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Research Article

Countably McCoy rings

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Abstract

The main goal of this paper is to study the class of countably $\mathcal{A}$-rings (or the countably McCoy rings) introduced by T. Lucas in [The diameter of a zero divisor graph, J. Algebra **301**, 174-193, 2006] which turns out to lie properly between the class of $\mathcal{A}$-rings (or McCoy rings) and the class of total-$\mathcal{A}$-rings. Also, we introduce and investigate the module theoretic version of the countably $\mathcal{A}$-ring notion, namely the countably $\mathcal{A}$-modules. Our focus is shed on the behavior of the countably $\mathcal{A}$-property vis-à-vis the polynomial ring, the power series ring, the idealization and the direct products. Numerous examples are provided to show the limits of the results.

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1. Introduction

Throughout this paper, all rings are supposed to be commutative with unit element and all R-modules are unital. Let R be a commutative ring and M an R-module. We denote by $\mathrm{Z}_R(M) = \{r \in R : rm = 0 \text{ for some nonzero element } m \in M\}$ the set of zero divisors of R on M and by $\mathrm{Z}(R) := \mathrm{Z}_R(R)$ the set of zero divisors of the ring R. In citeAC3, the notions of $\mathcal{A}$-module and $\mathcal{SA}$-module are extensively studied. In fact, an R-module M satisfies Property $(\mathcal{A})$, or M is an $\mathcal{A}$-module over R (or $\mathcal{A}$-module if no confusion is likely), if for every finitely generated ideal I of R with $I \subseteq \mathrm{Z}_R(M)$), there exists a nonzero $m \in M$ with $Im = 0$, or equivalently, $\mathrm{ann}_M(I) \neq 0$. M is said to satisfy strong Property $(\mathcal{A})$, or is an $\mathcal{SA}$-module over R (or an $\mathcal{SA}$-module if no confusion is likely), if for any $r_1, cdots, r_n \in \mathrm{Z}_R(M)$, there exists a nonzero $m \in M$ such that $r_1m = cdots = r_nm = 0$. The ring R is said to satisfy Property $(\mathcal{A})$, or an $\mathcal{A}$-ring, (respectively, $\mathcal{SA}$-ring) if R is an $\mathcal{A}$-module (resp., an $\mathcal{SA}$-module). One may easily check that M is an $\mathcal{SA}$-module if and only if M is an $\mathcal{A}$-module and $\mathrm{Z}_R(M)$ is an ideal of R. It is worthwhile reminding the reader that the Property $(\mathcal{A})$ for commutative rings was introduced by Quentel in citeQ who called it Property (C) and Huckaba used the term Property $(\mathcal{A})$ in citeH,HK. In citeF, Faith called rings satisfying Property $(\mathcal{A})$ McCoy rings. The Property $(\mathcal{A})$ for modules was introduced by Darani citeD who called such modules F-McCoy modules (for Faith McCoy

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terminology). He also introduced the strong Property ($\mathcal{A}$) under the name super coprimal and called a module M coprimal if $\mathrm{Z}_R(M)$ is an ideal. In citeMH, the strong Property ($\mathcal{A}$) for commutative rings was independently introduced by Mahdou and Hassani in citeMH and further studied by Dobbs and Shapiro in citeDS. Note that a finitely generated module over a Noetherian ring is an $\mathcal{A}$-module (for example, see cite[Theorem 82]K) and thus a Noetherian ring is an $\mathcal{A}$-ring. Also, it is well known that a zero-dimensional ring R is an $\mathcal{A}$-ring as well as any ring R whose total quotient ring $Q(R)$ is zero-dimensional. In fact, it is easy to see that R is an $\mathcal{A}$-ring if and only if so is $Q(R)$ [9, Corollary 2.6]. Any polynomial ring $R[X]$ is an $\mathcal{A}$-ring citeH as well as any reduced ring with a finite number of minimal prime ideals citeH. In citeB, we generalize a result of T.G. Lucas which states that if R is a reduced commutative ring and M is a flat R-module, then the idealization $R \ltimes M$ is an $\mathcal{A}$-ring if and only if R is an $\mathcal{A}$-ring cite[Proposition 3.5]L. In effect, we drop the reduceness hypotheses and prove that, given an arbitrary commutative ring R and any submodule M of a flat R-module F, $R \ltimes M$ is an $\mathcal{A}$-ring (resp., $\mathcal{SA}$-ring) if and only if R is an $\mathcal{A}$-ring (resp., $\mathcal{SA}$-ring). In citeBEK, we present an answer to a problem raised by D.D. Anderson and S. Chun in citeAC3 on characterizing when is the idealization $R \ltimes M$ of a ring R on an R-module M an $\mathcal{A}$-ring (resp., an $\mathcal{SA}$-ring) in terms of module-theoretic properties of R and M. Also, we were concerned with presenting a complete answer to an open question asked by these two authors which reads the following: What modules over a given ring R are homomorphic images of modules satisfying the strong Property ($\mathcal{A}$)? cite[Question 4.4 (1)]AC3. The main theorem of citeBE extends a result of Hong, Kim, Lee and Ryu in citeHKLR which proves that a direct product $\prod R_i$ of rings is an $\mathcal{A}$-ring if and only if so is any R_i cite[Proposition 1.3]HKLR. In this regard, we show that if $\{R_i\}_{i\in I}$ is a family of rings and $\{M_i\}_{i\in I}$ is a family of modules such that each M_i is an R_i-module, then the direct product $\prod\limits_{i\in I} M_i$ of the M_i is an $\mathcal{A}$-module over $\prod\limits_{i\in I} R_i$ if and only if each M_i is an $\mathcal{A}$-module over R_i, $i \in I$. Finally, our main concern in citeABE is to introduce and investigate a new class of rings lying properly between the class of $\mathcal{A}$-rings and the class of $\mathcal{SA}$-rings. The new class of rings, termed the class of $\mathcal{PSA}$-rings, turns out to share common characteristics with both $\mathcal{A}$-rings and $\mathcal{SA}$-rings. Numerous properties and characterizations of this class are given as well as the module-theoretic version of $\mathcal{PSA}$-rings is introduced and studied. For further works related to the Property ($\mathcal{A}$) and ($\mathcal{SA}$), we refer the reader to citeAC1,AC2,AC3,AC4,HEZ,HKLR,L,MM,MMZ.

