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Estimating the Parametric Functions and Reliability Measures for Exponentiated Lifetime Distributions Family

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Highlights

- This paper focuses on estimate the parameter functions of the exponentiated lifetime distribution.
- The stress-strength parameter has been estimate using two methods.
- The validity of our work has been conducted with the simulation study under the special sub model.

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Abstract

The paper aims are to extend the theory to estimate the parameters of the exponentiated lifetime distribution. For it, in this paper, we derived the probability density function, cumulative density, reliability function and the stress-strength parameter of the distribution. To estimate the parameters of such distribution, we considered the maximum likelihood and uniformly minimum variance unbiased methods. The validity of the proposed work has been conducted over the simulation study of both estimation methods under the special sub model as exponentiated inverse Gompertz distribution. Finally, some real data has been taken to conduct an analysis and to discuss the effectiveness and advantages of the established work by comparing with other methods.

1. INTRODUCTION

One of the most significant features of statistical investigation in applied sciences is the modeling and examining lifetime data. In the literature, numerous lifetime distributions such as the exponential, Weibull, Gamma, or their generalizations have been included. In reliability analysis, exponentiated distribution is a class of distribution that affords a more manageable model and widespread family of exponentiated distributions that holds the most common distributions in a lifetime. Mudholkar and Srivastava [1] proposed exponentiated Weibull where Mudholkar et al. [2] prolonged the applications of the exponentiated Weibull distribution in reliability and survival studies. Nadarajah and Kotz [3] acquainted four exponentiated type distributions namely the exponentiated gamma, exponentiated Weibull, exponentiated Gumbel, and the exponentiated Frechet distributions. Delgarm and Zadkarami [4] extend the three-parameter modified Weibull (MW) distribution to intend a four-parameter distribution named as the modified Weibull Poisson distribution, including such sub-models as Exponential Poisson, Weibull Poisson, and Rayleigh Poisson. Pourreza et al. [5] investigated some sub-models of the Gamma-X family of distributions.

Probability distributions are frequently employed in survival analysis that prepares intuition of numerous parameters and functions, especially the failure rate or hazard function, which manifests the necessity of estimation of the probability density function (PDF) and the cumulative distribution function (CDF). In the direction of estimating such PDF and CDF for some distribution, several researchers have put forwards

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different kinds of methods from the last decades. Among distributions, we refer to read the articles on the Pareto by Dixit and Jabbari Nooghabi [6]; Weibull extension by Bagheri et al. [7]; the exponential Gumbel by Bagheri et al. [8]; the exponentiated Weibull by Alizadeh et al. [9]; generalized exponential by Alizadeh et al. [10]; the Lindley by Maiti and Mukherjee [11] and the generalized inverted Weibull by Ghasemi Cherati et al. [12]. In the field of probability theory, the concept of the reliability analysis is one of the most emerging areas. The reliability function R(t) depicts the probability of the survival of the component until time t under the prescribed conditions which depends on the components failure rate. In these days, a concept of the reliability has been widely applied in many fields such as quality control, genetics, physics, biostatistics, psychology, technology, medicine and economics. The estimation of reliability measures to show system efficiency plays a crucial role in the reliability analysis.

To estimate the different parameters associated with the probabilities and reliability theory, Bekker and Roux [13] represented the maximum likelihood, Bayes and empirical Bayes estimators of the reliability measures of the Maxwell distribution. Krishna and Kumar [14] discussed Lindley reliability model and the reliability measures estimation by maximum likelihood and Bayesian approach for an incomplete set of data using various loss functions. They also, obtained interval estimation and coverage probability for the parameters. Rastogi and Tripathi [15] considered the exponentiated half-logistic distribution and obtained the reliability and hazard rate function estimators under progressive type II censoring. Abouei Ardakan et al. [16] investigated the reliability of components as a function of time since reliability is time dependent. The redundancy allocation problem in reliability optimization is adjusted by a new criterion as mission design life. Amirzadi et al. [17] represented the Bayes estimator of the reliability function of inverse generalized Weibull distribution under new loss function.

Recently, Dmitriev and Koshkin [18] designed a reliability estimator under the condition of nonparametric prior using auxiliary information and illustrated the asymptotic distribution of the optimal mean-square error estimators and adaptive optimal estimators. Sankaran and Kumar [19] proposed a new lifetime distribution as proportional hazards relevated Weibull and centered on the reliability properties of a proportional hazards relevation transform. Roy and Gupta [20] considered the renewal of a coherent system that may fail either on the failure of its first or second component and achieve the reliability function of the considered coherent system which is implemented with two cold standby components. A reliability model of a system with multiple correlated failure modes where Copula function is applied modeling the correlation among failure modes was introduced by Gu et al. [21]. Zhang et al. [22] acquainted a relative dispersion factor to modeling the failure dependence of components and a series system reliability allocation model contains three types of failure modes. They divided the factors influencing system reliability allocation into direct and indirect categories.

The stress-strength reliability of a system specifies the probability that the system will appropriately operate until the strength exceeds stress. In this case, the system fails if and only if, at any time, the applied stress is greater than its strength. The estimation of stress-strength reliability has been widely studied in many related fields such as Chaturvedi and Tomer [23], Kundu and Gupta [24], Turkkan and Pham-Gia [25]. Rezaei et al. [26] discussed the classical inference of reliability function, when strength and stress both are three-parameter generalized. Al-Mutairi et al. [27] investigated the estimation of the stress-strength parameter with Lindley random variables with different shape parameters. Nadar et al. [28] obtained the classical and Bayesian estimation of reliability measures for Kumaraswamy's distribution. Alghamdi and Percy [29] study survival equivalence factors and mean equivalence factors of a system of exponentiated Weibull components, which has subsystems. They improved the reliability of the system by reduction method and several duplication methods. Kizilaslan and Nadar [30] consider k strength components system, where each component is constructed by a pair of dependent bivariate Kumaraswamy elements and each element is exposed to a random Kumaraswamy stress variable. Rezaei et al. [31] obtained the maximum likelihood (ML) and uniformly minimum variance unbiased (UMVU) estimation of the stress-strength parameter, where the variables have generalized Lindley distribution type 5 Gamma distribution and the Bayes estimators derived under informative and noninformative prior.

Keeping in mind all the above discussion, the main aim of this study is to achieve the ML and UMVU estimators of the PDF, CDF, and reliability functions for exponentiated distributions class. Further, the performance of the ML and UMVU estimators is compared based on the mean square error (MSE). Accordingly, the primary objectives of the work are listed below

1) to obtain ML and UMVU estimators of PDF and their MSEs,
2) to obtain ML and UMVU estimators of CDF and their MSEs,
3) to obtain ML and UMVU estimators of reliability function and their MSEs,
4) to obtain ML and UMVU estimators of stress-strength parameter and their MSEs,
5) to compare the results by both Monte Carlo simulation and real data.

The rest of the paper is designed as follows: Section 2 describes the model of exponentiated distributions and obtains the PDF, CDF, and reliability function. In Section 3, ML estimators of the PDF, the CDF, and reliability function are calculated with their MSEs. In Section 4, UMVU estimators and MSE of the PDF, CDF, and reliability function are determined. In Section 5, ML and UMVU estimators and MSE of the stress-strength parameter are measured. A Monte Carlo simulation study and comparison of the ML and UMVU estimation methods are presented in Section 6. Two real data sets are investigated in Section 7. Finally, a concrete conclusion is given in Section 8.

2. THE MODEL

A method for generating new families of continuous distribution are introduced by Alzaatreh et al. [32], which a random transformer variable X is used to transform another random transformed variable T. A special case of the T-X family of distributions is shown as

$$G(x) = \int_0^{-\ln(1-\mathrm{F}(x))} z(t)dt = Z(-\ln(1-\mathrm{F}(x))),$$

where $F(x)$, $z(t)$ and $Z(t)$ are the CDF of X, PDF of T on t, $0 < t < \infty$ and CDF of t, respectively. If the random variable T follows exponential distribution with parameter α with PDF

$$z(t) = \alpha e^{-\alpha t} \ , \ \ t > 0 \ , \alpha > 0,$$

then CDF and PDF of exponential-X distribution with shape parameter α, named as exponentiated distribution, are given by

$$G(x) = 1 - \left(1 - F(x)\right)^{\alpha}, \qquad x > 0, \ \alpha > 0 \quad , \quad g(x) = \alpha f(x)(1 - F(x))^{\alpha-1},$$

where $\mathrm{F}(x)$ depend on the known parameter θ. The corresponding reliability function and the hazard function are shown respectively, as

$$R(x) = (1 - F(x))^{\alpha} = R_F(x)^{\alpha},$$
$$h(x) = \frac{\alpha f(x)}{1-F(x)} = \alpha h_F(x).$$

Note that $\mathrm{R_F}(x) < \mathrm{R}(x)$ and $\mathrm{h(x)} < h_{\mathrm{F}}(x)$ for $0 < \alpha < 1$ and $\mathrm{R}(x) < R_{\mathrm{F}}(x)$ and $h_{\mathrm{F}}(x) < \mathrm{h}(x)$ for $\alpha > 1$ in x.

