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On pricing variance swaps in discretely-sampled with high volatility model

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Abstract

In this paper, we investigate valuation of discretely-sampled variance swaps in a financial asset price model with increase in volatility. More precisely, we consider a stochastic differential equation model with an additional parameter which augments volatility. This is to cover the impact of financial crunches on pricing a given asset. Under these settings, calculation of annualized delivery price of a variance swap is not sure in a closed form. Following the literature, the delivery price can be written as a finite sum of conditional expectations. We focus on the computation of these expectations and obtain some interesting results. This leads to a semi-analytical solution to the variance swaps pricing problems. We also show the advantage of our model.

Keywords: stochastic differential equations, discretely-sampled variance swaps, high volatility model.

1. Introduction

Managing the risk is one of the most important research topic in financial mathematics. Financial derivatives are central tools utilized in risk management. There are different types of derivatives such as options, futures, forwards, and variance swaps. We are interested in this paper on variance swaps which are built on the volatility. Volatility can be seen as an indicator of discrepancies in prices of a product. We can distinguish two sort of volatilities, namely the implied volatility and the realized volatility. The implied volatility is the current market price of the volatility, which acts like the unbiased volatility price built on the expectation of the marketplace for movement over a period of time.

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The realized volatility is obtained from the variations in the underlying price over a stated stage. It is called the historical volatility when the time is in the past and the realized volatility if it is in the future.

Variance and volatility swaps are forward contracts. The underlying is not a financial asset but the future realized volatility of its returns. Variance swaps are forward contracts like volatility swaps, but built on realized variance which is equal to the square of the future volatility. We are interested in this work on the valuation of variance swaps. These derivatives allow traders to buy or to sell a volatility just as buying or selling any given asset. They allow to buy or sell future realized volatility in contrast with the actual value of the implied volatility.

However, volatility and variance swaps are not regular swaps that consist of a simple swap of money movements. Actually, the payoff for a long position of a volatility swap at settlement equals to the annualized realized volatility over a given period minus the volatility strike of the contract times a notional amount of the swap in dollars per annualized volatility point. The payoff at maturity is Notional Amount × (Volatility˘Volatility Strike).

There exist many articles dealing with managing the risk, see for instance on portfolio optimization or on calculating a stochastic hedge ratio [8]. On the other hand, the literature contains an abundant amount of research papers on pricing financial derivatives. About pricing variance swaps with different models predicting the underlying asset price trajectory S_t, we refer to [1], [2], [11] and [10], [13], [14] and [15] for more details about suggested models and methods in the literature. To the best of our knowledge, pricing variance swap under a high volatility model has previously never been addressed. In this paper, we price the variance swap in discretely-sampled for markets with high volatility. The model considered in this work allows the underlying asset price to have a usual increase of volatility which is more general than normal situations. An augmented volatility implies a higher risk. This is the importance of considering such a model, which could cover markets with crunch. We study the crisis model introduced in [3] first. In [5], the authors provide a closed form solution for the European option price under a particular function $g(t)$ which represents the increase in the volatility. Sensitivities for prices of the same crisis model has been addressed in the work of [4]. Recently, option pricing under an illiquid with increased volatility model has been shown in [6]. Then, we investigate the valuation of variance swaps under a high volatility model where the market is under stress in this paper. More precisely, the problem of determining the fair price for discretely sampled variance swap is explored. This price is known to be written as sum of expectations and the valuation problem turns out to calculate these expectations. In this paper, we have obtained a partial formula for these expectations.

The paper is organized as follows. Section 2 briefly presents a review on discretely-sampled variance swaps. We investigate the valuation of variance swaps with an increased volatility and we obtain our main result on calculating the fair variance delivery price in Section 3. In Section 4, a numerical application of our obtained formula is performed with a comparison to the price during normal situations. Section 5 concludes the paper.

2. Discretely-sampled variance swaps

We first review an approach in the case of the Heston stochastic volatility model on finding the value of a discretely-sampled variance swap. The method provided here can be found in [13] and [14].