The main goal of this paper is to study the class of countably $\mathcal{A}$-rings ($\mathcal{CA}$-rings for short) introduced by T. Lucas in citeL which turns out to lie properly between the class of $\mathcal{A}$-rings (or McCoy rings) and the class of total-$\mathcal{A}$-rings. Any Noetherian ring proved to be a $\mathcal{CA}$-ring. Furthermore, we introduce the module theoretic version of the countably $\mathcal{CA}$-ring notion, namely the countably $\mathcal{CA}$-modules. Our focus is shed on the behavior of the $\mathcal{CA}$-property vis-à-vis the polynomial ring, the power series ring, the idealization and the direct products. Numerous examples are also provided to show the limits of the results. It is known that the polynomial ring $R[X]$ is always an $\mathcal{A}$-ring. Now, the legitimate question which arises is whether this result remains true for the $\mathcal{CA}$-property. We prove that the polynomial ring $R[X]$ needs not be a $\mathcal{CA}$-ring, in general, and we give necessary and sufficient conditions for $R[X]$ to be a $\mathcal{CA}$-ring when the base ring R is an $\mathcal{A}$-ring. Regarding the power series ring, recall that a longstanding question, which is still open, asks whether $R[[X]]$ is always an $\mathcal{A}$-ring. In this aspect, recall that McCoy's theorem on polynomial rings don't carry over to power series ring $R[[X]]$ over R (see cite[Example 3^2]F). Then several authors showed interest in determining the commutative rings R that satisfy the extension of McCoy's theorem to $R[[X]]$ and that we will call throughout the $R[[X]]$-McCoy's theorem. In this regard, Fields proved that if R is Noetherian, then R satisfies the $R[[X]]$-McCoy's theorem cite[Theorem 5]Fi. Also, Gilmer, Grams and Parker proved that if either R is reduced or the total quotient ring of R is a von Neumann regular ring, then R satisfies the $R[[X]]$-McCoy's theorem (see citeGGP). We prove, in this context, that if R satisfies the $R[[X]]$-McCoy's theorem, then R is a $\mathcal{CA}$-ring implies that $R[[X]]$ is a $\mathcal{CA}$-ring and thus, in particular, an $\mathcal{A}$-ring. This stands as a partial answer to the above question on $\mathcal{A}$-property of $R[[X]]$. Moreover, we give an example of an $\mathcal{A}$-ring R such that $R[[X]]$ is a not a $\mathcal{CA}$-ring. In Section 4, we aim at seeking when an idealization $R \ltimes M$ of a ring R on an R-module M is a $\mathcal{CA}$-ring. We characterize the $\mathcal{CA}$-Property of $R \ltimes M$ in terms of properties of R and M. In particular, we prove that if R is a domain, then $R \ltimes M$ is a $\mathcal{CA}$-ring if and only if M is a $\mathcal{CA}$-module. Finally, in Section 5, we study the behavior of the $\mathcal{CA}$-property with respect to direct products of rings. In fact, it is known that the direct product $\prod_i R_i$ of rings $(R_i)_{i\in\Lambda}$ is an $\mathcal{A}$-ring if and only if so is each ring R_i. We aim next at characterizing when $\prod_i R_i$ is a $\mathcal{CA}$-ring. In this aspect, we prove that $\prod_{i\in\Lambda} R_i$ is a $\mathcal{CA}$-ring if and only if Λ is a finite set and R_i is a $\mathcal{CA}$-ring for each $i \in \Lambda$.

2. $\mathcal{CA}$-rings and $\mathcal{CA}$-modules

Recall that the countably McCoy rings were introduced by T. Lucas in citeL in his investigation on the graph of power series rings. In this section, we aim at studying this class of rings. Also, we introduce and investigate the countably McCoy modules. We prove that the class of $\mathcal{CA}$-rings is a proper intermediate class between the class of $\mathcal{A}$-rings and the class of total-$\mathcal{A}$-rings. Also, it is worth reminding that the polynomial ring $R[X]$ is always an $\mathcal{A}$-ring. Now, the legitimate question which arises is whether this result still remains true for the $\mathcal{CA}$-property. In this section, we prove that the polynomial ring $R[X]$ needs not be a $\mathcal{CA}$-ring, in general, and we give necessary and sufficient conditions for $R[X]$ to be a $\mathcal{CA}$-ring when the base ring R is an $\mathcal{A}$-ring.

Definition 2.1. Let R be a ring and M an R-module.

(1) R is said to be a countably McCoy ring or a countably $\mathcal{A}$-ring ($\mathcal{CA}$-ring for short), if any ideal $J \subseteq \mathrm{Z}(R)$ such that J is countably generated, $\mathrm{ann}_R(J) \neq 0$.

(2) M is said to be a countably McCoy module or a countably $\mathcal{A}$-module ($\mathcal{CA}$-module for short), if any ideal $I \subseteq \mathrm{Z}_R(M)$ such that I is countably generated, $\mathrm{ann}_M(I) \neq 0$.

