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Research Article

Some New Fourier and Jackson–Nikol'skii Type Inequalities in Unbounded Orthonormal Systems

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ABSTRACT. We consider the generalized Lorentz space $L_{\psi,q}$ defined via a continuous and concave function ψ and the Fourier series of a function with respect to an unbounded orthonormal system. Some new Fourier and Jackson-Nikol'skii type inequalities in this frame are stated, proved and discussed. In particular, the derived results generalize and unify several well-known results but also some new applications are pointed out.

Keywords: Inequalities, generalized Lorentz spaces, unbounded orthonormal system, Fourier inequalities, Jackson–Nikol'skii inequality.

2020 Mathematics Subject Classification: 42A16, 42B05, 26D15, 26D20, 46E30.

1. INTRODUCTION

Let the function ψ be continuous and concave by $[0,1]$, $\psi(0)=0$ and $0<q\leqslant\infty$. Such functions are called Φ functions. The generalized Lorentz space $L_{\psi,q}$ is the set of measurable functions f on $[0,1]$ for which

$$\|f\|_{\psi,q} := \left(\int_0^1 f^{*^q}(t)\psi^q(t)\frac{dt}{t}\right)^{1/q} < \infty,$$

where f^* is the non-increasing rearrangement of the function $|f|$ (see e.g. [36]).
For a given function $\psi(t), t\in[0,1]$, we define

$$\alpha_\psi := \underline{\lim}_{t\to 0}\frac{\psi(2t)}{\psi(t)}, \quad \beta_\psi := \overline{\lim}_{t\to 0}\frac{\psi(2t)}{\psi(t)}.$$

It is known that $1\leqslant\alpha_\psi\leqslant\beta_\psi\leqslant 2$ (see e.g. [35]) .
Note that for $\psi(t)=t^{1/p}$, the space $L_{\psi,q}$ coincides with the Lorentz space $L_{p,q}$, $0<q,p<\infty$, which consists of all functions f such that (see e.g. [38, p. 228])

$$\|f\|_{p,q} := \left(\int_0^1 f^{*^q}(t)t^{\frac{q}{p}-1}dt\right)^{1/q}.$$

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In particular, for the case $p = q$, we have the usual Lebesgue space with the norm (quasi-norm if $0 < q < 1$)

$$\|f\|_q := \left(\int_0^1 |f(x)|^q dx \right)^{1/q}, \quad 0 < q < \infty.$$

Let $q, p \in (0, +\infty)$ and $\alpha \in R = (-\infty, +\infty)$. The Lorentz-Zygmund space $L_{p,q}(\log L)^\alpha$ is the set of all functions f measurable on $[0, 1]$ for which (see e.g. [37])

$$\|f\|_{p,q,\alpha} := \left\{ \int_0^1 (f^*(t))^q (1 + |\log t|)^{\alpha q} t^{\frac{q}{p}-1} dt \right\}^{\frac{1}{q}} < +\infty.$$

For A, B the notation $A \asymp B$ means that there exits positive constants C_1, C_2 such that $C_1 A \leqslant B \leqslant C_2 A$.

We consider the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}} \subset L_2[0, 1]$ (see [22, p. 58]) satisfying the condition

$$\|\varphi_n\|_r := \left(\int_0^1 |\varphi_n(x)|^r dx \right)^{\frac{1}{r}} \leqslant M_n, \quad n \in \mathbb{N} \tag{1}$$

for some $r \in (2, +\infty]$. Here, we assume that $\{M_n\}$ is a non-decreasing sequence.

Let $\hat{f}(n)$ be the Fourier coefficients of the function f with respect to the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$.

J. Marcinkiewicz and A. Zygmund [22] proved some inequalities for the sums of the Fourier coefficients of the orthogonal system $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfying condition (1) and norms of the function $f \in L_p$, $1 < p < \infty$. Later, many authors investigated this problem in other functional spaces (for example, see [3], [6], [7], [8], [11], [13], [21], [30], [32], [33], [42] and bibliographic references in them).

In particular, the following statement is known (see S.V. Bochkarev [11]):

Theorem 1.1. *Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal system of complex-valued functions*

$$\|\varphi_n\|_\infty \leqslant M, \; n = 1, 2, ... \tag{2}$$

for some $M < \infty$. Then, for any $2 < q \leqslant \infty$ and $n = 2, 3, ...$, the following inequality holds:

$$\left[\sum_{k=1}^{n} (\hat{f}^*(k))^2 \right]^{\frac{1}{2}} \leqslant CM \|f\|_{2,q} (\log n)^{\frac{1}{2}-\frac{1}{q}}.$$

In the case $q = \infty$, Theorem 1.1 was previously proved by V.I. Ovchinnikov, V.D. Raspopova and V.A. Rodin [32].

In the case when $\{\varphi_n\}_{n\in\mathbb{N}}$ is a trigonometric system, in the Lorentz-Zygmund space $L_{2,q,}(\log L)^\alpha$ H. Oba , E. Sato and Y. Sato [30] stated and proved the following:

Theorem 1.2. *Let $2 < q \leqslant \infty$, $n \geqslant 3$ and $\alpha \in \mathbb{R}$. Then the following inequality holds:*

$$\left[\sum_{k=1}^{n} (\hat{f}^*(k))^2 \right]^{\frac{1}{2}} \leqslant CA_n \|f\|_{2,q,\alpha}$$

for some constant C which is independent of n and f, and A_n is as follows:

$$A_n = \begin{cases} (\log n)^{\frac{1}{2}-\frac{1}{q}-\alpha}, & \text{if } \alpha < \frac{1}{2} - \frac{1}{q}, \\ (\log(\log n))^\alpha, & \text{if } \alpha = \frac{1}{2} - \frac{1}{q}, \\ 1, & \text{if } \alpha > \frac{1}{2} - \frac{1}{q}. \end{cases}$$

A generalization of this theorem for the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfying condition (2) was proved by L.R.Ya. Doktorski (see [13]). Moreover, N. Tleukhanova and G. Mussabaeva [42] for the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfying condition (2) proved the inequality

$$\sup_{n\in\mathbb{N}} \frac{1}{n^{1/2}(\log(n+1))^{\frac{1}{2}-\frac{1}{q}}} \sum_{k=1}^{n} \hat{f}^*(k) \leqslant C\|f\|_{2,q} \tag{3}$$

for any function $f \in L_{2,q}, 2 < q \leqslant \infty$.

