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On the Spherical Indicatrices of a Timelike Curve as Generalized Helices in Minkowski 3-Space

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Abstract

In the present paper, we investigate new properties of the spherical indicatrices of a timelike curve in Minkowski 3-space $\mathbb{E}_1^3$. We focus on the conditions of the spherical indicatrix to be a generalized helix depending on its causal character. We also give some integral equations by defining the axis of the helix in the means of the local frame.

Keywords: *Timelike curves; spherical indicatrices; Integral equations, Minkowski space.*

2010 Mathematics Subject Classification: *53A35, 53B30, 53C50.*

1. Introduction

A passion with helical configuration in nature has long been maintained by natural scientists, often touching on supernatural obsession. Helices can be found in nanosprings, carbon nanotubes, DNA, sea shells and many more [11]. From the view of differential geometry, a generalized helix in Euclidean 3-space is defined by the property that the tangent makes a constant angle with a fixed straight line called the axis of the helix [1]. A classical result has been stated by M.A. Lancret in 1982 and first proved by B. de Saint Venant in 1845 (see [17]). For more understanding about the generalized helices in Euclidean space, we refer to some recent books such as [8] and [5].

On the other hand the theory of differential geometry in Minkowski space plays an important role both in mathematics and physics. One of its major role can be found in the development of general relativity theory and gravitation. There exist many mathematicians and physicists studying about theory of curves in Minkowski space such as [10] [12], [7] and [15]. Locally, curves in Minkowski space are categorized in three types based on their causal characteristics. A curve is timelike, spacelike or lightlike if the tangent vector on a point along the curves is timelike, spacelike or lightlike, respectively [13].

The Frenet-Serret frame on curves in a Lorentzian manifold helps many physicist in learning the motion of an accelerated particle. For example, Sygne [18] applied the formalism to investigate intrinsic geometric properties of the world lines of charged particles placed in an electromagnetic field and it shows a constant and uniform electromagnetic field of the point charged describes a timelike helix in Minkowski space. Theory of helices in Minkowski space can be seen in [6].

One of the theories in Riemannian geometry which can be extended to the semi-Riemannian spaces is the spherical indicatrices of regular curves. The idea has been existed for long time ago to the time of Gauss. The idea is that, one might construct and examine the related spherical indicatrix if there is a group of sets including lines in space in some organized relationship with others [14]. Theory of spherical indicatrix of curves in Minkowski space can be found in [1], [4] and [16] while in the case of Lorentz-Minkoski space can be seen in [3]. Furthermore, the casual characterization of timelike curves in Minkowski 3-space can be seen in [2].

In this study, we investigate the conditions for spherical indicatrices of a timelike curve to be generalized helix depending on their casual characters. For this, we express the axis of the helix in the means of the local frame at every point on the timelike curve and give some integral equations.

2. Preliminaries

Minkowski space $\mathbb{E}_1^3$ is the real vector space $\mathbb{R}^3$ equipped with the standard Lorentzian metric $\langle , \rangle$ defined by

$$\langle x,y\rangle = -x_1y_1 + x_2y_2 + x_3y_3, \tag{2.1}$$

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for any vectors $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$. The cross product in Minkowski 3-space is defined as

$$x\times y=(x_3y_2-x_2y_3,x_3y_1-x_1y_3,x_1y_2-x_2y_3). \tag{2.2}$$

In Minkowski space, any vector v can have only one of three following causal characters: timelike if $\langle v,v\rangle<0$, spacelike if $\langle v,v\rangle>0$ or $v=0$ and null (lightlike) if $\langle v,v\rangle=0$ and $v\neq 0$. The norm of a vector in $\mathbb{E}_1^3$ is defined by $\|v\|=\sqrt{|\langle v,v\rangle|}$.
Let $\alpha:I\to\mathbb{E}_1^3$ be a curve in Minkowski 3-space. Locally, α can be timelike, spacelike or null if its tangent vector is timelike, spacelike or null, respectively. For non-null curves, the arc length s is defined by $s=\int_0^t\sqrt{|\langle\alpha',\alpha'\rangle|}dt$. For null curves, the pseudo-arclength is defined by $s=\int_0^t\langle\alpha'',\alpha''\rangle^{\frac{1}{4}}dt$.
Denote $\{T(s),N(s),B(s)\}$ the moving Frenet frame along the curves $\alpha(s)$ which are respectively called, the tangent, the principal normal and binormal vector fields. Then, If α is a timelike curve then