Remark 1. The cumulative hazard function of X, can be expressed as $H(x) = -\ln\left(1 - \mathrm{F}(x)\right)$, so PDF and CDF of the exponential-X distribution can be rewritten as

$$G(x) = Z\left(H(\mathrm{x})\right), \quad g(x) = \alpha \mathrm{f}(x)\left(1 - \mathrm{F}(x)\right)^{\alpha-1} = h_{\mathrm{F}}(x) R_F(x)^{\alpha}.$$

Therefore, the exponential-X distribution can be considered as a family of distributions arising from a weighted hazard function. Let $X_1, X_2, \ldots, X_n$ be a random sample from the proposed exponentiated distributions, the joint PDF of $X_1, X_2, \ldots, X_n$ is

$$g(x_1, x_2, \dots, x_n, \alpha) = \prod_{i=1}^{n} \alpha f(x_i)(1 - F(x_i))^{\alpha-1},$$

which can be rewritten as

$$g(x_1, x_2, \dots, x_n, \alpha) = \alpha^n e^{\alpha \sum_{i=1}^{n} \ln(1-\mathrm{F}(x_i))} \prod_{i=1}^{n} \frac{f(x_i)}{1-\mathrm{F}(x_i)}.$$

Proposition 1. The UMVU estimator of the parameter α is represented as $\tilde{\alpha} = \frac{n-1}{S}$, where $S = -\sum_{i=1}^{n} \ln(1 - F(X_i))$ and $MSE(\tilde{\alpha}) = \frac{\alpha^2}{(n-2)}$.

Proof. By Neyman factorization theorem, $S = -\sum_{i=1}^{n} \ln(1 - F(X_i))$ is the sufficient complete statistic for the family of exponentiated distributions. The statistic S is minimal and gamma random variable with parameters $\left(n, \frac{1}{\alpha}\right)$ as $q(s) = \frac{\alpha^n}{\Gamma(n)} s^{n-1} e^{-\alpha s}$, $s > 0, \alpha > 0$. According to Lehmann-Scheffe theorem, the UMVU estimator of α is given by $\tilde{\alpha} = \frac{n-1}{S}$ with $MSE(\tilde{\alpha}) = \frac{\alpha^2}{(n-2)}$, $n > 2$. Since $MSE(\tilde{\alpha}) \xrightarrow{n\to\infty} 0$, then UMVU of α is consistent.

3. THE ML ESTIMATORS OF PROBABILITY DENSITY, CUMULATIVE DENSITY AND RELIABILTY FUNCTION

Let $X_1, X_2, \dots, X_n$ be a random sample of life time from an exponentiated distribution. So, the log-likelihood function is

$$\ln L(\alpha, \underline{x}) = n \ln \alpha + \sum_{i=1}^{n} \ln\big(f(x_i)\big) + (\alpha - 1) \sum_{i=1}^{n} \ln(1 - F(x_i)).$$

Evidently, the ML estimator of α, denoted by $\hat{\alpha}$ is obtained as $\hat{\alpha} = \frac{n}{-\sum_{i=1}^{n} \ln(1-F(x_i))} = \frac{n}{S}$.

Corollary 1. The PDF of the ML estimator of α has inverse gamma distribution with the parameters $(n, n\alpha)$ and the PDF is represented at given point w as

$$f_{\alpha}(w) = \frac{(n\alpha)^n}{w^{n+1}\Gamma(n)} e^{-\frac{n\alpha}{w}}, \qquad w > 0, \alpha > 0.$$

Proof. Since S has gamma distribution with $(n, \frac{1}{\alpha})$, then PDF of $\hat{\alpha}$ can be easily obtained.

Corollary 2. The ML estimator of α is biased and the MSE of $\hat{\alpha}$ is represented as

$$MSE(\hat{\alpha}) = \frac{(n+2)\alpha^2}{(n-1)(n-2)}, \qquad n > 2.$$

Proof. The r-th moment of $\hat{\alpha}$ is obtained as

$$E(\hat{\alpha}^r) = \int_0^{\infty} \frac{w^r (n\alpha)^n}{w^{n+1}\Gamma(n)} e^{-\frac{n\alpha}{w}} dw = \int_0^{\infty} \frac{y^{n-r-1}(n\alpha)^n}{\Gamma(n)} e^{-n\alpha y} dy = \frac{\Gamma(n-r)(n\alpha)^r}{\Gamma(n)}, \quad n > r.$$

Let $r = 1$, then $\hat{\alpha}$ is biased with $E(\hat{\alpha}) - \hat{\alpha} = \frac{\alpha}{n-1}$. The MSE of $\hat{\alpha}$ is obtained as

$$MSE(\hat{\alpha}) = \frac{(n\alpha)^2}{(n-1)(n-2)} - \frac{2n\alpha^2}{(n-1)} + \alpha^2 = \frac{(n+2)\alpha^2}{(n-1)(n-2)}, \quad n > 2.$$

It is clear that $\hat{\alpha}$ is a consistent estimator, since $MSE(\hat{\alpha}) \xrightarrow{n\to\infty} 0$. Based on the invariance property of ML estimators, the ML estimators of the PDF, CDF and reliability function at a specified point x are obtained as

$$\hat{g}(x) = \hat{\alpha} f(x)(1-\mathrm{F}(x))^{\hat{\alpha}-1}, \quad \hat{G}(x) = 1 - \left(1-\mathrm{F}(x)\right)^{\hat{\alpha}}, \quad \hat{R}(x) = (1-\mathrm{F}(x))^{\hat{\alpha}}, \quad x > 0,$$

respectively. Since $\hat{g}(x)$, $\hat{G}(x)$ and $\hat{R}(x)$ are continuous functions of consistent estimator $\hat{\alpha}$, so, they are also consistent. The bias and MSE of the $\hat{g}(x), \hat{G}(x)$ and $\hat{\mathrm{R}}$(x) are discussed in the next Theorems.

Theorem 1. The ML estimators $\hat{g}(x), \hat{G}(x)$ and $\hat{R}(x)$ are biased and

(i) $E(\hat{g}(x)^r) = \frac{2(n\alpha)^n}{\Gamma(n)} h_F(x)^r \psi_r(x) K_{r-n}\left(2\sqrt{-n\alpha r \ln(1-\mathrm{F}(x))}\right)$,

where $\psi_r(x) = [\frac{n\alpha}{-r\ln(1-\mathrm{F}(x))}]^{\frac{r-n}{2}}$, $\Psi_i(x) = 2(\frac{n\alpha}{-i\ln(1-\mathrm{F}(x))})^{-\frac{n}{2}}$, and $\mathrm{K}_\nu(.)$ denotes the modified Bassel function of the second kind of order ν that is defined as $K_\nu\left(2\sqrt{\beta\varphi}\right) = 0.5(\frac{\varphi}{\beta})^{\frac{\nu}{2}} \int_0^\infty x^{\nu-1} e^{-\frac{\beta}{x}} e^{-\varphi x} dx$,

(ii) $E\left(\hat{G}(x)^r\right) = 1 + \sum_{i=1}^r \binom{r}{i} (-1)^i \frac{(n\alpha)^n}{\Gamma(n)} \Psi_i(x) K_{-n}(2\sqrt{-n\alpha i \ln(1-\mathrm{F}(x))})$,

(iii) $E\left(\hat{R}(x)^r\right) = \frac{(n\alpha)^n}{\Gamma(n)} \Psi_r(x) K_{-n}(2\sqrt{-n\alpha r \ln(1-\mathrm{F}(x))})$.