Next, we exhibit some examples of $\mathcal{CA}$-rings.

Proposition 2.2. *Let R be a ring. Then*

(1) *If R is Noetherian, then R is a $\mathcal{CA}$-ring.*

(2) *Any Noetherian or Artinian module M is a $\mathcal{CA}$-module.*

(3) *Let Λ denote the set of all countably generated ideals of R. Then $\bigoplus_{I\in\Lambda} \frac{R}{I}$ is a $\mathcal{CA}$-module over R.*

Proof. 1) It is clear as any ideal of R is finitely generated.
2) In fact, any Noetherian or Artinian module over R is a total-$\mathcal{A}$-module cite[Theorem 2.2]AC3.
3) Let $I \subseteq \mathrm{Z}(R)$ be a countably generated ideal of R. Then $I\overline{1_{R/I}} = (\overline{0})$ and $\overline{1_{R/I}} \neq \overline{0}$. Hence $\bigoplus_{I\in\Lambda} \frac{R}{I}$ is a $\mathcal{CA}$-module over R.

□

The next proposition records the fact that the class of countably McCoy rings is a proper intermediate class between the class of total McCoy rings and the class of McCoy rings. Recall that a ring R is said to be a total-$\mathcal{A}$-ring if for any ideal $I \subseteq \mathrm{Z}(R)$, we have $\mathrm{ann}_R(I) \neq 0$.

Proposition 2.3. *total-$\mathcal{A}$-rings $\subsetneq$ $\mathcal{CA}$-rings $\subsetneq$ $\mathcal{A}$-rings.*

Proof. The large inclusions are clear from the definition of $\mathcal{CA}$-rings. The second strict inclusion is proved by Lucas in cite[Example 5.4]L. Ahead, Example 4.6 proves the first strict inclusion by providing an example of a $\mathcal{CA}$-ring which is not a total-$\mathcal{A}$-ring. □

In cite[Example 5.4]L, T. Lucas presented an example of a reduced $\mathcal{A}$-ring which is not a $\mathcal{CA}$-ring. In this aspect, the next example provides a countable 0-dimensional local ring R which is an $\mathcal{A}$-ring while it is not a $\mathcal{CA}$-ring.

Example 2.4. Let $R = \frac{\mathbb{Q}[X_n]_{n\in\mathbb{N}}}{(X_n^2)_n}$. Then:

(1) R is a countable local 0-dimensional ring.
(2) R is an $\mathcal{A}$-ring.
(3) R is not a $\mathcal{CA}$-ring.

Proof. R is a countable local ring of Krull dimension 0.
1) Since R is zero-dimensional, then R is an $\mathcal{A}$-ring.
2) Let $x_n = \overline{X_n}$ for each integer $n \geq 0$. Let $I = (x_n)_{n\in N}$ be the unique maximal ideal of R. Assume that $\text{ann}_R(I) \neq (0)$. Note that $I = \text{Z}(R)$. Let $f(X_1, X_2, cdots, X_p) \in \mathbb{Q}[X_1, X_2, cdots, X_p]$ such that $\overline{0} \neq \overline{f(X_1, cdots, X_p)} \in \text{ann}_R(I)$ for some positive integer p. Then, we may assume, without loss of generality, that

$$f(X_1, cdots, X_p) = \sum_{1\leq i_1, cdots, i_s \leq p} a_{i_1} cdots a_{i_s} X_{i_1} cdots X_{i_s},$$

that is, the degree of f on each indeterminate X_i is $\deg_{X_i}(f) \leq 1$. Now, $\overline{f}I = (\overline{0})$, then, in particular, $\overline{f}x_{p+1} = \overline{0}$. Thus $fX_{p+1} \in (\{X_n^2\}_{n\in\mathbb{N}})$. This leads to a contradiction since $\deg_{X_i}(f) \leq 1$ for each $i = 1, cdots, p$, so that $\deg_{X_i}(fX_{p+1}) \leq 1$ for each $i = 1, cdots, p, p+1$. It follows that R is not a $\mathcal{CA}$-ring. □

Let R be a commutative ring and M an R-module. Put $S_M := R\setminus \text{Z}_R(M)$ the set of non zero divisors of M. We define the total quotient ring of M over R to be the localization ring $Q_R(M) := S_M^{-1}R$ and the total quotient module of M to be the $Q_R(M)$-module $Q(M) := S_M^{-1}M$. It is well known that M is an $\mathcal{A}$-module over R if and only if $Q(M)$ is $\mathcal{A}$-module over $Q_R(M)$ cite[Theorem 2.1 (3)]AC3. We next prove an analog result of this theorem for the $\mathcal{CA}$-property.

Theorem 2.5. *Let R be a commutative ring and M be an R-module. Then the following assertions are equivalent.*

(1) *M is a $\mathcal{CA}$-module over R;*
(2) *$Q(M)$ is a $\mathcal{CA}$-module over $Q_R(M)$;*