Most results concerning Fourier inequalities are derived for bounded orthonormal systems. However, for several applications it is also important to derive such results for unbounded orthonormal systems like those described in our final Remark 4.11. One aim of this paper is to further complement our recent research in this direction (see [6], [7] and [8]) and also prove and discuss some new related Nikol'skii type inequalities of this type. Let us first mention that in [3] for an unbounded orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$, the following statement was proved (for the case $\alpha = 0$, see [2]).

Theorem 1.3. *Let the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2, +\infty]$ satisfy the condition (1). Then, for any function $L_{2,q}(\log L)^{\alpha}, 2 < q \leqslant \infty, \alpha < \frac{1}{2} - \frac{1}{q}, n \in \mathbb{N}$, the following inequality holds:*

$$\left[\sum_{k=1}^{n} |\hat{f}(k)|^2\right]^{\frac{1}{2}} \leqslant C\|f\|_{2,q,\alpha} \left[\ln(1 + \sum_{j=1}^{n} M_j^2)\right]^{\frac{1}{2}-\frac{1}{q}-\alpha}.$$

For a trigonometric polynomial

$$T_n(x) = \sum_{k=-n}^{n} a_k e^{ikx}, n \in \mathbb{N}$$

the following Jackson–Nikol'skii inequality is well known (see [17], [27])

$$\|T_n\|_q \leqslant 2n^{1/p}\|T_n\|_p \tag{4}$$

for $1 \leqslant p < q \leqslant \infty$. This inequality is also called the inequality of different metrics for a trigonometric polynomial.

For case $0 < p < q \leqslant \infty$, inequality (4) was proved in [16] and [10]. Moreover, for $p = 0 < q < \infty$, it was proved by V.V. Arestov [10].

Nowadays, there are various generalizations of the Jackson-Nikol'skii inequality (see [5], [12], [29] and the bibliography therein). One of the generalizations is its extension to polynomials in orthonormal systems of functions. In particular, M.F. Timan [40] proved the following statement:

Theorem 1.4. *Let $1 \leqslant p \leqslant 2, p < q \leqslant \infty$ and $\{\varphi_n\}_{n=1}^{\infty}$ be a uniformly bounded sequence of orthonormal systems of functions. Then for the polynomial*

$$f_n(x) = \sum_{k=1}^{n} c_k \varphi_k(x), n \in \mathbb{N},$$

holds the following inequality:

$$\|f_n\|_q \leqslant Cn^{1/p-1/q}\|f_n\|_p. \tag{5}$$

A multidimensional version of inequality (5) in the spaces L_p was established by R.J. Nessel and G. Wilmes [25], [26]. The Jackson-Nikol'skii inequality for polynomials in a uniformly bounded system of functions in some symmetric spaces was proved by V.A. Rodin [34]. Moreover, L.R.Ya. Doktorski and D.Gendler [14] proved the inequality of different metrics for polynomials in a uniformly bounded orthonormal system of functions in the Lorentz–Zygmund

space. Jackson–Nikol'skii inequality is also known for polynomials in an unbounded orthonormal system of functions (see, for example, [19], [20], [23], [24]).
In this paper, we complement the results above by proving some new Fourier and Jackson-Nikol'skii type inequalities in the generalized Lorentz space $L_{\psi,q}$ and in unbounded systems satisfying (1).
In Section 2, we present and discuss our main results. The announced generalizations and unifications of Fourier type inequalities can be found in Theorem 2.1 while the corresponding results concerning Jackson-Nikol'skii type inequalities are given in Theorem 2.2. These detailed proofs are presented; in Section 3 and Section 4 is reserved for some concluding remarks and result (see Proposition 4.1).

2. The main results

We denote by SVL (slowly varing) the set of all non-negative functions on $[0,1]$ of $\psi(t)$ for which $(\log 2/t)^{\varepsilon}\psi(t)\uparrow +\infty$ and $(\log 2/t)^{-\varepsilon}\psi(t)\downarrow 0$ for $t\downarrow 0$ (see e.g. [8]).
First, we formulate the following generalization and unification of Theorem 1.1, Theorem 1.2 for the case $\alpha < \frac{1}{2}-\frac{1}{q}$, assertion 1) of Theorem 1.3 and inequality (3):

Theorem 2.1. *Let ψ a function satisfying the conditions $1<\alpha_{\psi}=\beta_{\psi}=2^{1/2}$, $\frac{t^{1/2}}{\psi(t)}\in SVL$,*

$$\sup_{t\in(0,1]}\frac{\psi(t)}{t^{1/2}}<\infty,$$

and assume that the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r\in(2,+\infty]$ satisfies the condition (1). Then, for any function $f\in L_{\psi,q}$, $2<q\leqslant\infty$, the following inequality holds:

$$\Big[\sum_{k\in A}|\hat{f}(k)|^2\Big]^{\frac{1}{2}}\leqslant C\|f\|_{\psi,q}\Big[\ln(1+\sum_{j\in A}M_j^2)\Big]^{\frac{1}{2}-\frac{1}{q}}\frac{\sqrt{(1+\sum_{j\in A}M_j^2)^{-1}}}{\psi((1+\sum_{j\in A}M_j^2)^{-1})},$$

where A is a non-empty set in $\mathbb{N}$ and C is positive constant which depends only on q and r.