Proof. (i) Using PDF of ML $\hat{\alpha}$ at point w, The r-th moment of $\hat{g}(x)$ is obtained as

$$\begin{aligned} E(\hat{g}(x)^r) &= \int_0^\infty [wf(x)(1-\mathrm{F}(x))^{w-1}]^r \frac{(n\alpha)^n}{w^{n+1}\Gamma(n)} e^{-\frac{n\alpha}{w}} dw, \\ &= \frac{(n\alpha)^n}{\Gamma(n)} (f(x))^r (1-\mathrm{F}(x))^{-r} \int_0^\infty w^{r-n-1} (1-\mathrm{F}(x))^{rw} e^{-\frac{n\alpha}{w}} dw, \\ &= \frac{(n\alpha)^n}{\Gamma(n)} h_{\mathrm{F}}(x)^r \int_0^\infty w^{r-n-1} e^{rw\ln(1-\mathrm{F}(x))} e^{-\frac{n\alpha}{w}} dw, \\ &= \frac{2(n\alpha)^n}{\Gamma(n)} h_{\mathrm{F}}(x)^r [\frac{n\alpha}{-r\ln(1-\mathrm{F}(x))}]^{\frac{r-n}{2}} K_{r-n}(2\sqrt{-n\alpha r \ln(1-\mathrm{F}(x))}). \end{aligned}$$

By substituting $r = 1$, we obtain $E\left(\hat{g}(x)\right) \neq g(x)$. So, $\hat{g}(x)$ is a biased estimator of $g(x)$.

(ii) The r-th moment of $\hat{G}(x)$ is obtained as

$$\begin{aligned} E\left(\hat{G}(x)^r\right) &= \int_0^\infty [1-(1-\mathrm{F}(x))^w]^r \frac{(n\alpha)^n}{w^{n+1}\Gamma(n)} e^{-\frac{n\alpha}{w}} dw, \\ &= \int_0^\infty \sum_{i=0}^r (-1)^i \binom{r}{i} [1-\mathrm{F}(x)]^{iw} \frac{(n\alpha)^n}{w^{n+1}\Gamma(n)} e^{-\frac{n\alpha}{w}} dw, \\ &= \sum_{i=0}^r \binom{r}{i} (-1)^i \frac{(n\alpha)^n}{\Gamma(n)} \int_0^\infty w^{-n-1} (1-\mathrm{F}(x))^{iw} e^{-\frac{n\alpha}{w}} dw, \\ &= \sum_{i=0}^r \binom{r}{i} (-1)^i \frac{(n\alpha)^n}{\Gamma(n)} \int_0^\infty w^{-n-1} e^{iw\ln(1-\mathrm{F}(x))} e^{-\frac{n\alpha}{w}} dw, \\ &= 1 + \sum_{i=1}^r \binom{r}{i} (-1)^i \frac{(n\alpha)^n}{\Gamma(n)} [2(\frac{n\alpha}{-i\ln(1-\mathrm{F}(x))})^{-\frac{n}{2}}] K_{-n}(2\sqrt{-n\alpha i \ln(1-\mathrm{F}(x))}). \end{aligned}$$

By substituting $r = 1$, then $E\left(\hat{G}(x)\right) \neq G(x)$. So, $\hat{G}(x)$ is a biased estimator of $G(x)$.

(iii) The r-th moment of $\hat{R}(x)$ is calculated as

$$E\left(\hat{R}(x)^r\right) = \int_0^\infty (1 - F(x))^{rw} \frac{(n\alpha)^n}{w^{n+1}\Gamma(n)} e^{-\frac{n\alpha}{w}} dw,$$
$$= \frac{(n\alpha)^n}{\Gamma(n)} \int_0^\infty w^{-n-1} e^{rw \ln(1-\mathrm{F}(x))} e^{-\frac{n\alpha}{w}} dw,$$
$$= \frac{(n\alpha)^n}{\Gamma(n)} [2(\frac{n\alpha}{-r\ln(1-\mathrm{F}(x))})^{-\frac{n}{2}}] K_{-n}(2\sqrt{-n\alpha r \ln(1-\mathrm{F}(x))}).$$

By substituting $r = 1$, then $E\left(\hat{R}(x)\right) \neq R(x)$. So, $\hat{R}(x)$ is a biased estimator of $R(x)$ and the proof is completed.

Theorem 2. The MSEs of $\hat{g}(x)$,$\hat{G}(x)$ and $\hat{R}(x)$ are given by

$$MSE\left(\hat{g}(x)\right) = \frac{2(n\alpha)^n}{\Gamma(n)} h_F(x)^2 \psi_2(x) K_{2-n}\left(2\sqrt{-2n\alpha \ln(1-\mathrm{F}(x))}\right)$$
$$-\frac{4(n\alpha)^n}{\Gamma(n)} h_F(x)\psi_1(x) K_{1-n}\left(2\sqrt{-n\alpha\ln(1-\mathrm{F}(x))}\right)(\alpha f(x)(1-\mathrm{F}(x))^{\alpha-1})$$
$$+(\alpha f(x)(1-\mathrm{F}(x))^{\alpha-1})^2,$$
$$MSE\left(\hat{G}(x)\right) = \sum_{i=1}^{2} \binom{2}{i} (-1)^i \frac{(n\alpha)^n}{\Gamma(n)} \left[2\left(\frac{n\alpha}{-i\ \ln(1-\mathrm{F}(x))}\right)^{-\frac{n}{2}}\right] K_{-n}\left(2\sqrt{-n\alpha i\ \ln(1-\mathrm{F}(x))}\right)$$
$$+\frac{4(n\alpha)^n}{\Gamma(n)} (\frac{n\alpha}{-\ln(1-\mathrm{F}(x))})^{-\frac{n}{2}} K_{-n}(2\sqrt{-n\alpha\ln(1-\mathrm{F}(x))})(1-(1-\mathrm{F}(x))^\alpha)$$
$$+(1-(1-\mathrm{F}(x))^\alpha)^2,$$

and

$$MSE\left(\hat{R}(x)\right) = \frac{(n\alpha)^n}{\Gamma(n)} \Psi_2(x) K_{-n}(2\sqrt{-2n\alpha\ln(1-\mathrm{F}(x))})$$
$$-\frac{2(n\alpha)^n}{\Gamma(n)} \Psi_1(x) K_{-n}\left(2\sqrt{-n\alpha\ln\left(1-\mathrm{F}(x)\right)}\right)\left(1-\mathrm{F}(x)\right)^\alpha + \left(1-\mathrm{F}(x)\right)^{2\alpha}.$$

Proof. By using Theorem 1, the proof can be obtained easily.

4. THE UMVU ESTIMATORS OF PROBABILITY DENSITY, CUMULATIVE DENSITY AND RELIABILTY FUNCTION

Suppose $X_1, X_2, \ldots, X_n$ be a random sample from the family of exponentiated distributions. The UMVU estimators of the PDF, CDF and reliability function are discussed in the following.

Theorem 3. At a specified point *x*,

(i) The UMVU estimator of $g(x)$ is given by

$$\tilde{g}(x) = \begin{cases} \frac{(n-1)h_F(x)}{s} (1 + s^{-1}\, ln\,(1-F(x)))^{n-2}, & s > -ln\,(1-F(x)) \\ 0, & otherwise \end{cases},$$

(ii) The UMVU estimator of $\mathrm{G}(x)$ is given by

$$\tilde{G}(x) = \begin{cases} 1-(1+s^{-1}\,ln(1-F(x)))^{n-1}, & s > -\,ln(1-F(x)) \\ 0, & otherwise \end{cases},$$

(iii) The UMVU estimator of $R(x)$ is given by

$$\tilde{R}(\mathrm{x}) = \begin{cases} (1+\mathrm{s}^{-1}\ln(1-\mathrm{F}(x)))^{n-1}, & \mathrm{s} > -\ln(1-\mathrm{F}(x)) \\ 0, & otherwise \end{cases}.$$

Proof. (i) Consider complete and sufficient statistic S, function $K(S)$ will be the UMVUE of $g(x)$, if $K(\mathrm{S})$ be unbiased, i.e. $\mathrm{E}\big(K(\mathrm{S})\big) = g(x)$. So,

$$\frac{\alpha^n}{\Gamma(n)}\int_0^\infty K(s)s^{n-1}e^{-\alpha s}ds = \alpha f(x)(1-F(x))^{\alpha-1},$$

consequently

$$\frac{\alpha^{n-1}}{\Gamma(n)}\int_0^\infty K(s)s^{n-1}e^{-\alpha(s+\ln(1-\mathrm{F}(x)))}\mathrm{ds} = f(x)(1-\mathrm{F(x)})^{-1},$$

where the equality holds if consider $K(s)$ as

$$\tilde{g}(x) = K(s) = \begin{cases} \dfrac{(n-1)f(x)}{s(1-F(x))}(1+s^{-1}\,ln\,(1-F(x)))^{n-2}, & s > -ln\,(1-F(x)) \\ 0, & otherwise \end{cases}.$$

Hence this part is completed.