Proof. 1) ⇒ 2) Assume that M is a $\mathcal{CA}$-module over R. Let J be a countably generated ideal of $Q_R(M)$ such that

$$J = (a_1/s_1, a_2/s_2, cdots, a_n/s_n, cdots) \subseteq \text{Z}_{Q_R(M)}(Q(M))$$

with $a_n \in \text{Z}_R(M)$ and $s_n \in S_M$ for each integer $n \geq 1$ since, by cite[Lemma 2.2]BEK, $\text{Z}_{Q_R(M)}(Q(M)) = S_M^{-1}\,\text{Z}_R(M)$. Let $I = (a_1, a_2, cdots, a_n, cdots)$. Then I is a countably generated ideal of R. Let $x = r_1a_1 + r_2a_2 + cdots + r_na_n \in I$ with the $r_i \in R$. Then $x/1 = r_1a_1/1 + cdots + r_na_n/1 \in J \subseteq \text{Z}_{Q_R(M)}(Q(M))$. Hence there exists $s \in S_M$ such that $sx \in \text{Z}_R(M)$. Thus there exists $0 \neq t \in M$ such that $sxt = 0$ so that, since $s \in S_M$, $xt = 0$. Then $x \in \text{Z}_R(M)$. It follows that $I \subseteq \text{Z}_R(M)$. Hence there exists $0 \neq y \in M$ such that $Iy = 0$ since M is a $\mathcal{CA}$-module. Observe that $y/1 \neq 0$ in $Q(M)$. Hence $J(y/1) = (0)$ with $y/1 \neq 0$. Therefore $\text{ann}_{Q(M)}(J) \neq (0)$. It follows that $Q(M)$ is a $\mathcal{CA}$-module over $Q_R(M)$, as desired.
2) ⇒ 1) Assume that $Q(M)$ is a $\mathcal{CA}$-module over $Q_R(M)$. Let I be a countably generated ideal of R such that $I \subseteq \text{Z}_R(M)$. Then $J = S_M^{-1}I$ is a countably generated ideal of $Q_R(M)$.

Hence there exists $0 \neq x/s \in Q(M)$ such that $J(x/s) = 0$. Therefore it is easy to check that $Ix = (0)$ and that $x \in M$ with $x \neq 0$. It follows that M is a $\mathcal{CA}$-module over R completing the proof of the theorem. $\square$

It is worth reminding the reader that the property $\mathcal{A}$ of modules is not stable under direct sums and direct summands. The property $\mathcal{CA}$ turns out to be pathologic as well vis-à-vis direct sums and direct summands. In fact, Anderson and Chun gave an example, namely cite[Example 2.6]AC3, of an $\mathcal{A}$-module $A = A_1 \oplus A_2$ while neither A_1 nor A_2 is an $\mathcal{A}$-module. In this example, the two authors proved that A is a total-$\mathcal{A}$-module. Consequently, Example 2.6 of citeAC3 turns out to be an example of a $\mathcal{CA}$-ring $A = A_1 \oplus A_2$ while neither A_1 nor A_2 is a $\mathcal{CA}$-module. This proves that the $\mathcal{CA}$-property is not stable under direct summand. Also, Anderson and Chun proved that the $\mathcal{A}$-property is not stable under direct sum by providing an example of two modules A_1 and A_2 over a two-dimensional regular local ring R such that A_1 and A_2 are $\mathcal{A}$-modules while $A_1 \oplus A_2$ is not an $\mathcal{A}$-module see cite[Example 2.11]AC3. A_2 is proved to be a total-$\mathcal{A}$-module and thus A_2 is a $\mathcal{CA}$-module over R. Besides, $A_1 = R/(x)$ with $x \notin \mathrm{Z}(R)$. Then $\mathrm{Z}_R(A_1) = (x)$ as $R/(x)$ is a regular local domain and thus (x) is a prime ideal of R. Hence, if $J \subseteq \mathrm{Z}_R(A_1)$, then $J\overline{1}_{A_1} = \overline{0}$. It follows that A_1 is a total-$\mathcal{A}$-module and thus a $\mathcal{CA}$-module. Consequently, A_1 and A_2 are $\mathcal{CA}$-modules while $A_1 \oplus A_2$ is not a $\mathcal{CA}$-module.

Recall that the polynomial ring $R[X]$ over a ring R is always an $\mathcal{A}$-ring. It turns out that this is no longer true for the $\mathcal{CA}$-property. In fact, ahead, we give, via Example 2.7, an example of a countable ring R such that $R[X]$ is not a $\mathcal{CA}$-ring. Also, the next theorem provides a necessary and sufficient condition for the polynomial ring $R[X]$ to be a $\mathcal{CA}$-ring when the base ring R is an $\mathcal{A}$-ring. Given a ring R and an element $f \in R[X]$, we denote by $\mathrm{c}(f)$ the content of f, that is, the ideal of R generated by the coefficients of f.

Theorem 2.6. *Let R be a ring.*

(1) *If R is a $\mathcal{CA}$-ring, then $R[X]$ is a $\mathcal{CA}$-ring.*
(2) *Moreover, if R is an $\mathcal{A}$-ring, then $R[X]$ is a $\mathcal{CA}$-ring if and only if so is R.*

Proof. 1) Assume that R is a $\mathcal{CA}$-ring. Let $J = (f_1, cdots, f_n, cdots)$ be a countably generated ideal of $R[X]$ such that $J \subseteq \mathrm{Z}(R[X])$. Let $I = (\mathrm{c}(f_1), \mathrm{c}(f_2), cdots, \mathrm{c}(f_n), cdots)$ be the ideal of R generated by the contents of the f_n. Then I is a countably generated ideal of R. We prove that $I \subseteq \mathrm{Z}(R)$. In fact, it suffices to prove that $(\mathrm{c}(f_1), \mathrm{c}(f_2), cdots, \mathrm{c}(f_n)) \subseteq \mathrm{Z}(R)$ for each integer $n \geq 1$. Fix an integer $n \geq 1$. Observe that

$$g = f_1 + X^{d_1+1} f_2 + X^{d_1+d_2+2} f_3 + cdots + X^{d_1+d_2+cdots+d_{n-1}+n-1} f_n \in J \subseteq \mathrm{Z}(R[X]).$$