$$T'=\kappa N,\quad N'=\kappa T+\tau B,\quad B'=-\tau N \tag{2.3}$$

where

$$\langle T,T\rangle=-1,\quad \langle N,N\rangle=\langle B,B\rangle=1,\quad \langle T,N\rangle=\langle B,N\rangle=\langle T,B\rangle=0$$

and

$$T\times N=-B\quad N\times B=T\quad B\times T=-N$$

[10].
Let $C:I\to\mathbb{E}_1^3$ be a regular timelike curve parametrized by arc length with Frenet frame $\{T,N,B\}$. The curve $\alpha(s)=T(s)$ on the sphere of radius 1 about the origin is called the tangent indicatrix of $C(s)$. Similarly, $\beta(s)=N(s)$ and $\gamma(s)=B(s)$ are respectively called the principal normal indicatrix and binormal indicatrix of C(s).

Theorem 2.1. *[2] Let $\alpha:I\to\mathbb{E}_1^3$ be a tangent indicatrix of timelike curve C parametrized by arc length s. Then α is a spacelike curve and the tangent vector T^* of α equals to the normal vector N of C.*

Theorem 2.2. *[2] Let $\beta:I\to\mathbb{E}_1^3$ be a principal normal indicatrix of timelike curve C parametrized by arclength s. Then α is a spacelike or timelike curve and the tangent vector T^* of β can be expressed by*

$$T^*=\frac{\kappa}{\sqrt{|\lambda|}}T+\frac{\tau}{\sqrt{|\lambda|}}B \tag{2.4}$$

where $\lambda=-\kappa^2+\tau^2$, T and B are the tangent and binormal vector fields of C, respectively.

Theorem 2.3. *[2] Let $\gamma:I\to\mathbb{E}_1^3$ be a binormal indicatrix of timelike curve parametrized by arclength s. Then γ is a spacelike curve and the tangent vector T^* of γ equals to the normal vector N of C.*

3. Generalized Helix of Spherical Indicatrices of Timelike Curves

In this section, we give some conditions for the spherical indicatrices of timelike curve to be a generalized helix. We also express the axis of the helices in the means of local frame at every point.

Definition 3.1. *Let $\alpha(s)=T(s)$ be a tangent indicatrix of timelike curves in $\mathbb{E}_1^3$. If there exists a nonzero constant vector field V such that $\langle T^*,V\rangle$ is non zero constant where T^* is the tangent vector of $\alpha(s)$, then $\alpha(s)$ is said to be a generalized helix and V is called the axis of $\alpha(s)$.*

Theorem 3.2. *Let $r(s)$ be a timelike curve and $\alpha(s)=T(s)$ be the tangent indicatrix of r in $\mathbb{E}_1^3$. If $\alpha(s)$ is a generalized helix then it satisfies the integral equation*

$$\kappa\int\kappa ds-\tau\int\tau ds=0, \tag{3.1}$$

where κ and τ are curvature and torsion of $r(s)$, respectively.

Proof. Let $\alpha(s)$ be a generalized helix. Define a constant vector field V with respect to the local frame as follow

$$V=aT+bN+cB. \tag{3.2}$$

Since $\alpha(s)$ is a generalized helix, from theorem 2.1 we have $\langle T^*,V\rangle=\langle N,V\rangle=b=constant$.
If we take the derivative of V we obtain