(ii) The unbiased estimator of $G(x)$ at a specified point x is given by

$$\tilde{G}(x) = \int_0^x \tilde{g}(t)dt,$$

and by part (i), it follows that if $\mathrm{s} \le -\ln(1-\mathrm{F}(x))$ then $\widetilde{\mathrm{G}}(\mathrm{x}) = 0$, otherwise we have

$$\begin{aligned} \tilde{G}(x) &= \int_0^x \tilde{g}(t)\,dt, \\ &= \int_0^x \frac{(n-1)f(t)(1-F(t))^{-1}}{s}(1+s^{-1}\,ln\,(1-F(t)))^{n-2}\,dt, \\ &= \int_0^{-s^{-1}\,ln\,(1-F(x))}(n-1)(1-u)^{n-2}\,du, \\ &= 1-(1+s^{-1}\,ln(1-F(x)))^{n-1}, \qquad s > -\,ln(1-F(x)). \end{aligned}$$

which complete this part.

(iii) The unbiased estimator of R(t) is $\tilde{R}(x) = \int_{\mathrm{x}}^\infty \tilde{\mathrm{g}}(t)dt$, since

$$\begin{aligned} E\left(\tilde{R}(x)\right) &= \int_0^\infty \tilde{R}(x)q(s)ds, \\ &= \int_0^\infty(\int_{\mathrm{x}}^\infty \tilde{g}(t)dt)q(s)ds, \\ &= \int_x^\infty\int_0^\infty \tilde{g}(t)q(s)ds\,dt, \\ &= \int_x^\infty E\big(\tilde{g}(t)\big)dt, \end{aligned}$$

$$= \int_x^\infty g(t)dt = R(x).$$

Note that, the conditions of Fubini's theorem are satisfied for the change of order of integration.
By part (i), for $S \leq -\ln(1-F(x))$, we have $\tilde{R}(x) = 0$, otherwise

$$\begin{aligned}
\tilde{R}(x) &= \int_x^\infty \tilde{g}(t)\,dt, \\
&= \int_x^\infty \frac{(n-1)f(t)(1-F(t))^{-1}}{s}(1 + s^{-1}\,ln\,(1-F(t)))^{n-2}\,dt, \\
&= \int_{-s^{-1}\,ln(1-F(x))}^{1}(n-1)(1-u)^{n-2}\,du, \\
&= (1 + s^{-1}\,ln(1-F(x)))^{n-1}, \qquad s > -\ln(1-F(x)).
\end{aligned}$$

The moments and MSEs of the UMVU estimator of the PDF, CDF and reliability function are investigated in the following.

Theorem 4. The moment of the UMVU estimators are listed as

(i) The r-th moment of the UMVU estimator of the PDF is represented as

$$E(\tilde{g}(x)^r) = [(n-1)h_F(x)]^r \sum_{j=0}^{r(n-2)} \binom{r(n-2)}{j} (\ln\,(1-F(x)))^j \alpha^{r+j}\varphi_j(r),$$

where $\varphi_j(r) = \frac{\Gamma(n-r-j,-\alpha\ln\,(1-F(x))}{\Gamma(n)}$.

(ii) The second moment of the UMVU estimator of the CDF is given by

$$\begin{aligned}
E(\tilde{G}(x)^2) = \varphi_0(0) - \frac{2}{\Gamma(n)}\sum_{j=0}^{n-1}\binom{n-1}{j}(\alpha\ln(1-F(x)))^j\varphi_j(0) \\
+ \sum_{j=0}^{2n-2}\binom{2n-2}{j}(\alpha\ln(1-F(x)))^j\varphi_j(0).
\end{aligned}$$

(iii) The r-th moment of the UMVU estimator of the reliability function is shown as

$$E(\tilde{R}(x)^r) = \sum_{j=0}^{r(n-1)}\binom{r(n-1)}{j}(\alpha\ln\,(1-F(x)))^j\varphi_j(0).$$

Proof. (i) By the PDF of S, we have

$$\begin{aligned}
E(\tilde{g}(x)^r) &= \int_{-ln\,(1-F(x))}^{\infty}\left[\frac{(n-1)h_F(x)}{s}(1+s^{-1}\,ln\,(1-F(x)))^{n-2}\right]^r \frac{\alpha^n}{\Gamma(n)} s^{n-1}e^{-\alpha s}ds, \\
&= \frac{\alpha^n[(n-1)h_F(x)]^r}{\Gamma(n)}\int_{-ln\,(1-F(x))}^{\infty}\left(1+\frac{ln\,(1-F(x))}{s}\right)^{r(n-2)} s^{n-r-1}\,e^{-\alpha s}ds, \\
&= \frac{\alpha^n[(n-1)h_F(x)]^r}{\Gamma(n)}\int_{-ln\,(1-F(x))}^{\infty}\sum_{j=0}^{r(n-2)}\binom{r(n-2)}{j}\left(\frac{ln\,(1-F(x))}{s}\right)^j s^{n-r-1}e^{-\alpha s}ds, \\
&= \frac{\alpha^n[(n-1)h_F(x)]^r}{\Gamma(n)}\sum_{j=0}^{r(n-2)}\binom{r(n-2)}{j}(ln\,(1-F(x)))^j\int_{-ln\,(1-F(x))}^{\infty} s^{n-r-j-1}e^{-\alpha s}ds, \\
&= [(n-1)h_F(x)]^r\sum_{j=0}^{r(n-2)}\binom{r(n-2)}{j}(ln\,(1-F(x)))^j\alpha^{r+j}\frac{\Gamma(n-r-j,-\alpha ln\,(1-F(x))}{\Gamma(n)}.
\end{aligned}$$

(ii) The second moment of $\tilde{G}(x)$ is obtained as

$$E\big(\tilde{G}(x)^2\big) = \int_{-\,ln(1-F(x))}^{\infty} [1-(1+s^{-1}\,ln(1-F(x)))^{n-1}]^2 \frac{\alpha^n}{\Gamma(n)} s^{n-1}e^{-\alpha s}ds,$$
$$= \int_{-\,ln(1-F(x))}^{\infty} \frac{\alpha^n s^{n-1}e^{-\alpha s}}{\Gamma(n)} ds - 2\int_{-\,ln(1-F(x))}^{\infty} \left(1+\frac{ln(1-F(x))}{s}\right)^{n-1} \frac{\alpha^n s^{n-1}e^{-\alpha s}}{\Gamma(n)} ds,$$
$$+\int_{-\,ln(1-F(x))}^{\infty} \left(1+\frac{ln(1-F(x))}{s}\right)^{2n-2} \frac{\alpha^n s^{n-1}e^{-\alpha s}}{\Gamma(n)} ds,$$
$$= \frac{\Gamma(n,-\,\alpha ln(1-F(x)))}{\Gamma(n)} + I + II,$$

where

$$I = -2\frac{\alpha^n}{\Gamma(n)} \int_{-\,ln(1-F(x))}^{\infty} \left(1+\frac{ln(1-F(x))}{s}\right)^{n-1} s^{n-1}e^{-\alpha s}ds,$$
$$= -2\frac{\alpha^n}{\Gamma(n)} \int_{-\,\ln(1-\mathrm{F}(x))}^{\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{\ln(1-\mathrm{F}(x))}{s}\right)^{j} s^{n-1}e^{-\alpha s}ds,$$
$$= -2\frac{\alpha^n}{\Gamma(n)} \sum_{j=0}^{n-1} \binom{n-1}{j} \big(\ln(1-\mathrm{F}(x))\big)^{j} \int_{-\,\ln(1-\mathrm{F}(x))}^{\infty} s^{n-j-1}e^{-\alpha s}ds,$$
$$= \frac{-2}{\Gamma(n)} \sum_{j=0}^{n-1} \binom{n-1}{j} \big(\alpha\,ln(1-F(x))\big)^{j} \frac{\Gamma(n-j,-\,\alpha ln(1-F(x)))}{\Gamma(n)}.$$

and

$$II = \frac{\alpha^n}{\Gamma(n)} \int_{-\,\ln(1-\mathrm{F}(x))}^{\infty} \sum_{j=0}^{2n-2} \binom{2n-2}{j} \left(\frac{\ln(1-\mathrm{F}(x))}{s}\right)^{j} s^{n-1}e^{-\alpha s}ds,$$
$$= \frac{\alpha^n}{\Gamma(n)} \sum_{j=0}^{2n-2} \binom{2n-2}{j} (\,ln(1-F(x))^{j} \int_{-\,ln(1-F(x))}^{\infty} s^{n-j-1}e^{-\alpha s}ds,$$
$$= \sum_{j=0}^{2n-2} \binom{2n-2}{j} (\alpha ln(1-F(x)))^{j} \frac{\Gamma(n-j,-\,\alpha ln(1-F(x)))}{\Gamma(n)}.$$