Then, by McCoy's theorem, there exists $a \in R \setminus \{0\}$ such that $ag = 0$. Hence, by the construction of g, $af_i = 0$ for $i = 1, cdots, n$. It follows that $a\mathrm{c}(f_i) = 0$ for $i = 1, cdots, n$. Hence $(\mathrm{c}(f_1), \mathrm{c}(f_2), cdots, \mathrm{c}(f_n)) \subseteq \mathrm{Z}(R)$ for each integer $n \geq 1$ and thus $I \subseteq \mathrm{Z}(R)$ proving our claim. Since R is a $\mathcal{CA}$-ring, we get $\mathrm{ann}_R(I) \neq (0)$. Let $b \in R \setminus \{0\}$ such that $bI = (0)$. Then it is easy to see that $bJ = (0)$. This proves that $R[X]$ is a $\mathcal{CA}$-ring.
2) Assume that R is an $\mathcal{A}$-ring. Also, suppose that $R[X]$ is a $\mathcal{CA}$-ring. Let $J \subseteq \mathrm{Z}(R)$ be a countably generated ideal of R. Then, as R is an $\mathcal{A}$-ring, $JR[X] \subseteq \mathrm{Z}(R[X])$. Since $JR[X]$ is a countably generated ideal of $R[X]$ and $R[X]$ is a $\mathcal{CA}$-ring, then there exists $f \in R[X] \setminus \{0\}$ such that $f(0) \neq 0$ and $fJR[X] = 0$. Hence $f(0)J = (0)$ and $f(0) \neq 0$. It follows that R is a $\mathcal{CA}$-ring completing the proof of the theorem.

$\square$

We next exhibit an example of a countable local 0-dimensional ring R such that $R[X]$ is not a $\mathcal{CA}$-ring.

Example 2.7. Let $R = \dfrac{\mathbb{Q}[X_n]_n}{(X_n^2)_n}$. As R is a 0-dimensional ring, then R is an $\mathcal{A}$-ring. By Example 2.4, R is not a $\mathcal{CA}$-ring. It follows, by Theorem 2.6, that $R[X]$ is not a $\mathcal{CA}$-ring, as desired.

Corollary 2.8. *Let R be a ring and let $n \geq 0$ be an integer. Let $X, X_1, X_2, cdots, X_n$ be indeterminates over R. Then $R[X_1, X_2, cdots, X_n]$ is a $\mathcal{CA}$-ring if and only if $R[X]$ is a $\mathcal{CA}$-ring. Moreover, if R is an $\mathcal{A}$-ring, then $R[X_1, X_2, cdots, X_n]$ is a $\mathcal{CA}$-ring if and only if R is a $\mathcal{CA}$-ring*

Proof. By Theorem 2.6(1) and by iteration, if $R[X]$ is a $\mathcal{CA}$-ring then so is $R[X_1, cdots, X_n]$. Conversely, assume that $R[X_1, cdots, X_n]$ is a $\mathcal{CA}$-ring. As $R[X]$ is an $\mathcal{A}$-ring, applying Theorem 2.6(2), we get $R[X_1]$ is a $\mathcal{CA}$-ring and thus $R[X]$ is a $\mathcal{CA}$-ring, as desired. □

3. $\mathcal{CA}$-property and power series ring

This section aims at investigating the behavior of the power series ring $R[[X]]$ with respect to the property $\mathcal{CA}$. At this point, recall that, given a ring R, McCoy proved that if $f \in \mathrm{Z}(R[X])$, then there exists $a \in R \setminus \{0\}$ such that $af = 0$. This theorem don't carry over to power series ring $R[[X]]$ over R (see cite[Example 3^2]F). The question that arises is what are the commutative rings R that satisfy the extension of McCoy's theorem to $R[[X]]$ and that we will call throughout the $R[[X]]$-McCoy's theorem. In this regard, Fields proved that if R is Noetherian, then R satisfies the $R[[X]]$-McCoy's theorem. Also, Gilmer, Grams and Parker proved that if either R is reduced, or the total quotient ring of R is a von Neumann regular ring or each zero divisor f of $R[[X]]$ is annihilated by an element of $R[X]$, then R satisfies the $R[[X]]$-McCoy's theorem. On the other hand, it is an open question to know whether the power series ring $R[[X]]$ is an $\mathcal{A}$-ring. The main theorem of this section answers positively this question when R is a $\mathcal{CA}$-ring such that $\mathrm{Z}(R) = \mathrm{Rad}(R)$. Also, it permits to construct an example of a ring R such that $R[[X]]$ is not a $\mathcal{CA}$-ring.

We begin by announcing the main theorem of this section. Given a ring R, we denote by $\mathrm{Z}(R)[X]$ (resp., $\mathrm{Z}(R)[[X]]$) the subset of $R[X]$ (resp., of $R[[X]]$) consisting of elements f of $R[X]$ (resp., of $R[[X]]$) such that the coefficients of f are elements of $\mathrm{Z}(R)$.

Theorem 3.1. *Let R be a ring.*

(1) *Assume that $\mathrm{Z}(R[[X]]) \subseteq \mathrm{Z}(R)[[X]]$. If R is a $\mathcal{CA}$-ring, then $R[[X]]$ is a $\mathcal{CA}$-ring.*
(2) *Assume that $\mathrm{Z}(R) = \mathrm{Rad}(R)$. Then $R[[X]]$ is a $\mathcal{CA}$-ring if and only if R is a $\mathcal{CA}$-ring.*

It is worthwhile noting that in the case of a polynomial ring $R[X]$ over a ring R, we always have by McCoy's theorem that $\mathrm{Z}(R[X]) \subseteq \mathrm{Z}(R)[X]$. This is no longer true in the case of a power series ring $R[[X]]$, in the sense that, $\mathrm{Z}(R[[X]]) \nsubseteq \mathrm{Z}(R)[[X]]$, in general (see cite[Example $3.^2$]Fi).