$$V'=(a'+b\kappa)T+(a\kappa-c\tau)N+(b\tau+c')B.$$

It implies that

$$a'+b\kappa=0 \tag{3.3}$$

$$a\kappa-c\tau=0, \tag{3.4}$$

$$b\tau + c' = 0. \tag{3.5}$$

From equations (3.3) and (3.5) we have

$$a = -b\int \kappa\, ds$$

and

$$c = -b\int \tau\, ds.$$

If we substitute last equations into equation (3.4), we get

$$\kappa\int \kappa\, ds - \tau\int \tau\, ds = 0 \tag{3.6}$$

□

Theorem 3.3. *Let $\alpha(s) = T(s)$ be the tangent indicatrix of a timelike curve $r(s)$. If $\alpha(s)$ is a generalized helix then the axis of $\alpha(s)$ is*

$$V(s) = -b\frac{\tau^3 - \kappa^2\tau}{\kappa'\tau - \kappa\tau'}T(s) + bN(s) + b\frac{\kappa^3 - \kappa\tau^2}{\kappa'\tau - \kappa\tau'}B(s).$$

Here, κ and τ are curvature and torsion of timelike curve $r(s)$, respectively and b is a non zero constant.

Proof. Taking the first derivative of equation (3.6) yields

$$\kappa'\int \kappa\, ds + \kappa^2 - \tau'\int \tau\, ds - \tau^2 = 0. \tag{3.7}$$

If we solve the equations (3.6) and (3.7), we get

$$\int \kappa\, ds = \frac{\tau^3 - \kappa^2\tau}{\kappa'\tau - \kappa\tau'}$$

and

$$\int \tau\, ds = \frac{\tau^2\kappa - \kappa^3}{\kappa'\tau - \kappa\tau'} = -\frac{\kappa^3 - \tau^2\kappa}{\kappa'\tau - \kappa\tau'}.$$

Therefore, we have

$$a = -b\frac{\tau^3 - \kappa^2\tau}{\kappa'\tau - \kappa\tau'} \quad c = b\frac{\kappa^3 - \tau^2\kappa}{\kappa'\tau - \kappa\tau'}.$$

Substituting a, b, and c into equation (3.2) yields

$$V(s) = -b\frac{\tau^3 - \kappa^2\tau}{\kappa'\tau - \kappa\tau'}T(s) + bN(s) + b\frac{\kappa^3 - \kappa\tau^2}{\kappa'\tau - \kappa\tau'}B(s).$$

□

Definition 3.4. *Let $\beta(s) = N(s)$ be a non-null curve as the principal normal indicatrix of timelike curves in $\mathbb{E}_1^3$. If there exists a nonzero constant vector field V such that $\langle T^*, V\rangle$ is non zero constant where T^* is the tangent vector of $\beta(s)$, then $\beta(s)$ is said to be a generalized helix and V is called the axis of $\beta(s)$.*

Theorem 3.5. *Let $\beta(s) = N(s)$ be a principal normal indicatrix of timelike curve $r(s)$. If β is a generalized helix then it satisfies the following integral equation*

$$\kappa\int\int p\kappa\sqrt{|\lambda|}\, ds + \tau\int\int p\tau\sqrt{|\lambda|}\, ds^2 = p\sqrt{|\lambda|} \tag{3.8}$$

and its axis can be expressed by

$$V = -\int\int p\kappa\sqrt{|\lambda|}\, ds^2 T + \int p\sqrt{|\lambda|}\, ds N + \int\int p\tau\sqrt{|\lambda|}\, ds^2 B \tag{3.9}$$

for some non zero constant p.

Proof. From theorem 2.2, we have

$$T^* = \frac{\kappa}{\sqrt{|\lambda|}}T + \frac{\tau}{\sqrt{|\lambda|}}B.$$

Let we define the axis of α by

$$V = aT + bN + cB. \tag{3.10}$$

Since $\langle T^*, V\rangle = constant$ we get $-a\kappa + c\tau = p\sqrt{|\lambda|}$ for some non zero constant p. Taking the first derivative of V yields

$$\begin{aligned} V' &= a'T + a(\kappa N) + b'N + b(\kappa T + \tau B) + c'B + c(-\tau N) \\ &= (a' + b\kappa)T + (a\kappa + b' - c\tau)N + (b\tau - c')B. \end{aligned}$$