From now on, the following assumptions and notations are used (unless otherwise stated). We work on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{F}_T, Q)$, a standard Wiener process $(W_t)_{t\in[0,T]}$ with $\mathcal{F}_t := \sigma(W_t)$ for any $t \in [0, T]$ and a known risk-neutral probability Q. The underlying asset price is denoted by $(S_t)_{t\in[0,T]}$, the notional amount of the variance swap in dollars per annualized volatility point squared is L. Moreover, let σ_R be the realized volatility (in annual terms) of the underlying asset S computed using arithmetic return.

The formula for the realized variance defining the pay-off is then given by

$$\sigma_R^2 = \frac{AF}{N}\sum_{i=1}^{N}\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2 \times 100^2. \tag{1}$$

Assume that there are total of N closing prices S_{t_i} of the underlying asset observed at an equally-spaced time t_i. In this case, we multiply by $AF = N/T$ to get the annualized variance in the above formula. Let the strike be K_{var}. At time t the value of variance swap is

$$V_t = e^{-r(T-t)}E^Q\left[L(\sigma_R^2 - k_{var})|\mathcal{F}_t\right] = e^{-r(T-t)}E_t^Q[L(\sigma_R^2 - k_{var})],$$

where $E_t^Q = E^Q[.|\mathcal{F}_t]$ is the conditional expectation at time t. Since at inception $V_0 = 0$, we obtain

$$K_{var} = \frac{AF}{N}\sum_{i=1}^{N}E_0^Q\left[\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] \times 100^2 \tag{2}$$

by the definition of the fair variance delivery price and (1). The problem of pricing variance swap consists in finding the fair variance delivery price K_{var}. In other words, we need to compute all the conditional expectations

$$E_0^Q\left[\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] \tag{3}$$

for $i = 1, \cdots, N$, which depends essentially on the SDE employed to predict the future values of the underlying asset S_{t_i} for $i = 1, \cdots, N$. To compute the expectation (3), we use the same way as Rujivan and Zhu did for the case of Heston model [13]. We start first by expanding the above expectation into three parts in the following lemma.

Lemma 2.1. *We have*

$$E_0^Q\left[\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] = 1 + E_0^Q\left[\frac{1}{S_{t_{i-1}}^2}E_{t_{i-1}}^Q[S_{t_i}^2]\right] - 2E_0^Q\left[\frac{1}{S_{t_{i-1}}}E_{t_{i-1}}^Q[S_{t_i}]\right]. \tag{4}$$

Proof. By the tower property, the conditional expectation (3) can be transformed into equation (4), which has been shown by equation (2.5) in [13]. □

Now the key idea to compute the two right hand expectations of equation (4) can be summarized in two steps as follows

- Step 1: calculate the expectations $E_{t_{i-1}}^Q[S_{t_i}]$ and $E_{t_{i-1}}^Q[S_{t_i}^2]$.
- Step 2: plug the values of expectations obtained in step 1 into (4) then evaluate the outcome expectation in the form $E_0^Q[.]$.

Step 1 can be done by computing the conditional expectation of $Y_t := S_t^\gamma$, which is

$$E_{t_{i-1}}^Q[Y_t] = E_{t_{i-1}}^Q\left[Y_t|(Y_{t_{i-1}} = y, v_{t_{i-1}} = v)\right]$$

for all $t \in [t_{i-1}, t_i]$, where γ can be any non-zero real number especially 1 and 2 (cf. Proposition 2.1 in [13]). The second step is accomplished in Proposition 2.2 in [14].

3. Valuation of variance swaps in discretely-sampled for markets with increased volatility

The main result of this section is to address the issue of pricing discretely-sampled variance swaps under high volatile model. We derive partially the value of the discretely-sampled variance swap.

3.1. Variance swaps in an increased volatile model

To examine the impact of a high volatility on the value of fair variance delivery strike K_{var}, we look into the variance swap pricing problem when the underlying asset price has an augmented volatility. More precisely, here we assume that $(S_t)_{t\in[0,T]}$ is given by the SDE

$$dS_t = rS_t dt + (\sigma S_t + \beta e^{rt})dW_t, \tag{5}$$

where r is a fixed short-run risk free rate, σ is the volatility of the asset, β is a constant and the initial value of the asset $S_0 > 0$. This model presents some practical advantages such as accounting for crisis situations where the prices are suffering from unusual and sudden depreciation. Moreover, there exists a closed form solution for pricing European option in the case of this model. An additional advantage of using this model is that it is a stochastic volatility model that satisfies the leverage effect where the volatility and the asset price are inversely proportional. The SDE (5) has the solution

$$S_t = (S_0 + \frac{\beta}{\sigma})e^{(r-\frac{\sigma^2}{2})t+\sigma W_t} - \frac{\beta e^{rt}}{\sigma}. \tag{6}$$

(See [5] for more details on the above solution and derivation of prices for European options.)

