L^2(\Omega,H))}\lesssim \|\varphi\|_{L^2(\Omega,\dot H^\mu)}+\|f\|_{L^p(J;L^2(\Omega,\dot H^\nu))}+\|B\|_{L^q(J;L^2(\Omega,L_0^2(H,\dot H^s)))}.\tag{32}$$

Proof. To show the regularity property (32) holds, we bound the first term in (2) firstly. Using $|e^{-a}-e^{-b}|\lesssim b^\mu-a^\mu$, for $0<b<a$, we can see

$$e^{-\frac{(t+\delta)^\gamma\lambda_k}{\gamma}}-e^{-\frac{t^\gamma\lambda_k}{\gamma}}\lesssim \lambda_k^\mu((t+\delta)^{\gamma\mu}-t^{\gamma\mu})\le \lambda_k^\mu\delta^{\gamma\mu},$$

which leads to

$$\left\|\left(e^{-\frac{(t+\delta)^\gamma A}{\gamma}}-e^{-\frac{t^\gamma A}{\gamma}}\right)\varphi\right\|^2_{L^2(\Omega,H)}\lesssim \delta^{2\gamma\mu}\mathbb{E}\,\|\varphi\|^2_{\dot H^\mu}=\delta^{2\gamma\mu}\,\|\varphi\|^2_{L^2(\Omega,\dot H^\mu)}.\tag{33}$$

Part ii) of Lemma 3.1 and Part ii) of Lemma 3.2 imply that

$$\left\|\int_0^{t+\delta} z^{\gamma-1}e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}}f(z)dz-\int_0^t z^{\gamma-1}e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}f(z)dz\right\|_{L^2(\Omega,H)}\lesssim \delta^{\gamma-\frac{1}{p}}\,\|f\|_{L^p(J;L^2(\Omega,H^\nu))},\tag{34}$$

and

$$\begin{aligned}&\left\|\int_0^{t+\delta} z^{\gamma-1}e^{\frac{(z^\gamma-(t+\delta)^\gamma)A}{\gamma}}B(z)dW(z)-\int_0^t z^{\gamma-1}e^{\frac{(z^\gamma-t^\gamma)A}{\gamma}}B(z)dW(z)\right\|_{L^2(\Omega,H)}\\ &\qquad\lesssim \delta^{\gamma-\frac{q+2}{2q}}\left\|B\right\|_{L^q(J;L^2(\Omega,L_0^2(H,\dot H^s)))}.\end{aligned}\tag{35}$$

Now, combining (2), (33)-(35), we conclude

$$\|u(t+\delta)-u(t)\|_{L^2(\Omega,H)}\lesssim \delta^\alpha\left(\|\varphi\|_{L^2(\Omega,\dot H^\mu)}+\|f\|_{L^p(J;L^2(\Omega,H^\nu))}+\left\|B\right\|_{L^q(J;L^2(\Omega,L_0^2(H,\dot H^s)))}\right),$$

which follows the regularity property (32). □

Remark 3.1 (The continuity with respect to the initial function). *If u_1, u_2 are the solutions to Problem* (1) *with respect to $\varphi_1, \varphi_2 \in L^2(\Omega, \dot{H}^\mu)$ with $\mu \in (0,1]$ then*

$$\|u_1 - u_2\|_{C(\overline{J};L^2(\Omega,H))} \lesssim \|\varphi_1 - \varphi_2\|_{L^2(\Omega,\dot{H}^\mu)}. \tag{36}$$

Proof. It is clear to see that

$$\|u_1(t) - u_2(t)\|^2_{L^2(\Omega,H)} = \left\|e^{-\frac{t^\gamma A}{\gamma}}(\varphi_1 - \varphi_2)\right\|^2_{L^2(\Omega,H)} \le \left\|A^{-\mu}A^\mu(\varphi_1 - \varphi_2)\right\|^2_{L^2(\Omega,H)}.$$

Using the fact that $\|A^{-\mu}\rho\|^2_{L^2(\Omega,H)} = \sum_{k\ge1} \lambda_k^{-2\mu}|(\rho, e_k)|^2 \le \lambda_1^{-2\mu}\sum_{k\ge1}|(\rho, e_k)|^2 = \lambda_1^{-2\mu}\|\rho\|^2_{L^2(\Omega,H)}$, for $\rho \in L^2(\Omega, H)$, one obtains

$$\|u_1(t) - u_2(t)\|^2_{L^2(\Omega,H)} \le \lambda_1^{-2\mu}\left\|A^\mu(\varphi_1 - \varphi_2)\right\|^2_{L^2(\Omega,H)} = \lambda_1^{-2\mu}\left\|\varphi_1 - \varphi_2\right\|^2_{L^2(\Omega,\dot{H}^\mu)},$$

which leads to the continuity results (36) □

4. Conclusion

In the present article, a forward problem (or called initial value problem) for a sub-diffusion equation perturbed by standard Brownian motion is investigated. With the support of stochastic analysis, we obtain some sufficient conditions ensuring the existence, regularity and continuity of the mild solution of such problem.

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