Therefore,

$$a' + b\kappa = 0 \tag{3.11}$$

$$a\kappa + b' - c\tau = 0 \tag{3.12}$$

$$b\tau - c' = 0. \tag{3.13}$$

Using the equation (3.12) we have

$$b' = p\sqrt{|\lambda|} \implies b = \int p\sqrt{|\lambda|}\,ds.$$

From the equations (3.11) and (3.13), we find

$$a = -\int\int b\kappa\,ds^2 = -\int\int p\kappa\sqrt{|\lambda|}\,ds^2$$

and

$$c = \int\int b\tau\,ds^2 = \int\int p\tau\sqrt{|\lambda|}\,ds^2.$$

Substituting, a, b and c into equation (3.12) yields equation (3.8) and substituting a, b and c into equation (3.10) obtain equation (3.9). □

Theorem 3.6. *Let $\beta(s) = N(s)$ be a non-null indicatrix of timelike curve $r(s)$. If β is a helix 1-type then it satisfies the equations*

$$\int\int p\tau\sqrt{|\lambda|}\,ds^2 = \xi\left(p\kappa'\sqrt{|\lambda|} - p\kappa(\sqrt{|\lambda|})' + \kappa^2\phi + \kappa\tau\eta\right)$$

and

$$\int\int p\kappa\sqrt{|\lambda|}\,ds^2 = -\xi\left(p\tau'\sqrt{|\lambda|} - p\tau(\sqrt{|\lambda|})' + \kappa\tau\phi + \tau^2\eta\right),$$

where $\xi = \frac{1}{\kappa'\tau - \kappa\tau'}$.

Proof. From equation (3.8), we have

$$\kappa\int\int p\kappa\sqrt{|\lambda|}\,ds^2 + \tau\int\int p\tau\sqrt{|\lambda|}\,ds^2 = p\sqrt{|\lambda|}.$$

If we take derivative of the above equation we get

$$\kappa'\int\int p\kappa\sqrt{|\lambda|}\,ds^2 + \tau'\int\int p\tau\sqrt{|\lambda|}\,ds^2 = p(\sqrt{|\lambda|})' - \kappa\int p\kappa\sqrt{|\lambda|}\,ds - \tau\int p\tau\sqrt{|\lambda|}\,ds. \tag{3.14}$$

By solving the equations (3.8) and (3.14), we find

$$\int\int p\tau\sqrt{|\lambda|}\,ds^2 = \frac{1}{\kappa'\tau - \kappa\tau'}\left(p\kappa'\sqrt{|\lambda|} - p\kappa(\sqrt{|\lambda|})' + \kappa^2\int p\kappa\sqrt{|\lambda|}\,ds + \kappa\tau\int p\tau\sqrt{|\lambda|}\,ds\right) \tag{3.15}$$

and

$$\int\int p\kappa\sqrt{|\lambda|}\,ds^2 = -\frac{1}{\kappa'\tau - \kappa\tau'}\left(p\tau'\sqrt{|\lambda|} - p\tau(\sqrt{|\lambda|})' + \kappa\tau\int p\kappa\sqrt{|\lambda|}\,ds + \tau^2\int p\tau\sqrt{|\lambda|}\,ds\right). \tag{3.16}$$

Let $\xi = \frac{1}{\kappa'\tau - \kappa\tau'}$, then we take derivative of the equation (3.15) and we obtain

$$\begin{aligned} \int p\tau\sqrt{|\lambda|}\,ds &= \left(\xi p\kappa'\sqrt{|\lambda|} - \xi p\kappa(\sqrt{|\lambda|})'\right)' + \left(p\xi\kappa\sqrt{|\lambda|}\right)(\kappa^2 + \tau^2) \\ &\quad + (\xi\kappa^2)'\int p\kappa\sqrt{|\lambda|}\,ds + (\xi\kappa\tau)'\int p\tau\sqrt{|\lambda|}\,ds. \end{aligned}$$