3.2. Valuation of discretely-sampled variance swap

This subsection deals with pricing variance swap under the high volatility model (5) by investigating the valuation of strike K_{var} given by (2) which is reduced to the calculation of the conditional expectations (3). Our way to do this is inspired from the method of [13] employed for Heston model stated in Lemma 2.1 and steps 1 and 2 in the previous section.

Theorem 3.1. *Let AF be the annualized factor and N be the number of observations. Assume that*

$$C = \frac{\left(S_0 + \frac{\beta}{\sigma}\right)^2}{S_0 + \frac{\beta}{2\sigma}} e^{2r\Delta t} - \frac{2\beta}{\sigma} e^{r(2t_i - t_{i-1})} \quad \text{and} \quad D = \frac{\left(S_0 + \frac{\beta}{\sigma}\right)^2}{S_0 + \frac{\beta}{2\sigma}} \frac{\beta}{2\sigma} e^{2rt_i} - \frac{\beta^2}{\sigma^2} e^{2rt_i}.$$

Then, the annualized delivery price for the variance swap of the high volatile model is given by

$$K_{var} = \frac{AF}{N} \sum_{i=1}^{N} \left[1 + E_0^Q \left[\frac{1}{S_{t_{i-1}}^2} E_{t_{i-1}}^Q [S_{t_i}^2] \right] - 2E_0^Q \left[\frac{1}{S_{t_{i-1}}} E_{t_{i-1}}^Q [S_{t_i}] \right] \right] \times 100^2,$$

where

$$E_0^Q \left[\frac{1}{S_{t_{i-1}}} E_{t_{i-1}}^Q [S_{t_i}] \right] = e^{r(t_i - t_{i-1})}$$

and

$$E_0^Q \left[\frac{1}{S_{t_{i-1}}^2} E_{t_{i-1}}^Q [S_{t_i}^2] \right] = C E_0^Q \left[\frac{1}{S_{t_{i-1}}} \right] + D E_0^Q \left[\frac{1}{S_{t_{i-1}}^2} \right].$$

To prove the Theorem 3.1, we start by applying Itô formula to have the SDE of stochastic processes power of S_t as provided in the next corollary.

Lemma 3.2. *Let γ be a non-zero real number and let $(Y_t)_{t\in[0,T]}$ be the process defined by $Y_t = S_t^\gamma$. Then, we have*

$$dY_t = \gamma Y_t \left[\left(r + \frac{\gamma - 1}{2}(\sigma + \beta e^{rt} Y_t^{-\frac{1}{\gamma}})^2 \right) dt + \left(\sigma + \beta e^{rt} Y_t^{-\frac{1}{\gamma}} \right) dW_t \right]. \tag{7}$$

Proof. Applying Itô formula with the function $f(x) = x^\gamma$ and S_t given by (5). Then

$$\begin{aligned} dY_t &= d(S_t^\gamma) = d(f(S_t)) = f'(S_t)dS_t + \frac{1}{2}f''(S_t)d\langle S_t, S_t\rangle \\ &= \gamma(S_t)^{\gamma-1}\left[rS_t dt + (\sigma S_t + \beta e^{rt})dW_t\right] + \frac{1}{2}\gamma(\gamma-1)(S_t)^{\gamma-2}\left[(\sigma S_t + \beta e^{rt})^2 dt\right]. \end{aligned}$$

Since

$$\begin{aligned} \langle dS_t, dS_t\rangle &= \langle rS_t dt + (\sigma S_t + \beta e^{rt})dW_t, rS_t dt + (\sigma S_t + \beta e^{rt})dW_t\rangle \\ &= (\sigma S_t + \beta e^{rt})^2 dt. \end{aligned}$$