(iii) Eventually, the r-th moment of the $\tilde{R}(x)$ is computed as following

$$E\big(\tilde{R}(x)^r\big) = \int_{-ln\,(1-F(x))}^{\infty} (1+s^{-1}\,ln(1-F(x)))^{r(n-1)} \frac{\alpha^n}{\Gamma(n)} s^{n-1}e^{-\alpha s}ds,$$
$$= \frac{\alpha^n}{\Gamma(n)} \int_{-ln\,(1-F(x))}^{\infty} \left(1+\frac{ln\,(1-F(x))}{s}\right)^{r(n-1)} s^{n-1}\,e^{-\alpha s}ds,$$
$$= \frac{\alpha^n}{\Gamma(n)} \int_{-ln\,(1-F(x))}^{\infty} \sum_{j=0}^{r(n-1)} \binom{r(n-1)}{j} \left(\frac{ln\,(1-F(x))}{s}\right)^{j} s^{n-1}e^{-\alpha s}ds,$$
$$= \frac{\alpha^n}{\Gamma(n)} \sum_{j=0}^{r(n-1)} \binom{r(n-1)}{j} (ln\,(1-F(x)))^{j} \int_{-ln\,(1-F(x))}^{\infty} s^{n-j-1}e^{-\alpha s}ds,$$
$$= \sum_{j=0}^{r(n-1)} \binom{r(n-1)}{j} (\alpha ln\,(1-F(x)))^{j} \frac{\Gamma(n-j,-\alpha ln\,(1-F(x)))}{\Gamma(n)}.$$

which complete the proof.

Theorem 5. The MSEs of $\tilde{g}(x)$, $\tilde{\mathrm{G}}(x)$ and $\tilde{R}(x)$ are given respectively as

$$MSE\big(\tilde{g}(x)\big) = [(n-1)h_F(x)]^2 \sum_{j=0}^{2(n-2)} \binom{2(n-2)}{j} (ln\,(1-F(x)))^{j} \alpha^{2+j} \varphi_j(2),$$
$$-(\alpha f(x)(1-F(x))^{\alpha-1})^2,$$

$$MSE\left(\tilde{G}(x)\right)=\frac{\Gamma\left(n,-\alpha ln\left(1-F(x)\right)\right)}{\Gamma(n)}-\frac{2}{\Gamma(n)}\sum_{j=0}^{n-1}\binom{n-1}{j}\left(\alpha ln\left(1-F(x)\right)\right)^{j}\varphi_j(0)$$
$$+\sum_{j=0}^{2n-2}\binom{2n-2}{j}(-\alpha ln(1-F(x)))^{j}\varphi_j(0)-(1-(1-F(x))^{\alpha})^2,$$

and

$$MSE\left(\tilde{R}(x)\right)=\sum_{j=0}^{2(n-1)}\binom{2(n-1)}{j}(\alpha\ln(1-\mathrm{F}(x)))^{j}\varphi_j(0)-(1-\mathrm{F}(x))^{2\alpha}.$$

Proof. The proof can be easily obtained.

5. THE ESTIMATOR OF THE STRESS STRENGTH PARAMETER

Consider Y as "stress" experienced by the component and random variable X representing "strength" of a component, the stress-strength parameter can be interpreted as an assessment of reliability to dominate the possible stress. In the reliability analyzes, the stress strength parameter represented as $P=\mathrm{P}(\mathrm{X}>Y)$. Suppose random strength X and random stress Y be two independent random variables following exponentiated distributions with different shape parameters and PDFs are shown as $g_1(x,\alpha_1)$ and $g_2(y,\alpha_2)$, respectively. Let $X_1,X_2,\dots,X_n$ and $Y_1,Y_2,\dots,Y_m$ be two independent random samples, $S=-\sum_{i=1}^{n}\ln(1-\mathrm{F}(X_i))$ and $V=-\sum_{i=1}^{m}\ln(1-\mathrm{F}(Y_i))$, then

$$g(x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_m,\alpha_1,\alpha_2)=\alpha_1{}^{n}\alpha_2{}^{m}e^{\alpha_1\sum_{i=1}^{n}\ln(1-\mathrm{F}(x_i))+\alpha_2\sum_{i=1}^{m}\ln(1-\mathrm{F}(y_i))}$$
$$\times\prod_{i=1}^{n}\frac{f(x_i)}{1-\mathrm{F}(\mathrm{x}_i)}\prod_{i=1}^{m}\frac{f(y_i)}{1-\mathrm{F}(\mathrm{y}_i)}.$$

Since $g(x,y,\alpha_1,\alpha_2)=g_1(x,\alpha_1)g_2(y,\alpha_2)$ belongs to full-rank exponential family, therefore (S,V) is sufficient and complement statistic for exponential family of $g(x,y,\alpha_1,\alpha_2)$. Also, stress-strength parameter is computed as

$$\mathrm{P}=\mathrm{P}(\mathrm{X}>\mathrm{Y})=\int_0^{\infty}\mathrm{R}_1(y)g_2(y)\,dy=\int_0^{\infty}\alpha_2 f(y)(1-\mathrm{F}(\mathrm{y}))^{\alpha_1+\alpha_2-1}\,dy=\frac{\alpha_2}{\alpha_1+\alpha_2}.$$

The ML estimator of the stress-strength parameter is shown as $\widehat{\mathrm{P}}=\frac{\widehat{\alpha}_2}{\widehat{\alpha}_1+\widehat{\alpha}_2}$ where $\hat{\alpha}_1=\frac{n}{S}$ and $\hat{\alpha}_2=\frac{m}{V}$. The expectation and MSEs of $\hat{\alpha}_1$and $\hat{\alpha}_2$ are represented in the following

$$E(\hat{\alpha}_1)=\frac{n\alpha_1}{n-1},\ MSE(\hat{\alpha}_1)=\frac{(n+2)\alpha_1^2}{(n-1)(n-2)},\qquad n>2,$$
$$E(\hat{\alpha}_2)=\frac{m\alpha_2}{m-1},\ MSE(\hat{\alpha}_2)=\frac{(m+2)\alpha_2^2}{(m-1)(m-2)},\qquad m>2.$$

By the Taylor expansion up to order two, the mean squared error of $\hat{P}$ can be represented as (see Nadarajah et al. [33])

$$MSE(\hat{P})=E(\hat{P}-P)^2\cong\sum_{i=1}^{2}\left(\frac{\partial P}{\partial\alpha_i}\right)^2E(\hat{\alpha}_i-\alpha_i)^2+\sum_{\substack{i,j=1\\(i\neq j)}}^{2}\left(\frac{\partial P}{\partial\alpha_i}\right)\left(\frac{\partial P}{\partial\alpha_j}\right)E(\hat{\alpha}_i-\alpha_i)(\hat{\alpha}_j-\alpha_j),$$

Hence

$$MSE(\hat{P}) \cong \frac{\alpha_2^2\alpha_1^2}{(\alpha_1+\alpha_2)^4}\left[\frac{n^2}{(n-1)(n-2)} - \frac{2n}{(n-1)} + 1\right] + \frac{\alpha_2^2\alpha_1^2}{(\alpha_1+\alpha_2)^4}\left[\frac{m^2}{(m-1)(m-2)} - \frac{2m}{(m-1)} + 1\right] + \frac{2\alpha_2^2\alpha_1^2}{(\alpha_1+\alpha_2)^4}\left[\frac{nm}{(n-1)(m-1)} - \frac{n}{n-1} - \frac{m}{m-1} + 1\right]$$
$$= \frac{\alpha_2^2\alpha_1^2}{(\alpha_1+\alpha_2)^4}\left[\frac{n+2}{(n-1)(n-2)} + \frac{m+2}{(m-1)(m-2)} + \frac{2}{(n-1)(m-1)}\right].$$

Theorem 6. The UMVU estimator of stress-strength parameter P, is represented as

$$\widetilde{P} = \begin{cases} \sum_{i=0}^{n-1}(-1)^i \frac{(m-1)!(n-1)!}{(m+i-1)!(n-i-1)!}(\frac{v}{s})^i, & s > v \\ \sum_{i=0}^{m-2}(-1)^i \frac{(m-1)!(n-1)!}{(m-i-2)!(n+i)!}(\frac{s}{v})^{i+1}, & v > s \end{cases}.$$