Corollary 2.1. *Let ψ be a function satisfying the conditions of Theorem 2.1 and the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r\in(2,+\infty]$ satisfying the condition (2). Then, for any function $f\in L_{\psi,q}, 2<q\leqslant\infty$, we have the inequality*

$$\Big[\sum_{k=1}^{|A|}(\hat{f}^*(k))^2\Big]^{\frac{1}{2}}\leqslant C\|f\|_{\psi,q}\Big[\log(1+|A|M^2)\Big]^{\frac{1}{2}-\frac{1}{q}}\frac{\sqrt{(1+|A|M^2)^{-1}}}{\psi((1+|A|M^2)^{-1})},$$

where $|A|$ is the number of elements in the set $A\subset\mathbb{N}$.

Corollary 2.2. *Let ψ be a function satisfying the conditions of Theorem 2.1 and let the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r\in(2,+\infty]$ satisfying the condition (2). Then, for any function $f\in L_{\psi,q}, 2<q\leqslant\infty$, the following inequality holds:*

$$\sup_{n\in\mathbb{N}} n^{-1/2}\Big[\log(1+nM^2)\Big]^{\frac{1}{q}-\frac{1}{2}}\left(\frac{\sqrt{(1+nM^2)^{-1}}}{\psi((1+nM^2)^{-1})}\right)^{-1}\sum_{k=1}^{n}\hat{f}^*(k)\leqslant C\|f\|_{\psi,q}.$$

Remark 2.1. *In the case $\psi(t)=t^{1/2}$ from Corollary 2.1 and Corollary 2.2, we accordingly obtain the statement of Theorem 1.1 and inequality (3).*

Remark 2.2. *In the case $\psi(t)=t^{1/2}(1+|\log t|)^{\alpha}$ and $\{\varphi_n\}$ the trigonometric system from Corollary 2.2, we obtain the statement in Theorem 1.2 for $\alpha<\frac{1}{2}-\frac{1}{q}$.*

Remark 2.3. *If $\psi(t) = t^{1/2}(1 + |\log t|)^{\alpha}$ and the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2, +\infty]$ satisfies condition (2), then from Corollary 2.2, we obtain assertion 1) of Theorem 1.3.*

Remark 2.4. *In the case $\psi(t) = t^{1/2}$ and $A = \{1, ..., n\}$, it was proved in [11] that the inequality in Corollary 2.1 is exact for the multiplicative Crestenson–Levy system . This fact for a trigonometric system in the Lorentz–Zygmund space $L_{2,q,}(\log L)^{\alpha}$ was proved in [30]. By also using Theorem 2 in [5], we obtain the following statement:*

Corollary 2.3. *Let ψ be a function satisfying the conditions of Theorem 2.1, $2 < q < \infty$ and $\{e^{inx}\}_{n\in\mathbb{Z}}$ be the trigonometric system. Then*

$$\sup_{f\neq 0} \frac{\left(\sum_{k=1}^{2n+1}(\hat{f}^*(k))^2\right)^{1/2}}{\|f\|_{\psi,q}} \asymp \frac{\sqrt{(1+n)^{-1}}}{\psi((1+n)^{-1})}\left[\log(1+n)\right]^{\frac{1}{2}-\frac{1}{q}}.$$

Next, we state a Jackson–Nikol'skii type inequality which generalizes some results for the trigonometric system in [17] and [27], [28] (for a complementary bibliography see also [4], [5]).

Theorem 2.2. *Let the function ψ satisfy the conditions $1 < \alpha_\psi = \beta_\psi = 2^{1/2}$, $\frac{\psi(t)}{t^{1/2}} \in SVL$,*

$$\sup_{t\in(0,1]} \frac{t^{1/2}}{\psi(t)} < \infty, \tag{6}$$

let the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2, +\infty]$ satisfy the condition (1) and $f_n(x) = \sum_{k=1}^{n} c_k\varphi_k(x)$.
1) If $1 < q < 2$, then

$$\|f_n\|_{\psi,q} \leqslant C\left(\frac{\sqrt{\left(1+\sum_{j=1}^{n} M_j^2\right)^{-1}}}{\psi((1+\sum_{j=1}^{n} M_j^2)^{-1})}\right)^{-1}\left(\log\left(1+\sum_{k=1}^{n} M_k^2\right)\right)^{\frac{1}{q}-\frac{1}{2}}\|f_n\|_2$$

for some constant C depending only on q.
2) If $1 < p < 2 < q < +\infty$, then

$$\|f_n\|_{\psi,p} \leqslant C(p,q)\|f_n\|_{\psi,q}\left(\log(1+\sum_{k=1}^{n} M_k^2)\right)^{\frac{1}{p}-\frac{1}{q}}$$

for some constant C depending only on p and q.
3) If $2 < p < q < +\infty$, then

$$\|f_n\|_{\psi,p} \leqslant C(p,q)\|f_n\|_{\psi,q}\left(\log(1+\sum_{k=1}^{n} M_k^2)\right)^{\frac{1}{p}-\frac{1}{q}}$$

for some constant C depending only on p and q.