Therefore, we have

$$\begin{aligned} dY_t &= d(S_t^\gamma) = \gamma r S_t^\gamma dt + \gamma(S_t)^{\gamma-1}(\sigma S_t + \beta e^{rt})dW_t + \frac{1}{2}\gamma(\gamma-1)(S_t)^{\gamma-2}(\sigma S_t + \beta e^{rt})^2 dt \\ &= \left[\left(r + \frac{1}{2}(\gamma-1)S_t^{-2}(\sigma S_t + \beta e^{rt})^2\right)\gamma S_t^\gamma\right]dt + \left[\gamma(S_t)^{\gamma-1}(\sigma S_t + \beta e^{rt})\right]dW_t. \\ &= \left[\left(r + \frac{1}{2}(\gamma-1)\left(Y_t^{\frac{1}{\gamma}}\right)^{-2}(\sigma Y_t^{\frac{1}{\gamma}} + \beta e^{rt})^2\right)\gamma\left(Y_t^{\frac{1}{\gamma}}\right)^\gamma\right]dt \\ &\quad + \left[\gamma\left(Y_t^{\frac{1}{\gamma}}\right)^{\gamma-1}(\sigma Y_t^{\frac{1}{\gamma}} + \beta e^{rt})\right]dW_t. \end{aligned}$$

The last equation can be simplified to get (7). This ends the proof. □

Next proposition shows that conditional expectation of the form $E^Q_{t_{i-1}}[Y_t]$, where $t \in [t_{i-1}, t_i]$ is the solution of the PDE.

Proposition 3.3. *Let $t \in [t_{i-1}, t_i]$, then there exists a function $U_i^\gamma \in C^{1,2}([t_{i-1}, t_i]\times]0,\infty[)$ such that $E_{t_{i-1}}[Y_t] = U_i^\gamma(t, Y_t)$. Moreover $U_i^\gamma(t, y)$ is solution of the following PDE*

$$\begin{cases} \partial_t U_i^\gamma + \gamma Y_t \partial_y U_i^\gamma\left(r + \dfrac{\gamma-1}{2}(\sigma + \beta e^{rt}Y_t^{-\frac{1}{\gamma}})^2\right) + \dfrac{1}{2}\gamma^2 Y_t^2\left(\sigma + \beta e^{rt}Y_t^{-\frac{1}{\gamma}}\right)^2 \partial_{yy}U_i^\gamma = 0, \\ U_i^\gamma(t_i, y) = y. \end{cases} \tag{8}$$

Proof. Using the Markov property of $(Y_t)_{t\in[0,T]}$ we have for any $s \in [0, t]$, $E_s[f(Y_t)] = E\left[f(Y_t)|Y_s = y\right] = U^\gamma(s, y)$, with $U \in C^{1,2}([0,t]\times]0,\infty[)$. Consider the function

$$\begin{aligned} U_i^\gamma : [t_{i-1}, t_i]\times]0,\infty[&\longrightarrow \mathbb{R} \\ U_i^\gamma(t, y) &= E\left[Y_{t_i}|Y_t = y\right]. \end{aligned}$$

We have $U_i^\gamma(t_{i-1}, y) = E\left[Y_{t_i}|Y_{t_{i-1}} = y\right]$ and $U_i^\gamma(t_i, y) = E\left[Y_{t_i}|Y_{t_i} = y\right] = E[y] = y$. Applying Itô formula to $U_i^\gamma(t, Y_t)$ we obtain

$$dU_i^\gamma = \partial_t U_i^\gamma dt + \partial_y U_i^\gamma dY_t + \frac{1}{2}\partial_{yy}U_i^\gamma d\langle Y_t, Y_t\rangle \tag{9}$$

Using (7) we have

$$d\langle Y_t, Y_t\rangle = \gamma^2 Y_t^2\left(\sigma + \beta e^{rt}Y_t^{-\frac{1}{\gamma}}\right)^2 dt.$$

Therefore (9) becomes

$$\begin{aligned} dU_i^\gamma &= \left[\partial_t U_i^\gamma + \gamma Y_t \partial_y U_i^\gamma \left(r + \frac{\gamma-1}{2}(\sigma + \beta e^{rt} Y_t^{-\frac{1}{\gamma}})^2\right)\right. \\ &\quad \left. + \frac{1}{2}\gamma^2 Y_t^2 \left(\sigma + \beta e^{rt} Y_t^{-\frac{1}{\gamma}}\right)^2 \partial_{yy} U_i^\gamma\right] dt \\ &\quad + \gamma Y_t \partial_y U_i^\gamma \left(\sigma + \beta e^{rt} Y_t^{-\frac{1}{\gamma}}\right) dW_t. \end{aligned} \tag{10}$$

Since the process $(E[Y_t|\mathcal{F}_s])_{s\in[t_{i-1},t_i]}$ is a martingale, then by the martingale representation theorem, the term in dt of the above equation must vanish. This leads to the PDE (8). □

Now we can get the third term of (4).