To prove Theorem 3.1, we need the following proposition.

Proposition 3.2. *Let R be a ring such that $\mathrm{Z}(R[[X]]) \subseteq \mathrm{Z}(R)[[X]]$. Let $J \subseteq \mathrm{Z}(R[[X]])$ be an ideal of $R[[X]]$. Then $I = (\{\mathrm{c}(f) : f \in J\}) \subseteq \mathrm{Z}(R)$.*

Proof. It suffices to handle the case where $J = (f_1, f_2, cdots, f_n)$ is a finitely generated ideal of $R[[X]]$. Let $f_i = a_{i0} + a_{i1}X + cdots + a_{im}X^m + cdots$ and thus $\mathrm{c}(f_i) = (a_{i0}, a_{i1}, cdots, a_{im}, cdots)$ for $i = 0, cdots, n$ and let $I = (\mathrm{c}(f_1), cdots, \mathrm{c}(f_n))$ be the ideal of R generated by the contents of the f_i. Let $y = (r_{10}a_{10} + cdots + r_{1m_1}a_{1m_1}) + cdots + (r_{n0}a_{n0} + cdots + r_{nm_n}a_{nm_n})$, with the $r_{ij} \in R$, be an arbitrary element of I such that

$a_{im_i} \neq 0$ and $r_{im_i} \neq 0$ for $i = 1, cdots, n$. Without loss of generality, we may assume that $m_1 \leq m_2 \leq cdots \leq m_n$. Put

$$g = (r_{10}X^{m_n} + r_{11}X^{m_n-1} + cdots + r_{1m_1}X^{m_n-m_1})f_1 + cdots + (r_{n0}X^{m_n} + r_{n1}X^{m_n-1} + cdots + r_{nm_n})f_n$$

and note that $g \in J$ so that $g \in \mathrm{Z}(R[[X]])$. Then, by hypotheses, $g \in \mathrm{Z}(R)[[X]]$. Hence, since y is the m_nth coefficient of g, we get $y \in \mathrm{Z}(R)$. Therefore $I \subseteq \mathrm{Z}(R)$, as contended. □

Proof of Theorem 3.1. 1) Assume that R is a $\mathcal{CA}$-ring. Let $J = (\{f_n\}_{n\in\mathbb{N}})$ be a countably generated ideal of $\mathrm{Z}(R[[X]])$. By Proposition 3.2, $K = (\{\mathrm{c}(f_n)\}_{n\in\mathbb{N}}) \subseteq \mathrm{Z}(R)$. Now, as R is a $\mathcal{CA}$-ring and K is a countably generated ideal of $\mathrm{Z}(R)$, we get $\mathrm{ann}_R(K) \neq \{0\}$. It follows that $\mathrm{ann}_{R[[X]]}(J) \neq \{0\}$ yielding that $R[[X]]$ is a $\mathcal{CA}$-ring.
2) Suppose that $\mathrm{Z}(R) = \mathrm{Rad}(R)$ and $R[[X]]$ is a $\mathcal{CA}$-ring. Note that, By cite[Theorem 3]Fi, $\mathrm{Z}(R[[X]]) \subseteq \mathrm{Z}(R)[[X]]$. Then, applying (1), if R is a $\mathcal{CA}$-ring, $R[[X]]$ is so. Conversely, assume that $R[[X]]$ is a $\mathcal{CA}$-ring and let $I = (\{a_n\}_{n\in\mathbb{N}})$ be a countably generated ideal contained in $\mathrm{Z}(R) = \mathrm{Rad}(R)$. Observe that, as $\mathrm{Rad}(R) \subseteq \mathrm{Rad}(R[[X]])$, we get that

$$(\mathrm{Rad}(R))R[[X]] \subseteq \mathrm{Rad}(R[[X]]) \subseteq \mathrm{Z}(R[[X]]).$$

Then $IR[[X]] \subseteq (\mathrm{Rad}(R))R[[X]] \subseteq \mathrm{Z}(R[[X]])$. Hence, as $R[[X]]$ is a $\mathcal{CA}$-ring, there exists $g \in R[[X]]$ with $g(0) \neq 0$ such that $gIR[[X]] = (0)$ and thus $g(0)I = (0)$. It follows that R is a $\mathcal{CA}$-ring completing the proof of the theorem. □

It is easy to see that if R satisfies the $R[[X]]$-McCoy's theorem, in particular, if R is reduced, then $\mathrm{Z}(R[[X]]) \subseteq \mathrm{Z}(R)[[X]]$. The next result is a direct consequence of Theorem 3.1.

Corollary 3.3. *Let R be a ring.*

(1) *Assume that R satisfies the $R[[X]]$-McCoy's theorem. If R is a $\mathcal{CA}$-ring, then $R[[X]]$ is a $\mathcal{CA}$-ring.*
(2) *Assume that R is a reduced ring. If R is a $\mathcal{CA}$-ring, then $R[[X]]$ is a $\mathcal{CA}$-ring.*

The following example shows that there exists an $\mathcal{A}$-ring R such that $R[[X]]$ is not a $\mathcal{CA}$-ring.