So that we have following equation

$$(\xi\kappa^2)'\int p\kappa\sqrt{|\lambda|}\,ds+((\xi\kappa\tau)'-1)\int p\tau\sqrt{|\lambda|}\,ds=\left(\xi p\kappa(\sqrt{|\lambda|})'-\xi p\kappa'\sqrt{|\lambda|}\right)' -\left(p\xi\kappa\sqrt{|\lambda|}\right)(\kappa^2+\tau^2) \tag{3.17}$$

Similarly, by using (3.16), we have following equation

$$((\xi\kappa\tau)'+1)\int p\kappa\sqrt{|\lambda|}\,ds+(\xi\tau^2)'\int p\tau\sqrt{|\lambda|}\,ds=\left(\xi p\tau(\sqrt{|\lambda|})'-\xi p\tau'\sqrt{|\lambda|}\right)' -\left(p\xi\tau\sqrt{|\lambda|}\right)(\kappa^2+\tau^2). \tag{3.18}$$

Solving the equations (3.17) and (3.18), we have

$$\int p\tau\sqrt{|\lambda|}\,ds=\frac{\left[\left(\xi p\kappa(\sqrt{|\lambda|})'-\xi p\kappa'\sqrt{|\lambda|}\right)'-\left(p\xi\kappa\sqrt{|\lambda|}\right)(\kappa^2+\tau^2)\right]((\xi\kappa\tau)'+1)}{((\xi\kappa\tau)')^2-1-(\xi\kappa^2)'(\xi\tau^2)'} -\frac{\left[\left(\xi p\tau(\sqrt{|\lambda|})'-\xi p\tau'\sqrt{|\lambda|}\right)'-\left(p\xi\tau\sqrt{|\lambda|}\right)(\kappa^2+\tau^2)\right](\xi\kappa^2)'}{((\xi\kappa\tau)')^2-1-(\xi\kappa^2)'(\xi\tau^2)'}.$$

We also find

$$\int p\tau\sqrt{|\lambda|}\,ds=\frac{\left[\left(\xi p\kappa(\sqrt{|\lambda|})'-\xi p\kappa'\sqrt{|\lambda|}\right)'-\left(p\xi\kappa\sqrt{|\lambda|}\right)(\kappa^2+\tau^2)\right](\xi\tau^2)'}{(\xi\kappa^2)'(\xi\tau^2)'-((\xi\kappa\tau)')^2+1} -\frac{\left[\left(\xi p\tau(\sqrt{|\lambda|})'-\xi p\tau'\sqrt{|\lambda|}\right)'-\left(p\xi\tau\sqrt{|\lambda|}\right)(\kappa^2+\tau^2)\right]((\xi\kappa\tau)'-1)}{(\xi\kappa^2)'(\xi\tau^2)'-((\xi\kappa\tau)')^2+1}.$$

Substituting last two equations into the equations (3.15) and (3.16) completes the proof. □

Definition 3.7. *Let $\gamma(s)=B(s)$ be the binormal indicatrix of timelike curves in $\mathbb{E}^3_1$. If there exists a nonzero constant vector field V such that $\langle T^*,V\rangle$ is non zero constant where T^* is the tangent vector of $\gamma(s)$, then $\gamma(s)$ is said to be a generalized helix and V is called the axis of $\gamma(s)$.*

Theorem 3.8. *Let $r(s)$ be a timelike curve and $\gamma(s)=B(s)$ be the binormal indicatrix of r in $\mathbb{E}^3_1$. Then $\gamma(s)$ is a generalized helix if $\kappa\int\kappa ds-\tau\int\tau ds=0$, where κ and τ are curvature and torsion of $r(s)$, respectively.*

Theorem 3.9. *Let $\gamma(s)=B(s)$ be the binormal indicatrix of a timelike curve $r(s)$. If $\gamma(s)$ is a generalized helix then the axis is*

$$V(s)=-b\frac{\tau^3-\kappa^2\tau}{\kappa'\tau-\kappa\tau'}T(s)+bN(s)+b\frac{\kappa^3-\kappa\tau^2}{\kappa'\tau-\kappa\tau'}B(s),$$

where b is a non zero constant.

The proof of Theorem 3.8 and Theorem 3.9 similar to the proof of Theorem 3.2 and Theorem 3.3.

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