Lemma 3.4. *We have $E_{t_{i-1}}[S_{t_i}] = S_{t_{i-1}} e^{r(t_i - t_{i-1})}$ and*

$$E_0^Q\left[\frac{S_{t_i}}{S_{t_{i-1}}}\right] = e^{r(t_i - t_{i-1})}, \tag{11}$$

where S_{t_i} is the closing price of the asset at the i-th observation time.

Proof. If we take we apply Proposition 3.3 for $\gamma = 1$ then $Y_t = S_t$ and $E_{t_{i-1}}[S_{t_i}]$ satisfies the PDE

$$\begin{cases} \partial_t U_i + ry\partial_y U_i + \frac{1}{2}(\sigma_y + \beta e^{rt})^2 \partial_{yy} U_i = 0 \\ U_i(y, t_i) = y. \end{cases}$$

Let $\tau = t_i - t$. Assume that the solution of the PDE is of the form $U_i(t, y) = ye^{c(\tau)}$. Then,

$$\partial_t U_i = \frac{\partial v_i}{\partial t} = \frac{\partial v_i}{\partial \tau}\frac{\partial \tau}{\partial t} = -ye^{c(\tau)}\frac{dc}{d\tau}, \; \partial_y U_i = e^{c(\tau)} \text{ and } \partial_{yy} U_i = 0.$$

Substituting the above PDE to get the ODE

$$e^{c(\tau)}\frac{dc(\tau)}{d\tau} + re^{c(\tau)} = 0$$

subject to the initial condition $c(0) = 0$. This gives $dc(\tau) = rd\tau$ and $c(\tau) = r\tau$. Thus, $U_i(t, y) = ye^{r\tau}$. Therefore,

$$E_0^Q\left[\frac{S_{t_i}}{S_{t_{i-1}}}\right] = E_0^Q\left[\frac{1}{S_{t_{i-1}}} U_i(t_{i-1}, y)\right] = E_0^Q\left[\frac{1}{S_{t_{i-1}}} S_{t_{i-1}}.e^{r(t_i - t_{i-1})}\right] = e^{r\Delta t}.$$

The proof is complete. □

The second term in (4) can't be computed using the PDE in Proposition 3.3, however it can be reduced to the computation of conditional expectation in S_t^{-1}.

Proposition 3.5. *Let*

$$C = \frac{\left(S_0 + \frac{\beta}{\sigma}\right)^2}{S_0 + \frac{\beta}{2\sigma}} e^{2r\Delta t} - \frac{2\beta}{\sigma} e^{r(2t_i - t_{i-1})} \text{ and } D = \frac{\left(S_0 + \frac{\beta}{\sigma}\right)^2}{S_0 + \frac{\beta}{2\sigma}} \frac{\beta}{2\sigma} e^{2rt_i} - \frac{\beta^2}{\sigma^2} e^{2rt_i}.$$

Then we have

$$E_0^Q\left[\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2\right] = C E_0^Q\left[\frac{1}{S_{t_{i-1}}}\right] + D E_0^Q\left[\frac{1}{S_{t_{i-1}}^2}\right]. \tag{12}$$

Proof. By equation (4), we have

$$E_0^Q\left[\frac{S_{t_i}^2}{S_{t_{i-1}}^2}\right] = E_0^Q\left[\frac{1}{S_{t_{i-1}}^2}E_{t_{i-1}}^Q[S_{t_i}^2]\right].$$