Proof. The unbiased estimator of $P = P(X > Y)$ is given by

$$\widetilde{P} = \int_0^\infty \widetilde{R}_1(y)\widetilde{g}_2(y)\,dy$$

which can be checked as follows

$$E(\tilde{P}) = \int_v \int_s \left(\int_y \tilde{R}_1(y)\tilde{g}_2(y)\,dy\right) q_1(s)q_2(v)ds\,dv = \int_v(\int_y(\int_s \tilde{R}_1(y)\,q_1(s)ds)\tilde{g}_2(y)dy)q_2(v)dv,$$

where q_1 and q_2 are the PDF of S and V, respectively. Using the unbiasedness of UMVU estimators of $\tilde{g}(.)$ and $\widetilde{R}(.)$, we have

$$E(\tilde{P}) = \int_v(\int_y R_1(y)\tilde{g}_2(y)dy)q_2(v)dv = \int_y R_1(y)(\int_v \tilde{g}_2(y))q_2(v)dv)\,dy = \int_y R_1(y)g_2(y)\,dy = P.$$

Since (S, V) is sufficient and complement for the family of distribution $g(x, y, \alpha_1, \alpha_2)$, so $\widetilde{P}$ is UMVUE of P. The close form of $\widetilde{P}$ can be computed as follows. Let $e_1 = F^{-1}(1 - e^{-s})$ and $e_2 = F^{-1}(1 - e^{-v})$, then

$$\tilde{P} = \int_0^\infty \tilde{R}_1(y)\tilde{g}_2(y)\,dy,$$
$$= \int_0^{\min(e_1,e_2)}(1 + s^{-1}\ln(1 - F(y)))^{n-1}\frac{(m-1)f(y)}{v(1-F(y))}(1 + v^{-1}\ln(1 - F(y)))^{m-2}\,dy.$$

If $e_2 < e_1$, then

$$\tilde{P} = \int_0^{F^{-1}(1-e^{-v})}(1 + s^{-1}\ln(1 - F(y)))^{n-1}\frac{(m-1)f(y)}{v(1-F(y))}(1 + v^{-1}\ln(1 - F(y)))^{m-2}\,dy,$$
$$= \int_0^1(1 - s^{-1}uv)^{n-1}(m-1)(1-u)^{m-2}\,du,$$
$$= \int_0^1(m-1)(1-u)^{m-2}\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i(\frac{v}{s})^i u^i\,du,$$
$$= \sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i\left(\frac{v}{s}\right)^i\int_0^1(m-1)u^i(1-u)^{m-2}\,du,$$
$$= \sum_{i=0}^{n-1}(-1)^i\frac{(m-1)!(n-1)!}{(m+i-1)!(n-i-1)!}(\frac{v}{s})^i,$$

else

$$\begin{aligned}\tilde{P} &= \int_0^{F^{-1}(1-e^{-s})}(1+s^{-1}\,ln(1-F(y)))^{n-1}\frac{(m-1)f(y)}{v(1-F(y))}(1+v^{-1}\,ln\,(1-F(y)))^{m-2}\,dy,\\ &= \int_0^1\frac{s}{v}(1-u)^{n-1}(m-1)(1-v^{-1}\,su)^{m-2}\,du,\\ &= \int_0^1(m-1)(1-u)^{n-1}\sum_{i=0}^{m-2}\binom{m-2}{i}(-1)^i(\frac{s}{v})^{i+1}u^i\,du,\\ &= \sum_{i=0}^{m-2}\binom{m-2}{i}(-1)^i\left(\frac{s}{v}\right)^{i+1}\int_0^1(m-1)u^i(1-u)^{n-1}\,du,\\ &= \sum_{i=0}^{m-2}(-1)^i\frac{(m-1)!(n-1)!}{(m-i-2)!(n+i)!}(\frac{s}{v})^{i+1},\end{aligned}$$

which complete the proof.

6. SIMULATION STUDY

A simulation study is carried out to study the performance of UMVUE and MLE of PDF, CDF and reliability measures where baseline distribution of F(x) in the proposed exponentiated distributions is inverse Gompertz distribution as

$$F(x)=exp\,(-\frac{1}{\lambda}(e^{\frac{\lambda}{x}}-1)),\qquad x>0\,,\ \lambda>0.$$

The corresponding PDF and hazard function shown as

$$\begin{aligned}f(x)&=\frac{e^{\frac{\lambda}{x}}}{x^2}exp\,(-\frac{1}{\lambda}(e^{\frac{\lambda}{x}}-1)),\quad x>0\,,\ \lambda>0,\\ h_F(x)&=\frac{e^{\frac{\lambda}{x}}exp\,(-\frac{1}{\lambda}(e^{\frac{\lambda}{x}}-1))}{x^2(1-exp\,(-\frac{1}{\lambda}(e^{\frac{\lambda}{x}}-1))},\end{aligned}$$

respectively. The CDF, PDF and reliability function of exponentiated inverse Gompertz (EIG) distribution are given by

$$\begin{aligned}G(x)&=1-(1-exp\,(-\frac{1}{\lambda}(e^{\frac{\lambda}{x}}-1)))^{\alpha},\qquad x>0\ ,\alpha>0,\\ g(x)&=\frac{\alpha e^{\frac{\lambda}{x}}}{x^2}exp\,(-\frac{1}{\lambda}(e^{\frac{\lambda}{x}}-1))(1-exp\,(-\frac{1}{\lambda}(e^{\frac{\lambda}{x}}-1)))^{\alpha-1},\qquad x>0\ ,\alpha>0,\\ R(x)&=(1-exp\,(-\frac{1}{\lambda}(e^{\frac{\lambda}{x}}-1)))^{\alpha}.\end{aligned}$$

For generate data from exponentiated inverse Gompertz distribution, first a random data is generated u_i from uniform distribution (U) for the period (0,1). Then u_i are converted to exponentiated inverse Gompertz distribution with parameters λ and α through the adoption of cumulative distribution function by using the method of inverse transformation as

$$x_i=\lambda(ln(1-\lambda\,ln(1-(1-u_i)^{\frac{1}{\alpha}})))^{-1},\qquad i=1,2,\dots,n.$$

The density plot of EIG distribution for different value of the parameter α and λ are represented in Figure 1. Simulation is carried out for $(\alpha\,,\lambda)=(0.5,2),(1.5,2)$ where, $n\ =15,20,50{,}100$ and we consider that λ to be known. The process is repeated $h=1000$ times to obtain 1000 independent samples of size n. Then calculated the estimations of $\hat{\theta}_j=\hat{R}_j(t),\ \tilde{R}_j(t),\ \hat{P}_j,\ \tilde{P}_j,\ \hat{\alpha}_j=\frac{n}{S_j},\ \tilde{\alpha}_j=\frac{n-1}{S_j},\ j=1,2,\dots,1000$ where

$$S_j=-\sum_{i=1}^{n}ln(1-exp\,(-\frac{1}{\lambda}(e^{\frac{\lambda}{x_{ij}}}-1))),\qquad j=1,2,\dots,1000.$$

Finally, the average estimates and mean squared error is computed to compare the estimation methods as

$\hat{\theta} = \frac{1}{1000}\sum_{j=1}^{1000}\hat{\theta}_j, \qquad MSE(\hat{\theta}_j) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\theta}_j - \theta)^2.$

For $t = 1,2,\dots,5$, the reliability estimates are compared under integrated mean squared error (IMSE) which is an integration of the total area for t_i and $IMSE\left(\hat{R}(t)\right) = \frac{1}{n_t}\sum_{i=1}^{n_t} MSE\left(\hat{R}(t_i)\right)$, where n_t is the count of t.