Example 3.4. Let k be a field and $R = \dfrac{k[X_n]_n}{(X_n^2)_n} = k[x_n]_{n\in\mathbb{N}}$. Then $\mathrm{Z}(R) = \mathrm{Rad}(R) = (\{x_n\}_{n\in\mathbb{N}})$. Also, note, as proved in Example 2.4, that R is not a $\mathcal{CA}$-ring. Then, by Theorem 3.1, $R[[X]]$ is not a $\mathcal{CA}$-ring.

4. $\mathcal{CA}$-property and idealization

This section aims at seeking when the idealization $R \ltimes M$ of a ring R on an R-module M is a $\mathcal{CA}$-ring. We characterize the $\mathcal{CA}$-property of $R \ltimes M$ in terms of properties of R and M. In particular, we prove that if R is a domain, then $R \ltimes M$ is a $\mathcal{CA}$-ring if and only if M is a $\mathcal{CA}$-module.

Our first results investigate some characterizations of modules satisfying the $\mathcal{CA}$-property.

Theorem 4.1. *Let R be a ring. Let M be an R-module and N a submodule of M such that $\mathrm{Z}_R(M) = \mathrm{Z}_R(N)$. If N is a $\mathcal{CA}$-module, then M is so.*

Proof. Let J be a countably generated ideal of R with $J \subseteq \mathrm{Z}_R(M) = \mathrm{Z}_R(N)$. Suppose that N is a $\mathcal{CA}$-module. Then $\mathrm{ann}_N(J) \neq 0$. Hence, as $N \subseteq M$, $\mathrm{ann}_M(J) \neq 0$. It follows that M is a $\mathcal{CA}$-module. □

Corollary 4.2. *Let M and N be R-modules such that $\mathrm{Z}_R(N) \subseteq \mathrm{Z}_R(M)$. If M is a $\mathcal{CA}$-module, then $M \oplus N$ is so.*

Proof. Note that $\mathrm{Z}_R(M)=\mathrm{Z}_R(M\oplus N)$ and M is a submodule of $M\oplus N$. Then apply Theorem 4.1 to get the desired result. □

Corollary 4.3. *Let M be an R-module. Then M is a $\mathcal{CA}$-module if and only if $\bigoplus_I M$ is so.*

Proof. Assume that M is a $\mathcal{CA}$-module. We have $\mathrm{Z}_R(\bigoplus_I M)=\mathrm{Z}_R(M)$ and $M\subseteq\bigoplus_I M$. Then, by Theorem 4.1, $\bigoplus_I M$ is a $\mathcal{CA}$-module. Conversely, assume that $\bigoplus_I M$ is a $\mathcal{CA}$-module. Let J be a countably generated ideal of R such that $J\subseteq \mathrm{Z}_R(M)=\mathrm{Z}_R(\bigoplus_I M)$. Then $\mathrm{ann}_{\bigoplus_I M}(J)\neq 0$. So there exists $0\neq m=(m_i)_i\in\bigoplus_I M$ such that $(Jm_i)_i=Jm=0$. Therefore $Jm_i=0$ for each $i\in I$. Then, there exists $k\in I$ such that $m_k\neq 0$ and $Jm_k=0$. It follows that $\mathrm{ann}_M(J)\neq 0$. Hence M is a $\mathcal{CA}$-module completing the proof. □

Anderson and Chun proved in citeAC3 that if R is an integral domain and M is an R-module, then the idealization $R\ltimes M$ is an $\mathcal{A}$-ring (resp., an $\mathcal{SA}$-ring) if and only if M is an $\mathcal{A}$-module (resp., $\mathcal{SA}$-module) cite[Theorem 2.12]AC3. Also, we proved in citeBEK that $R\ltimes m$ is an $\mathcal{A}$-ring if and only if $R\oplus M$ is an $\mathcal{A}$-module cite[Theorem 2.1]BEK. The next theorem examines this result for the $\mathcal{CA}$-property.

Theorem 4.4. *Let R be a commutative ring and M an R-module. Then $R\ltimes M$ is a $\mathcal{CA}$-ring if and only if $R\oplus M$ is a $\mathcal{CA}$-module over R.*

Proof. Assume that $T:=R\ltimes M$ is a $\mathcal{CA}$-ring. Let I be a countably generated ideal of R such that $I\subseteq \mathrm{Z}_R(R\oplus M)$. Then $J:=I\ltimes(0)$ is a countably generated ideal of $R\ltimes M$. Since $I\subseteq \mathrm{Z}_R(R\oplus M)=\mathrm{Z}(R) cup\, \mathrm{Z}_R(M)$, we get $J\subseteq \mathrm{Z}(R\ltimes M)$ (see cite[Theorem 3.5]AW). Therefore $\mathrm{ann}_{R\ltimes M}(J)\neq 0$ as T is a $\mathcal{CA}$-ring. Hence there exists nonzero $(r,m)\in R\ltimes M$ such that $(r,m)J=(0,0)$. Let a be an arbitrary element of I. Then $(0,0)=(r,m)(a,0)=(ra,am)$, so that $ra=0$ and $am=0$. If $r=0$, then $(0,m)\neq(0,0)$ and $a(0,m)=(0,0)$, that is, $I(0,m)=(0)$. Hence $\mathrm{ann}_{R\oplus M}(I)\neq 0$, as desired. Now, suppose that $r\neq 0$. Then $a(r,0)=(0,0)$ and $(r,0)\neq(0,0)$. Hence $\mathrm{ann}_{R\oplus M}(I)\neq 0$. It follows that $R\oplus M$ is a $\mathcal{CA}$-module, as desired. Conversely, assume that $R\oplus M$ is a $\mathcal{CA}$-module. Let $J=((a_1,m_1), cdots,(a_n,m_n), cdots)T$ be a countably generated ideal of T such that $J\subseteq \mathrm{Z}(T)$. Let $I:=(a_1, cdots,a_n, cdots)R$. Next, we prove that $I\subseteq \mathrm{Z}_R(R\oplus M)$. In fact, let $x=\sum_{i=1}^{n}a_ir_i\in I$ with the $r_i\in R$. Then, as J is an ideal of T contained in $\mathrm{Z}(T)$,

$$\sum_{i=1}^{n}(a_i,m_i)(r_i,0)=\sum_{i=1}^{n}(a_ir_i,m_ir_i)=(x,\sum_{i=1}^{n}m_ir_i)\in \mathrm{Z}(T).$$