3. Proofs

Proof of Theorem 2.1. Let $f \in L_{\psi,q}$. This function can be represented as $f(x) = f_1(x) + f_2(x)$, where

$$f_1(x) = \begin{cases} f(x), & \text{when } |f(x)| \leqslant f^*(\tau), \\ 0, & \text{when } |f(x)| > f^*(\tau), \end{cases}$$
$$f_2(x) = f(x) - f_1(x), \quad 0 < \tau < 1.$$

Then, by the Minkowski inequality, we have that

$$\Big[\sum_{k\in A}|\hat{f}(k)|^2\Big]^{1/2}\leqslant\Big[\sum_{k\in A}|\hat{f}_1(k)|^2\Big]^{1/2}+\Big[\sum_{k\in A}|\hat{f}_2(k)|^2\Big]^{1/2}. \tag{7}$$

Now, we prove that each of the functions $f_i, i=1,2$, satisfies the inequality

$$\Big[\sum_{k\in A}|\hat{f}_i(k)|^2\Big]^{1/2}\leqslant C(q,r)\Big(\ln(1+\sum_{k\in A}M_k^2)\Big)^{\frac{1}{2}-\frac{1}{q}}\frac{\sqrt{(1+\sum_{j\in A}M_j^2)^{-\frac{r}{2(r-2)}}}}{\psi((1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}})}\|f\|_{\psi,q}. \tag{8}$$

According to the Parseval equality for an orthonormal system and Hölder's inequality for $\theta=\frac{q}{2}>1$, $\frac{1}{\theta}+\frac{1}{\theta'}=1$ for the function f_1, we find that

$$\sum_{k\in A}|\hat{f}_1(k)|^2\leqslant\|f_1\|_2^2\leqslant\int_\tau^1 {f^*}^2(t)dt\leqslant\|f\|_{\psi,q}^2\Big[\int_\tau^1\Big(\frac{t^{1/2}}{\psi(t)}\Big)^{2\theta'}t^{-1}dt\Big]^{\frac{1}{\theta'}}. \tag{9}$$

Since $\frac{t^{1/2}}{\psi(t)}\in SVL$, then $\frac{t^{1/2}}{\psi(t)}\log^\varepsilon 2/t\leqslant\frac{\tau^{1/2}}{\psi(\tau)}\log^\varepsilon 2/\tau$ for $t\in[\tau,1]$, $\forall\varepsilon>0$. Therefore

$$\Big[\int_\tau^1\Big(\frac{t^{1/2}}{\psi(t)}\Big)^{2\theta'}t^{-1}dt\Big]^{\frac{1}{\theta'}}\leqslant\Big(\frac{\tau^{1/2}}{\psi(\tau)}\Big)^2\log^{2\varepsilon}2/\tau\Big[\int_\tau^1(\log 2/t)^{-2\varepsilon\theta'}t^{-1}dt\Big]^{\frac{1}{\theta'}}. \tag{10}$$

Choose the number $\varepsilon\in(0,\frac{1}{2}-\frac{1}{q})$. Then, $1-2\varepsilon\theta'>0$ so that

$$\int_\tau^1(\log 2/t)^{-2\varepsilon\theta'}t^{-1}dt=\frac{1}{1-2\varepsilon\theta'}\Big[(\log 2/t)^{1-2\varepsilon\theta'}-1\Big].$$

Therefore, from inequality (10), it follows that

$$\Big[\int_\tau^1\Big(\frac{t^{1/2}}{\psi(t)}\Big)^{2\theta'}t^{-1}dt\Big]^{\frac{1}{\theta'}}\leqslant\frac{1}{1-2\varepsilon\theta'}\Big(\frac{\tau^{1/2}}{\psi(\tau)}\Big)^2(\log 2/t)^{\frac{1}{\theta'}}. \tag{11}$$

Now by using inequalities (9) and (11), we obtain that

$$\Big(\sum_{k\in A}|\hat{f}_1(k)|^2\Big)^{\frac{1}{2}}\leqslant\frac{1}{1-2\varepsilon\theta'}\frac{\tau^{1/2}}{\psi(\tau)}(\log 2/\tau)^{\frac{1}{2}-\frac{1}{q}}\|f\|_{\psi,q}. \tag{12}$$

In this formula, we put $\tau=(1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}}$. Then, for the function f_1 from (12), we can conclude that

$$\Big(\sum_{k\in A}|\hat{f}_1(k)|^2\Big)^{\frac{1}{2}}$$
$$\leqslant C\Big(\ln(1+\sum_{k\in A}M_k^2)\Big)^{\frac{1}{2}-\frac{1}{q}}\frac{\sqrt{(1+\sum_{j\in A}M_j^2)^{-\frac{r}{2(r-2)}}}}{\psi((1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}})}\Big(\ln(1+\sum_{k\in A}M_k^2)\Big)^{\frac{1}{2}-\frac{1}{q}}\|f\|_{\psi,q},$$

so (8) holds with $i=1$. For the function $f_2\in L_{r'}$ by the definition of the coefficient expansions and Hölder's inequality ($2<r<+\infty, r'=\frac{r}{r-1}$), we have that

$$|\hat{f}_2(k)|=\Big|\int_0^1 f_2(x)\varphi_k(x)dx\Big|\leqslant\|f_2\|_{r'}\|\varphi_k\|_r\leqslant M_k\|f\|_{r'}.$$

Hence,

$$\sum_{k\in A}|\hat{f}_2(k)|^2\leqslant\|f_2\|_{r'}^2\sum_{k\in A}M_k^2=\Big(\int_0^\tau {f^*}^{r'}(t)dt\Big)^{2/r'}\sum_{k\in A}M_k^2. \tag{13}$$