Since

$$S_{t_i} = \left(S_0 + \frac{\beta}{\sigma}\right)e^{(r-\frac{\sigma}{2})t_i+\sigma W_{t_i}} - \frac{\beta}{\sigma}e^{rt_i} = \alpha\xi_{t_i} - \frac{\beta}{\sigma}e^{rt_i},$$

where $\alpha = S_0 + \frac{\beta}{\sigma}$ and $\xi_{t_i} = e^{(r-\frac{\sigma}{2})t_i+\sigma W_{t_i}}$. Then,

$$S_{t_i}^2 = \alpha^2\xi_{t_i}^2 + \frac{\beta^2}{\sigma^2}e^{2rt_i} - 2\frac{\beta}{\sigma}\alpha\xi_{t_i}e^{rt_i}.$$

Moreover,

$$E_{t_{i-1}}^Q[S_{t_i}^2] = E_{t_{i-1}}^Q[\alpha^2\xi_{t_i}^2] + \frac{\beta^2}{\sigma^2}e^{2rt_i} - 2\frac{\beta}{\sigma}e^{rt_i}E_{t_{i-1}}^Q[\alpha\xi_{t_i}].$$

Now we investigate the calculation of these two conditional expectations in the above equality. The second expectation can be easily computed using (11). In fact, we have

$$E_{t_{i-1}}^Q[\alpha\xi_{t_i}] = E_{t_{i-1}}^Q[S_{t_i} + \frac{\beta}{\sigma}e^{rt_i}] = E_{t_{i-1}}^Q[S_{t_i}] + \frac{\beta}{\sigma}e^{rt_i} = S_{t_{i-1}}e^{r\Delta t} + \frac{\beta}{\sigma}e^{rt_i}.$$

To find the first conditional expectation $E_{t_{i-1}}^Q[\alpha^2\xi_{t_i}^2]$, we first notice that

$$\begin{aligned}\alpha^2\xi_{t_i}^2 &= \alpha^2 e^{2(r-\frac{\sigma}{2})t_i+2\sigma W_{t_i}} = \alpha^2 e^{(r_2-\frac{\sigma_2}{2})t_i+\sigma_2 W_{t_i}}\\ &= \left(\frac{\alpha^2}{S_0+\frac{\beta}{2\sigma}}\right)\left[\left(S_0+\frac{\beta}{\sigma_2}\right)e^{(r_2-\frac{\sigma_2}{2})t_i+\sigma_2 W_{t_i}} + \gamma_2(t_i)\right] - \frac{\alpha^2}{S_0+\frac{\beta}{2\sigma}}\gamma_2(t_i).\end{aligned}$$

Then

$$E_{t_{i-1}}^Q[\alpha^2\xi_{t_i}^2] = \frac{\alpha^2}{S_0+\frac{\beta}{2\sigma}}\left[E_{t_{i-1}}^Q[S_{2t_i}] - \gamma_2(t_i)\right],$$

where $S_{2t_i} = S_{t_i}(r_2, \sigma_2)$ and $\gamma_2(t_i) = \frac{-\beta}{2\sigma}e^{2rt_i}$. Let $\gamma(t_i) = \frac{-\beta}{\sigma}e^{rt_i}$. Then,

$$E_{t_{i-1}}^Q[\alpha^2\xi_{t_i}^2] = \frac{\alpha^2}{S_0+\frac{\beta}{2\sigma}}\left(S_{t_{i-1}}e^{2r\Delta t} + \frac{\beta}{2\sigma}e^{2rt_i}\right),$$

and

$$\begin{aligned}E_{t_{i-1}}^Q[S_{t_i}^2] &= AS_{t_{i-1}} + B + \gamma^2(t_i) + 2\gamma(t_i)[S_{t_{i-1}}e^{r\Delta t} - \gamma(t_i)]\\ &= [A + 2\gamma(t_i)e^{r\Delta t}]S_{t_{i-1}} + B - \gamma^2(t_i),\end{aligned}$$

where $A = \frac{\alpha^2}{S_0+\frac{\beta}{2\sigma}}e^{2r\Delta t}$ and $B = \frac{\beta}{2\sigma}e^{2rt_i}\frac{\alpha^2}{S_0+\frac{\beta}{2\sigma}}$. Thus, we have