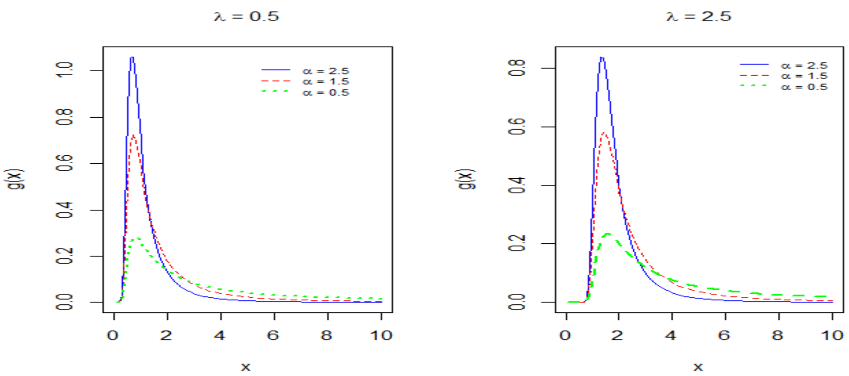

Figure 1. *The density plots of the EIG for different value of parameters*

Table 1. *Estimations and mean square error of MLE and UMVUE of α*

α	0.5		1.5	
n	$\hat{\alpha}$ MSE($\hat{\alpha}$)	$\tilde{\alpha}$ MSE($\tilde{\alpha}$)	$\hat{\alpha}$ MSE($\hat{\alpha}$)	$\tilde{\alpha}$ MSE($\tilde{\alpha}$)
15	0.540991 (0.025665)	0.504924 (0.020917)	1.624144 (0.221979)	1.515867 (0.180195)
20	0.518975 (0.014193)	0.493027 (0.012533)	1.564799 (0.127537)	1.486559 (0.111493)
50	0.511422 (0.005757)	0.501194 (0.005405)	1.52797 (0.050062)	1.497417 (0.047335)
100	0.503414 (0.002619)	0.498381 (0.002558)	1.515395 (0.023685)	1.500241 (0.022981)

The simulation results of estimations and MSEs are summarized in Tables 1-3 which verified the consistency properties of all the methods. From Table 1, for all values of n and α, the MSE of UMVUE of α is less than that of MLE, which confirmed the superiority of UMVU method. Also, as sample size n increased, the MSEs of both methods are decreased. The average estimation, MSEs and IMSEs of the estimators of reliability function are presented in Table 2. Based on the Table 2, the IMSE has its minimum for UMVU estimators that indicated preference of UMVUE. By increasing n, IMSE values decreased for both estimation methods. The average estimations and MSEs of the estimators of stress-strength parameter are presented in Table 3. Based on the MSE, the ML is more efficient than UMVU estimators. Clearly, by increasing n, the MSEs decreased for both MLE and UMVUE.

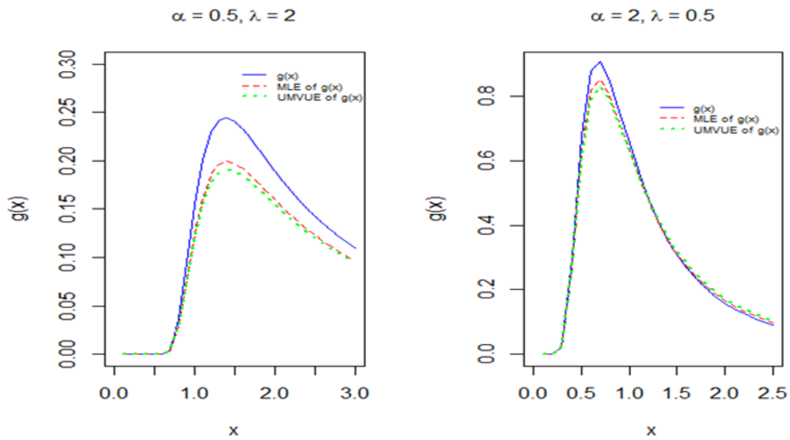

Figure 2. *The comparison of density function and its estimations when* $n = 20$

Table 2. *Estimations and mean square error of MLE and UMVUE of* $R(t)$

α		0.5			1.5		
n	t	R(t)	$\hat{R}(t)$ MSE($\hat{R}(t)$)	$\tilde{R}(t)$ MSE($\tilde{R}(t)$)	R(t)	$\hat{R}(t)$ MSE($\hat{R}(t)$)	$\tilde{R}(t)$ MSE($\tilde{R}(t)$)
15	1	0.979292	0.9775282 (4.2467e-05)	0.9791851 (3.1721e-05)	0.9391555	0.935373 (0.002192)	0.939401 (0.001825)
	2	0.759259	0.743827 (3.8532e-03)	0.7601901 (3.1817e-03)	0.4376932	0.424588 (0.120004)	0.438219 (0.111483)
	3	0.614337	0.595071 (7.1681e-03)	0.6146522 (6.3484e-03)	0.2318569	0.226395 (0.156993)	0.231726 (0.153671)
	4	0.526317	0.506764 (8.5912e-03)	0.5277853 (7.9530e-03)	0.1457956	0.145097 (0.149737)	0.145368 (0.150121)
	5	0.466914	0.4480462 (9.1281e-03)	0.4684436 (8.7121e-03)	0.1017917	0.103581 (0.135081)	0.101257 (0.137159)
IMSE			0.005756	0.005245		0.112802	0.110852
20	1	0.979292	0.978107 (2.7494e-05)	0.979178 (2.3678e-05)	0.9391555	0.935256 (0.000241)	0.938281 (0.000208)
	2	0.759259	0.748852 (2.5784e-03)	0.7580967 (2.2972e-03)	0.4376932	0.422439 (0.006791)	0.432633 (0.006853)
	3	0.614337	0.601326 (4.9071e-03)	0.612668 (4.7111e-03)	0.2318569	0.223249 (0.005231)	0.227191 (0.005643)
	4	0.526317	0.513116 (5.9662e-03)	0.524423 (5.8911e-03)	0.1457956	0.141791 (0.003434)	0.141921 (0.003769)
	5	0.466914	0.454191 (6.4021e-03)	0.464908 (6.4443e-03)	0.1017917	0.100378 (0.002337)	0.098557 (0.002557)
IMSE			0.003976	0.003887		0.003607	0.003806

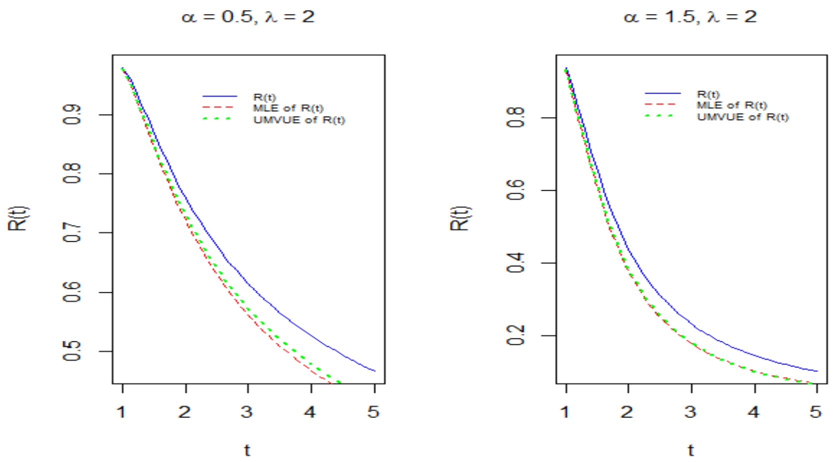

Figure 3. *The comparison of reliability function and its estimations when* $n = 20$

The performance of ML and UMVU estimators for PDF are checked by Figure 2, for $n = 20$. As can be seen, the ML is closer to density plot than UMVU estimation, so MLE is more efficient. Note that for large value of n, the difference between two method is insignificant and we omit the plot for large sample size. The ML and UMVU estimations of reliability function and their MSEs are shown in Figures 3 and 4, respectively. Both plots, confirmed the efficiency of UMVUE method than MLE. The MSEs of ML and UMVU estimations of the stress-strength parameter are depicted in Figure 5 for different value of (α_1, α_2). According to Figure 5, the MSE of the ML method is less than the UMVU estimators.

Table 3*. Estimations and mean square error of MLE and UMVUE of stress-strength parameter for different pair of* (α_1, α_2)

(α_1, α_2)	(0.5,1.5)		(1.5,0.5)	
P	0.75		0.25	
(n, m)	$\hat{P}$ $MSE(\hat{P})$	$\tilde{P}$ $MSE(\tilde{P})$	$\hat{P}$ $MSE(\hat{P})$	$\tilde{P}$ $MSE(\tilde{P})$
(15,15)	0.741734 (0.004858)	0.747906 (0.004892)	0.258106 (0.005068)	0.251948 (0.005114)
(20,20)	0.74581 (0.003424)	0.750481 (0.003458)	0.255801 (0.003588)	0.251149 (0.003611)
(50,50)	0.748302 (0.001545)	0.750168 (0.001550)	0.251242 (0.001352)	0.249368 (0.001358)
(100,100)	0.748715 (0.000691)	0.749699 (0.000694)	0.250055 (0.000704)	0.250995 (0.000708)
(α_1, α_2)	(0.5,2)		(2,0.5)	
P	0.8		0.2	
(15,15)	0.793261 (0.003969)	0.799636 (0.003987)	0.209751 (0.003446)	0.203308 (0.003574)
(20,20)	0.795549 (0.002702)	0.800350 (0.002731)	0.206917 (0.002643)	0.202103 (0.002699)
(50,50)	0.798229 (0.001034)	0.800151 (0.001069)	0.203291 (0.000989)	0.201367 (0.000999)
(100,100)	0.799262 (0.000508)	0.799736 (0.000509)	0.201339 (0.000490)	0.200366 (0.000492)