Since the function f^* is non-increasing and ψ is non-decreasing, then

$$\|f\|_{\psi,q} \geqslant \Big(\int\limits_{x/2}^{x} {f^*}^q(t)\psi^q(t)\frac{dt}{t}\Big)^{1/q}$$
$$\geqslant f^*(x)\psi(x/2)\Big(\int\limits_{x/2}^{x} \frac{dt}{t}\Big)^{1/q} = f^*(x)\psi(x/2)(\ln 2)^{1/q},\ x \in (0,1].$$

Therefore, from inequality (13), it follows that

$$\sum_{k\in A} |\hat{f}_2(k)|^2 \leqslant \|f\|^2_{\psi,q}\Big(\int\limits_0^{\tau} \psi^{-r'}(t/2)dt\Big)^{2/r'} \sum_{k\in A} M_k^2. \tag{14}$$

Since $\frac{t^{1/2}}{\psi(t)} \in SVL$, then

$$\Big(\int\limits_0^{\tau} \psi^{-r'}(t/2)dt\Big)^{2/r'} = \Big(\int\limits_0^{\tau}\Big(\frac{\sqrt{t/2}}{\psi(t/2)}\Big)^{r'}(t/2)^{-r'/2}dt\Big)^{2/r'}$$
$$\leqslant \Big(\frac{\sqrt{\tau/2}}{\psi(\tau/2)}\log^{\varepsilon}\frac{2}{\tau/2}\Big)^2\Big(\int\limits_0^{\tau}(\log\frac{2}{t/2})^{-\varepsilon r'}(t/2)^{-r'/2}dt\Big)^{2/r'}. \tag{15}$$

If $0<t<\tau$, then $(\log\frac{2}{t/2})^{-\varepsilon} < (\log\frac{2}{\tau/2})^{-\varepsilon}$, for $\varepsilon > 0$. Therefore, by using (15), we obtain that

$$\Big(\int\limits_0^{\tau} \psi^{-r'}(t/2)dt\Big)^{2/r'} \leqslant \Big(\frac{\sqrt{\tau/2}}{\psi(\tau/2)}\Big)^2\Big(\int\limits_0^{\tau}(t/2)^{-r'/2}dt\Big)^{2/r'}$$
$$= (\frac{2}{2-r'})^{2/r'}\Big(\frac{\sqrt{\tau/2}}{\psi(\tau/2)}\Big)^2(\tau/2)^{\frac{2}{r'}-1} = (\frac{2}{2-r'})^{2/r'}\Big(\frac{1}{\psi(\tau/2)}\Big)^2 2^{-\frac{2}{r'}}\tau^{\frac{2}{r'}}. \tag{16}$$

Now, it follows from inequalities (14) and (16) that

$$\Big(\sum_{k\in A}|\hat{f}_2(k)|^2\Big)^{1/2} \leqslant C\|f\|_{\psi,q}\frac{1}{\psi(\tau)}\tau^{\frac{1}{r'}}\Big(\sum_{k\in A}M_k^2\Big)^{1/2}.$$

In this formula, we put $\tau = (1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}}$. Then

$$\Big(\sum_{k=1}^{n}|\hat{f}_2(k)|^2\Big)^{1`/2} \leqslant C\|f\|_{\psi,q}\frac{1}{\psi((1+\sum_{j=1}^{n}M_j^2)^{-\frac{r}{r-2}})}\Big(1+\sum_{j=1}^{n}M_j^2\Big)^{-\frac{r}{r'(r-2)}}\Big(\sum_{k=1}^{n}M_k^2\Big)^{1/2}$$
$$= C\frac{1}{\psi((1+\sum_{j=1}^{n}M_j^2)^{-\frac{r}{r-2}})}\Big(1+\sum_{j=1}^{n}M_j^2\Big)^{-\frac{r}{2(r-2)}}\|f\|_{\psi,q}.$$

Now, taking into account that $1/2-1/q>0$, we get from here that

$$\Big(\sum_{k\in A}|\hat{f}_2(k)|^2\Big)^{1`/2}$$
$$\leqslant C\frac{1}{\psi((1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}})}\Big(1+\sum_{j\in A}M_j^2\Big)^{-\frac{r}{2(r-2)}}\Big(\log\Big(1+\sum_{j\in A}M_j^2\Big)\Big)^{1/2-1/q}\|f\|_{\psi,q},$$

so (8) holds also for $i=2$. From inequalities (7) and (8), it follows that

$$\Big(\sum_{k\in A}|\hat{f}(k)|^2\Big)^{1`/2}$$
$$\leqslant C\frac{1}{\psi((1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}})}\Big(1+\sum_{j\in A}M_j^2\Big)^{-\frac{r}{2(r-2)}}\Big(\log\Big(1+\sum_{j\in A}M_j^2\Big)\Big)^{1/2-1/q}\|f\|_{\psi,q}. \tag{17}$$