$$E_{t_{i-1}}^Q[S_{t_i}^2] = CS_{t_{i-1}} + D,$$

where $C = \frac{(S_0+\frac{\beta}{\sigma})^2}{S_0+\frac{\beta}{2\sigma}}e^{2r\Delta t} - \frac{2\beta}{\sigma}e^{r(2t_i-t_{i-1})}$ and $D = \frac{(S_0+\frac{\beta}{\sigma})^2}{S_0+\frac{\beta}{2\sigma}}\frac{\beta}{2\sigma}e^{2rt_i} - \frac{\beta^2}{\sigma^2}e^{2rt_i}$ Finally, we get

$$E_0^Q\left[\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2\right] = E_0^Q\left[\frac{1}{S_{t_{i-1}}^2}(CS_{t_{i-1}}+D)\right] = CE_0^Q\left[\frac{1}{S_{t_{i-1}}}\right] + DE_0^Q\left[\frac{1}{S_{t_{i-1}}^2}\right].$$

This ends the proof. □

Combining all aforementioned results, we finish the proof of Theorem 3.1.

4. Applications

In this section, we apply our result from previous section on valuation of variance swaps. An illustration for the sensitivity of fair delivery price to the additional parameter β is provided. It shows the volatility increase's impact on the a variance swap delivery price.

Let the annualized risk-free interest rate $r = 0.02$, the number of observations $N = 252$ (trading days) and the life time $T = 1$ (year). The annualized factor $AF = N/T = 252$. Assume that the parameter σ of the underlying asset S is 0.01.

Then, we use the Monte Carlo method to simulate the annualized delivery price for the variance swap of the high volatile model with respect to β. After running 5000 times for each β, we have the following Figure 1. From this figure, we can see that the annualized delivery price for the variance swap of the high volatile model is proportional to the value of β.

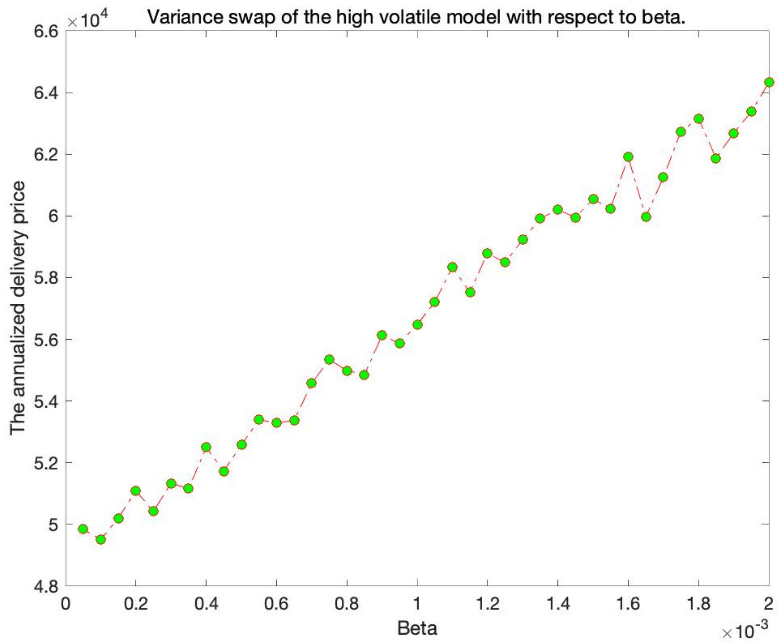

Figure 1: The annualized variance delivery price K_{var} with respect to β.

Now we fix the parameter $\beta = 0.0001$. After running 5000 times for each σ of the underlying asset S, we have the following Figure 2. From this figure, we can see that there is no significant change in the annualized delivery price for the variance swap of the high volatile model with small difference of the value of σ .