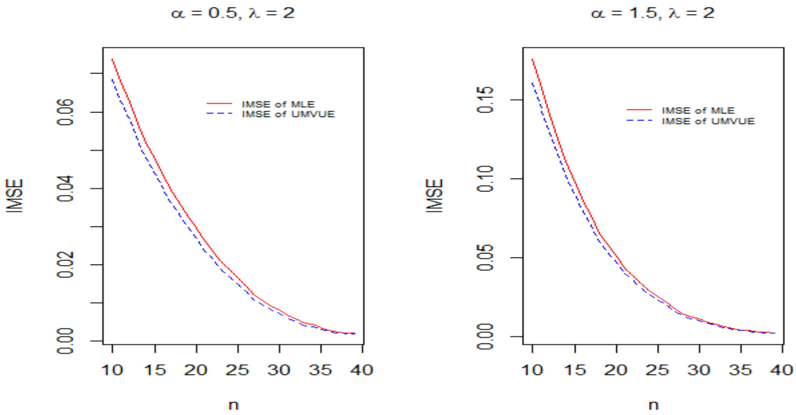

Figure 4. *The comparison of ML and UMVU estimators of reliability function with respect to of IMSE*

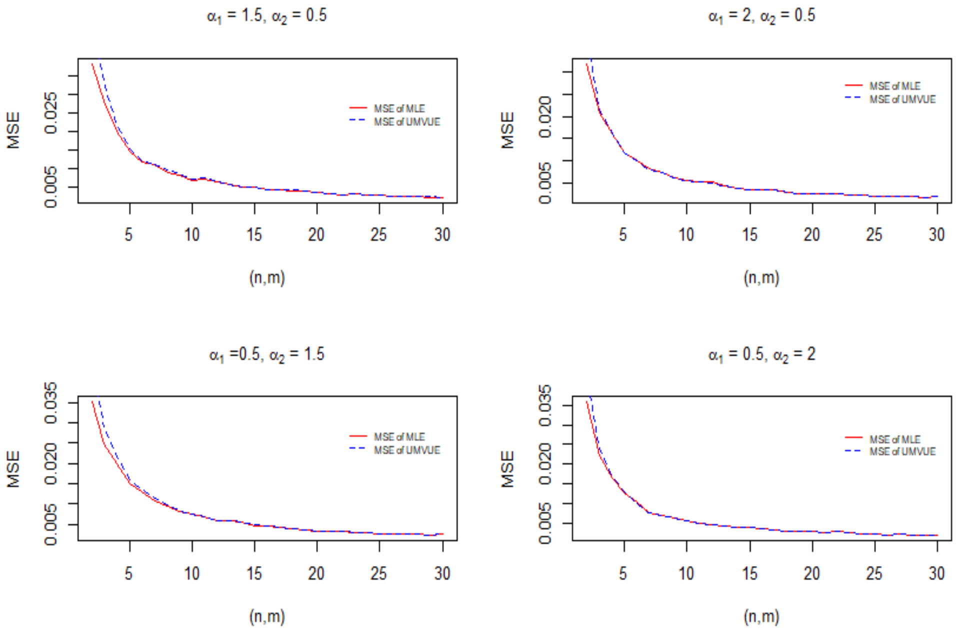

Figure 5. *The comparison of ML and UMVU estimators of stress-strength with respect to MSE*

7. REAL DATA

In this section, we illustrate the exponentiated inverse Gompertz distribution to model the real data sets and compare the EIG distribution with some competitive models with respect to goodness of fit measures.

First data set. We present the analysis of the data set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark [34].

Table 4. *Descriptive statistics of the real data*

Mean	Median	variance	Skewness	Kurtosis
1.9	1.7	0.4957	1.71975	5.924108

Table 4 gives some statistic measures for data, which indicate that the empirical distribution is skewed to the right and leptokurtic. The maximum likelihood method is applied for estimating the parameter of the EIG distribution, where the log likelihood function is obtained as follows

$$ln\, L(X,\alpha,\lambda) = n\, ln\, \alpha + \sum_{i=1}^{n} \frac{\lambda}{x_i} - 2\sum_{i=1}^{n} ln\, x_i - \frac{1}{\lambda}\sum_{i=1}^{n}\left(e^{\frac{\lambda}{x_i}} - 1\right) + (\alpha - 1)\sum_{i=1}^{n} ln\left(1 - exp\left(-\frac{1}{\lambda}\left(e^{\frac{\lambda}{x_i}} - 1\right)\right)\right).$$

The ML estimator can be easily obtained by maximizing log likelihood function numerically by statistical package such as R. The ML estimators and Kolmogorov-Smirnov (K-S) distances between the fitted and the empirical distribution function results are presented in Table 5.

Table 5. *Results of the real data analysis*

Data set	$\hat{\alpha}_{ML}$	$\hat{\lambda}_{ML}$	K-S distance	P-value
AC	3.3406	3.1524	0.0991	0.9849

Based on the P-value of K-S test, we conclude that the EIG distributions provides good fit for the given data set at significance level 0.05. We compare the EIG distribution with the Gompertz, inverse Gompertz (Eliwa et al. [35]) and extended Gompertz (El-Gohary et al. [36]) distributions for fitting lifetime data. Tables 6 shows the ML estimates of the parameters and goodness of fit statistics (Akaike information criterion (AIC)), for the relief times data. Based on the table, the EIG distribution has the smallest value of AIC, which confirm that the EIG distribution works well among the other distributions to modeling the data set.

Table 6. *MLEs of the fitted models and goodness of fit measures for the relief times data*

Models	ML estimations	AIC
Gompertz	$\hat{\alpha} = 0.14531$, $\hat{\beta} = 0.89443$	53.181
Inverse Gompertz	$\hat{\alpha} = 0.11034$, $\hat{\beta} = 6.14541$	36.783
Extended Gompertz	$\hat{\alpha} = 0.40073$, $\hat{\lambda} = 4.80408$, $\hat{\theta} = 275.2289$	37.003
EIG	$\hat{\alpha} = 3.34068$, $\hat{\lambda} = 3.15246$	34.811

Now the ML and UMVU estimates of reliability function are given in Table 7 for $t = 1, 2, 3, 4, 5$.

Table 7. *The ML and UMVU estimators of the reliability function*

t	$\hat{R}(t)$	$\tilde{R}(t)$
1	0.9972237	0.9973621
2	0.3094013	0.3171177
3	0.0673259	0.0637031
4	0.0214704	0.0173886
5	0.0089241	0.0060161

Second data set. In this section, we illustrate the proposed EIG distribution and estimation of the parameters and stress strength reliability by real data sets. The data from Crowder [37] give the lifetimes of the steel specimens tested at two different stress levels. We fit EIG distribution to the two datasets separately. The estimated parameters, K-S and the corresponding P-values are presented in Table 8. From the table, the EIG distribution fits quite well at significance level 0.05 to both data.

Table 8. *Results of the real data analysis*

Data set	n	$\hat{\alpha}_{ML}$	$\hat{\lambda}_{ML}$	K-S distance	P-value
1	10	0.61547	365.758	0.1886	0.8287
2	10	0.49751	437.863	0.2936	0.2989

Based on the two data sets and Table 8, the stress-strength parameter by two ML and UMVU estimation methods are obtained as follows $\hat{P} = 0.4470071;\ \tilde{P} = 0.4788577$

8. CONCLUSION

In this article, we investigated a new class of T-X family as exponential-X lifetime distribution. We considered the ML and UMVU estimations method to estimate the parameter, PDF and CDF of exponential-X distribution. The ML and UMVU estimations of reliability function and stress-strength parameters are viewed where variables are independent of the exponentiated distributions with varying shape parameters. As a member of this family, exponentiated inverse Gompertz distribution is considered, and through Monte Carlo simulation result, the performance of the ML and UMVU estimators are appraised. The simulation results shown that for all values of n and α, the MSE of UMVUE of α is less than that of MLE, which confirmed the superiority of UMVU method. Also, as sample size n increased, the MSEs of both methods are decreased. Also the IMSE has its minimum for UMVU estimators of reliability function that indicated preference of UMVUE. Based on the MSEs of the estimators of stress-strength parameter, the ML is more efficient than UMVU estimators. All the estimators endorsed the asymptotic behavior and convergent to real values when the sample size rise. The proposed model has sufficient versatility that can be practiced quite effectively for modeling lifetime data. The goodness of fit measures confirm that the EIG distribution works well among the other distributions as Gompertz and inverse Gompertz and extended Gompertz to modeling the data set. Finally, the real data analysis has confirmed the presented results.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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