Since $\frac{t^{1/2}}{\psi(t)}\in SVL$ and $\Big(1+\sum_{j\in A}M_j^2\Big)^{-\frac{r}{2(r-2)}}<\Big(1+\sum_{j\in A}M_j^2\Big)^{-1}$, then

$$\frac{\sqrt{\Big(1+\sum_{j\in A}M_j^2\Big)^{-\frac{r}{(r-2)}}}}{\psi((1+\sum_{j\in A}M_j^2)^{-\frac{r}{r-2}})}$$
$$\leqslant\frac{\sqrt{\Big(1+\sum_{j\in A}M_j^2\Big)^{-1}}}{\psi((1+\sum_{j\in A}M_j^2)^{-1})}\Big(\log\frac{2}{\Big(1+\sum_{j\in A}M_j^2\Big)^{-1}}\Big)^{-\varepsilon}\Big(\log\frac{2}{\Big(1+\sum_{j\in A}M_j^2\Big)^{-\frac{r}{(r-2)}}}\Big)^{\varepsilon}$$
$$\leqslant\frac{\sqrt{\Big(1+\sum_{j\in A}M_j^2\Big)^{-1}}}{\psi((1+\sum_{j\in A}M_j^2)^{-1})}\Big(\log 2\Big(1+\sum_{j\in A}M_j^2\Big)\Big)^{-\varepsilon}\Big(\frac{r}{r-2}\log 2\Big(1+\sum_{j\in A}M_j^2\Big)\Big)^{\varepsilon}$$
$$=\frac{r}{r-2}\frac{\sqrt{\Big(1+\sum_{j\in A}M_j^2\Big)^{-1}}}{\psi((1+\sum_{j\in A}M_j^2)^{-1})}. \tag{18}$$

It follows from inequalities (17) and (18) that

$$\Big(\sum_{k\in A}|\hat{f}(k)|^2\Big)^{1`/2}\leqslant\frac{r}{r-2}\frac{\sqrt{\Big(1+\sum_{j\in A}M_j^2\Big)^{-1}}}{\psi((1+\sum_{j\in A}M_j^2)^{-1})}\Big(\log\Big(1+\sum_{j\in A}M_j^2\Big)\Big)^{1/2-1/q}\|f\|_{\psi,q}.$$

The proof is complete. □

Proof of Corollary 2.1. In view of the fact that $M_j=M,\ j=1,2,...$ and the property of non-increasing rearrangement of numbers, it yields that

$$\sum_{k\in A}|\hat{f}(k)|^2=\sum_{k=1}^{|A|}(\hat{f}^*(k))^2,$$

so the proof follows by just applying Theorem 2.1. □

Proof of Corollary 2.2. According to Hölder's inequality, we have that

$$\sum_{k=1}^{n}\hat{f}^*(k)\leqslant n^{1/2}\Big(\sum_{k=1}^{n}(\hat{f}^*(k))^2\Big)^{1/2}.$$

Therefore, the assertion of Corollary 2.2 follows by applying Corollary 2.1 with $A=\{1,2,...,n\}$. □

Proof of Corollary 2.3. For the set $A = \{-n, ..., -1, 0, 1, ..., n\}$ from Corollary 2.1, we get

$$\sup_{f \neq 0} \frac{\left(\sum_{k=1}^{2n+1} (\hat{f}^*(k))^2\right)^{1/2}}{\|f\|_{\psi,q}} \leqslant C \frac{\sqrt{(1+n)^{-1}}}{\psi((1+n)^{-1})} \left[\log(1+n)\right]^{\frac{1}{2}-\frac{1}{q}}.$$

To prove the reversed inequality, we consider the trigonometric polynomial

$$f_n(x) = \sum_{k=-n}^{n} a_k e^{ikx}.$$

Then, by using Theorem 2 in [5] for $\psi_1(t) = t^{1/2}$, $\tau_1 = 2$, $\psi_2(t) = \psi(t)$, $\tau_2 = q$, we have that

$$\sup_{f_n \neq 0} \frac{\|f_n\|_2}{\|f_n\|_{\psi,q}} \geqslant C \frac{\sqrt{(1+n)^{-1}}}{\psi((1+n)^{-1})} \left[\log(1+n)\right]^{\frac{1}{2}-\frac{1}{q}}.$$

Therefore

$$\sup_{f \neq 0} \frac{\left(\sum_{k=1}^{2n+1} (\hat{f}^*(k))^2\right)^{1/2}}{\|f\|_{\psi,q}} \geqslant \sup_{f_n \neq 0} \frac{\|f_n\|_2}{\|f_n\|_{\psi,q}} \geqslant C \frac{\sqrt{(1+n)^{-1}}}{\psi((1+n)^{-1})} \left[\log(1+n)\right]^{\frac{1}{2}-\frac{1}{q}}.$$

The proof is complete. □

Proof of Theorem 2.2. For the generalized Lorentz space $L_{\psi,q}$, we have the relation (see [2])

$$\|f\|_{\psi,q} \asymp \sup_{\|f\|_{\bar{\psi},q'} \leqslant 1} \left| \int_0^1 f(x) g(x) dx \right|, \tag{19}$$

where $\bar{\psi}(t) = \frac{t}{\psi(t)}$, $t \in (0,1]$, $1 < q < \infty$, $q' = \frac{q}{q-1}$. Since the system $\{\varphi_n\}$ is orthonormal, then

$$\int_0^1 f_n(x) g(x) dx = \sum_{k=1}^{n} c_k \hat{g}(k), \;\; g \in L_{\bar{\psi},q'}$$

for any $n \in \mathbb{N}$.
Note that condition (6) implies that

$$\sup_{t \in (0,1]} \frac{\bar{\psi}(t)}{t^{1/2}} < \infty.$$

By applying Hölder's inequality, Theorem 2.1, and Parseval's equality, we obtain that

$$\begin{aligned} \left| \int_0^1 f_n(x) g(x) dx \right| &\leqslant \left(\sum_{k=1}^{n} |c_k|^2 \right)^{1/2} \left(\sum_{k=1}^{n} |\hat{g}(k)|^2 \right)^{1/2} \\ &\leqslant C \frac{\sqrt{\left(1 + \sum_{j=1}^{n} M_j^2\right)^{-1}}}{\bar{\psi}((1 + \sum_{j=1}^{n} M_j^2)^{-1})} \left(\log\left(1 + \sum_{j=1}^{n} M_j^2\right)\right)^{1/2 - 1/q'} \|g\|_{\bar{\psi},q'} \|f_n\|_2. \end{aligned}$$

Therefore, in virtue of relation (19), we have that

$$\|f_n\|_{\psi,q} \leqslant C \frac{\psi((1 + \sum_{j=1}^{n} M_j^2)^{-1})}{\sqrt{\left(1 + \sum_{j=1}^{n} M_j^2\right)^{-1}}} \left(\log\left(1 + \sum_{j=1}^{n} M_j^2\right)\right)^{1/q - 1/2} \|f_n\|_2$$

and 1) is proved.