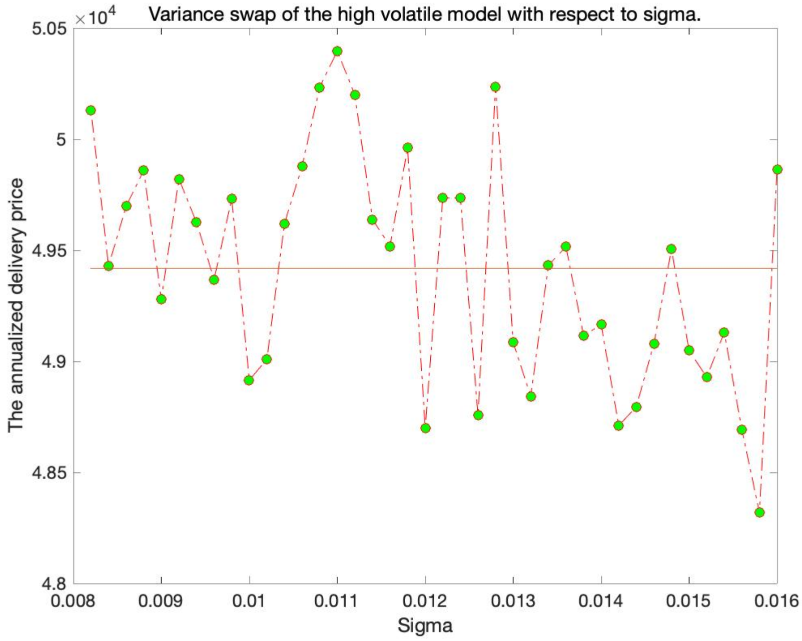

Figure 2: The annualized variance delivery price K_{var} with respect to σ.

Note that when we take $\beta = 0$, the high volatile model degenerates to the Black Scholes model.

model	B-S model	high volatile model				
	$\beta = 0$	$\beta = 0.0001$	$\beta = 0.00015$	$\beta = 0.0002$	$\beta = 0.00025$	$\beta = 0.0003$
K_{var}	4.953×10^4	4.957×10^4	4.993×10^4	5.031×10^4	5.115×10^4	5.146×10^4

Table 1: Comparison of the annualized variance delivery price K_{var} between the high volatile model and the Black-Scholes model.

The above table shows the annualized delivery price for the variance swap of these two models under the same assumptions.

Finally, we compare the annualized delivery price for the variance swap of the Black Scholes model and the high volatile model with respect to σ by the following figure.

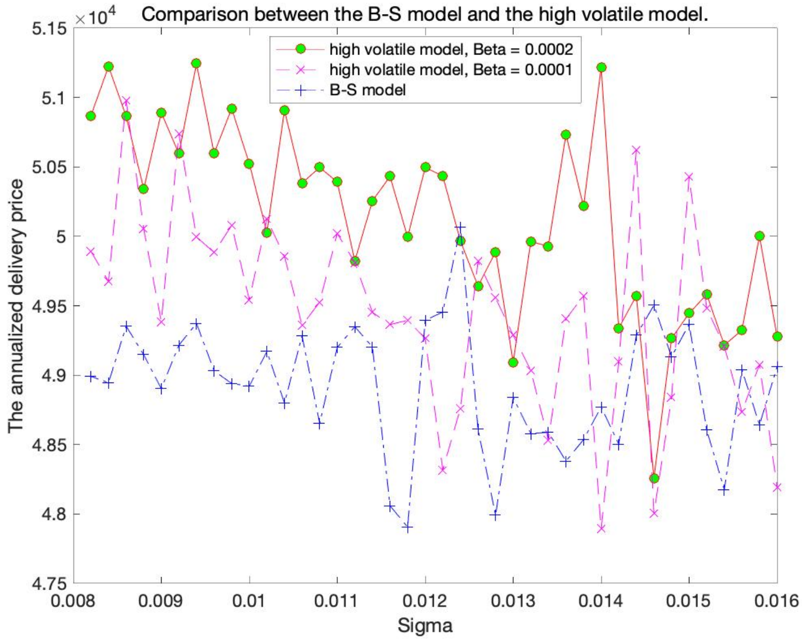

Figure 3: Comparison of the annualized variance delivery price K_{var} between the Black Scholes model and the high volatile model with respect to σ.

5. Conclusions

Pricing financial derivatives is an important problem in mathematical finance since these instruments are used extensively in hedging against risk. Therefore, obtaining an accurate price for a financial derivative product is decisive in risk management. However, the valuation of such products is primary depending on modeling its underlying asset.

In this paper, we study the evaluation of variance swaps during situations where the volatility is abnormally increased. A financial crunch is a typical example of such situation. During this period when risk is higher, an accurate price for the swaps is essential. The paper investigates a solution to the pricing of discretely-sampled variance swaps in markets with high volatility. The obtained result is expected to improve the accuracy of the fair delivery price during financial crisis. Numerical simulations show that variance swap prices are proportional to volatility increase.

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