We will now prove the second statement. Since $1 < p < 2$, according to item 1), it yields that

$$\|f_n\|_{\psi,p} \leqslant C \left(\frac{\sqrt{\left(1+\sum_{j=1}^{n} M_j^2\right)^{-1}}}{\psi((1+\sum_{j=1}^{n} M_j^2)^{-1})} \right)^{-1} \left(\log\left(1+\sum_{k=1}^{n} M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{2}} \|f_n\|_2. \tag{20}$$

Moreover, since $2 < q < \infty$, by Theorem 2.1 and Parseval's equality, we find that

$$\|f_n\|_2 \leqslant \frac{\sqrt{\left(1+\sum_{j=1}^{n} M_j^2\right)^{-1}}}{\psi((1+\sum_{j=1}^{n} M_j^2)^{-1})} \left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/2-1/q} \|f\|_{\psi,q}. \tag{21}$$

Now from inequalities (20) and (21), it follows that

$$\|f_n\|_{\psi,p} \leqslant C \left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/p-1/q} \|f\|_{\psi,q}$$

and 2) is proved.

Finally, let $2 < p < q < +\infty$. In the generalized Lorentz space $L_{\psi,q}$, the following inequality hold (see [36], p. 491):

$$\|g\|_{\psi,p} \leqslant \|g\|_{\psi,q}^{\frac{\frac{1}{\tau}-\frac{1}{p}}{\frac{1}{\tau}-\frac{1}{q}}} \|g\|_{\psi,\tau}^{\frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{\tau}-\frac{1}{q}}} \tag{22}$$

for $1 < \tau < p < q < +\infty$. Choose the number $\tau \in (1, 2)$. Then, according to the second statement, we have that

$$\|f_n\|_{\psi,\tau} \leqslant C \left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/\tau-1/q} \|f\|_{\psi,q}. \tag{23}$$

Now by in equality (22) setting $g = f_n$ and taking into account (23), we obtain that

$$\begin{aligned} \|f_n\|_{\psi,p} &\leqslant \|f_n\|_{\psi,q}^{\frac{\frac{1}{\tau}-\frac{1}{p}}{\frac{1}{\tau}-\frac{1}{q}}} \left\{ C \left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/\tau-1/q} \|f\|_{\psi,q} \right\}^{\frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{\tau}-\frac{1}{q}}} \\ &= C \left(\log\left(1+\sum_{j=1}^{n} M_j^2\right)\right)^{1/p-1/q} \|f\|_{\psi,q} \end{aligned}$$

and also 3) is proved. The proof is complete. □

4. Concluding remarks result

Remark 4.5. *In the case $\psi(t) = t^{1/p}(1 + |\log t|)^{\alpha}$, $1 < p < \infty$, Theorem 2.2 was previously proved in* [3]. *For the case $\alpha = 0$ see also* [2].

Remark 4.6. *In the case $\psi(t) = t^{1/p}(1 + |\log t|)^{\alpha}$, $0 < p < 2$, Theorem 2.2 for polynomials in a uniformly bounded system was proved in* [14], *Theorem 3 i).*

Remark 4.7. *A similar statement as that in Theorem 2.1 was recently proved and discussed in* [8].

Remark 4.8. *It is well-known that each concave function $\psi = \psi(t)$ has the quasi-monotonicity properties that $\frac{\psi(t)}{t}$ is non-increasing and $\psi(t)$ is non-decreasing. Moreover, the definition of the SVL clam means that the functions satisfy two quasi-monotonicity conditions but now on a logarithmic scale.*

These facts opens the possibility that some of the results in this paper can be further generalized in this direction.

From Theorem 2.1 and Theorem 2.2, we can also derive the following generalization of a result in [5]:

Proposition 4.1. *Let the functions ψ_1 and ψ_2 satisfy the conditions $1 < \alpha_{\psi_1} = \beta_{\psi_2} = 2^{1/2}$, $\frac{t^{1/2}}{\psi_1(t)} \in SVL$, $\frac{t^{1/2}}{\psi_2(t)} \in SVL$,*

$$\sup_{t\in(0,1]} \frac{\psi_2(t)}{\psi_1(t)} < \infty \tag{24}$$

and assume that the orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for some $r \in (2, +\infty]$ satisfies condition (1). If $1 < p \leqslant 2 < q < \infty$, then for any polynomial

$$f_n(x) = \sum_{k=1}^{n} c_k \varphi_k(x),$$

the following inequality holds:

$$\|f_n\|_{\psi_1,p} \leqslant C \frac{\psi_1((1+\sum_{j=1}^n M_j^2))^{-1}}{\psi_2((1+\sum_{j=1}^n M_j^2)^{-1})} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{q}} \|f_n\|_{\psi_2,q}.$$

Proof. Since $\frac{t^{1/2}}{\psi_1(t)} \in SVL$ and $1 < p \leqslant 2$, according to the first statement of Theorem 2.2, the following inequality holds:

$$\|f_n\|_{\psi_1,p} \leqslant C \left(\frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi_1((1+\sum_{j=1}^n M_j^2)^{-1})} \right)^{-1} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{2}} \|f_n\|_2.$$

Taking into account that $\frac{t^{1/2}}{\psi_2(t)} \in SVL$ and $2 < q < \infty$ by Theorem 2.1, we have that

$$\|f_n\|_2 \leqslant C \left(\frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi_2((1+\sum_{j=1}^n M_j^2)^{-1})} \right) \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{2}-\frac{1}{q}} \|f_n\|_{\psi_2,q}.$$

From these inequalities, it follows that

$$\begin{aligned}
\|f_n\|_{\psi_1,p} &\leqslant C \left(\frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi_1((1+\sum_{j=1}^n M_j^2)^{-1})} \right)^{-1} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{2}} \\
&\quad \times \frac{\sqrt{\left(1+\sum_{j=1}^n M_j^2\right)^{-1}}}{\psi_2((1+\sum_{j=1}^n M_j^2)^{-1})} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{2}-\frac{1}{q}} \|f_n\|_{\psi_2,q} \\
&= \frac{\psi_1((1+\sum_{j=1}^n M_j^2)^{-1})}{\psi_2((1+\sum_{j=1}^n M_j^2)^{-1})} \left(\log\left(1+\sum_{k=1}^n M_k^2\right)\right)^{\frac{1}{p}-\frac{1}{q}} \|f_n\|_{\psi_2,q}
\end{aligned}$$

for $1 < p \leqslant 2 < q < \infty$. The proof is complete. $\square$

Remark 4.9. *To investigate a statement as that Proposition 4.1 in the case of* $1 < p < q \leqslant 2$ *is an interesting open question. This case for polynomials in the trigonometric system was investigated in* [5]. *Furthermore, it seems to be possible to consider Proposition 4.1 also in the more general case* $1 \leqslant \beta_{\psi_2} < \alpha_{\psi_1} \leqslant 2$.

Remark 4.10. *In* [4], *it was proved that condition (24) implies that* $L_{\psi_1,p} \subset L_{\psi_2,q}$, $1 < p < q < \infty$, *in the case* $\psi_1 = \psi_2$ *see* [36].

Remark 4.11 (Final Remark)**.** *Most results concerning Fourier and Jackson–Nikol'skii type inequalities are derived for the case with bounded orthonormal systems. But since there are many important unbounded orthonormal systems, it is of importance to develop the theory to cover such cases too. Examples of such unbounded systems are the following:*
(a) $\{\chi_n\}$*–orthonormal system of Haar functions (see e.g.* [9]*). The functions* $\chi_n(t)$ *are defined as follows:* $\chi_1(t) := 1$ *for* $t \in [0,1]$ *and for* $n = 2^m + k$, $k = 1,\dots,m$ *and* $m = 0,1,\dots$ *put*

$$\chi_n(t) = \begin{cases} \sqrt{2^m}, & t \in (\frac{2k-2}{2^{m+1}}, \frac{2k-1}{2^{m+1}}), \\ -\sqrt{2^m}, & t \in (\frac{2k-1}{2^{m+1}}, \frac{2k}{2^{m+1}}), \\ 0, & t\overline{\in} \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right]. \end{cases}$$

The value of $\chi_n(t)$ *in a discontinuity point t is defined as*

$$\chi_n(t) = \frac{1}{2}\lim_{\varepsilon\to 0}[\chi_n(t+\varepsilon) + \chi_n(t-\varepsilon)].$$

(b) Let there be given an infinite sequence of integers $\{p_n\}$ *such that* $p_n \geqslant 2 \quad (n = 1,2,...)$. *We put* $m_n = p_1...p_n$, $n \geqslant 1$. *Then for any point* $t \in [0,1] \setminus A$, *there exists the unique expansion*

$$t = \sum_{k=1}^{\infty} \frac{\alpha_k(t)}{m_k}, \quad \alpha_k(t) = 0,1,...,p_k - 1,$$

where $A = \{\frac{l}{m_k}\}, l = 0,1,\dots,m_k$. *The generalized Haar system* $\chi\{p_k\} := \{\chi_n(t)\}$ *on* $[0,1]$ *is defined as follows (see* [15]*):*
$\chi_1(t) = 1$ *for* $t \in [0,1]$ *and if* $n \geqslant 2$, *then* $n = m_k + r(p_{k+1} - 1) + s$, *where* $m_0 = 1$ *and* $m_k = p_1p_2...p_k; k = 1,...; r = 0,1,...,m_k - 1; s = 1,2,...,p_{k+1} - 1$.
We put

$$\chi_n(t) := \chi_{k,r}^{(s)}(t) := \begin{cases} \sqrt{m_k} exp^{\frac{2\pi i s\alpha_{k+1}(t)}{p_{k+1}}} & ,t \in \left(\frac{r}{m_k}, \frac{r+1}{m_k}\right) \cap B, \\ 0 & ,t\overline{\in} \left[\frac{r}{m_k}, \frac{r+1}{m_k}\right], \end{cases}$$

where $B := [0,1] \setminus A$. *At the remaining points of the interval* $(0,1)$, $\chi_n(t)$ *is equal to the half-sum of its right-hand and left-hand limits on the set* $[0,1] \setminus A$, *and at the endpoints of* $[0,1]$, *to the limits from within the interval.*
(c) Other generalizations of the Haar system were defined by A.M. Olevskii [31] *and A. Kamont* [18]. *Jackson–Nikol'skii inequalities for polynomials in the* $\chi\{p_n\}$ *system in the Lebesgue spaces* L_p *and Lorentz spaces* $L_{p,\tau}$ *were proved in* [1], [19], [39] *and* [41].

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