Hence, by cite[Theorem 3.5]AW, $x\in \mathrm{Z}(R) cup\, \mathrm{Z}_R(M)=\mathrm{Z}_R(R\oplus M)$. It follows that $I\subseteq \mathrm{Z}_R(R\oplus M)$. As $R\oplus M$ is a $\mathcal{CA}$-module over R, there exists $(a,m)\in R\oplus M$ such that $(a,m)\neq(0,0)$ and $I(a,m)=(0,0)$. Thus $a_ia=0$ and $a_im=0$ for each $i\in\mathbb{N}\setminus\{0\}$. Hence, for each $i\in\mathbb{N}\setminus\{0\}$,

$$(a_i,m_i)(a,m)=(a_ia,a_im+am_i)=(0,am_i).$$

If $am_i=0$ for each $i\in\mathbb{N}\setminus\{0\}$, then $(a_i,m_i)(a,m)=(0,0)$ and thus $J(a,m)=(0,0)$ which means that $\mathrm{ann}_T(J)\neq(0,0)$. Assume that there exists $j\in\mathbb{N}\setminus\{0\}$ such that $am_j\neq 0$. Then

$$(a_i,m_i)(0,am_j)=(0,aa_im_j)=(0,0)$$

for each $i\in\mathbb{N}\setminus\{0\}$. It follows that $\mathrm{ann}_T(J)\neq(0,0)$. Consequently, T is a $\mathcal{CA}$-ring completing the proof.

□

Corollary 4.5. *Let R be an integral domain and M an R-module. Then $R \ltimes M$ is a $\mathcal{CA}$-ring if and only if M is a $\mathcal{CA}$-module.*

Proof. Assume that $R \ltimes M$ is a $\mathcal{CA}$-ring. Then, by Theorem 4.4, $R \oplus M$ is a $\mathcal{CA}$-module. Let I be a nonzero countably generated ideal of R such that $I \subseteq \mathrm{Z}_R(M) = \mathrm{Z}_R(R \oplus M)$ since R is an integral domain. Then $\mathrm{ann}_{R\oplus M}(I) \neq 0$ and so there is a nonzero element $(r, m) \in R \oplus M$ such that $Ir = 0$ and $Im = 0$. Now, since R is an integral domain and $I \neq (0)$, then $r = 0$ and thus $m \neq 0$. It follows that $\mathrm{ann}_M(I) \neq 0$. Hence M is a $\mathcal{CA}$-module, as desired. Conversely, assume that M is a $\mathcal{CA}$-module. Then, since $\mathrm{Z}_R(R \oplus M) = \mathrm{Z}_R(M)$, by Theorem 4.1, $R \oplus M$ is a $\mathcal{CA}$-module. Finally, apply Theorem 4.4 to complete the proof.

□

Now, we are able to present a $\mathcal{CA}$-ring which is not a total-$\mathcal{A}$-ring. First, we provide a local domain R admitting a $\mathcal{CA}$-module which is not a total-$\mathcal{A}$-module.

Example 4.6. Let k be a field, Λ an uncountable set and $\{X_i\}_{i\in\Lambda}$ be a set of indeterminates over k. Let $R = k[[\{X_i\}_{i\in\Lambda}]]$ and note that R is a local domain of maximal ideal $m = (X_i)_{i\in\Lambda}$. Let Ω be the set of all countable subsets of Λ and, for each $A \in \Omega$, let $P_A = (X_j)_{j\in A}$ be the countably generated prime ideal of R generated by the X_j with $j \in A$. Consider the R-module $M = \bigoplus_{A\in\Omega} \dfrac{R}{P_A}$. Observe that $\mathrm{Z}_R(M) = \bigcup_{A\in\Omega} P_A$ and that the maximal ideal $m = (X_i)_{i\in\Lambda} \subseteq \bigcup_{A\in\Omega} P_A = \mathrm{Z}_R(M)$. Let $I \subseteq \mathrm{Z}_R(M)$ be a countably generated ideal of R. Note that there exists $A \in \Omega$ such that $I \subseteq P_A$. As $P_A(\overline{1}_{R/P_A}) = \overline{0}$, we get that $I(\overline{1}_{R/P_A}) = \overline{0}$. It follows that M is a $\mathcal{CA}$-module over R. We prove that M is not a total-$\mathcal{A}$-module. In effect, assume that there exists $0 \neq a = (\overline{a_1}, cdots, \overline{a_n}) \in M$ such that $ma = (0)$. Let $\overline{a}_1 \neq \overline{0}$ and $\overline{a_1} \in \dfrac{R}{P_A}$ for some $A \in \Omega$. Hence $ma_1 \subseteq P_A$ and thus $m \subseteq P_A$ as $a_1 \notin P_A$. Therefore $m = P_A$. This leads to a contradiction since m is not a countably generated ideal. It follows that $\mathrm{ann}_M(m) = (0)$. Consequently, M is a $\mathcal{CA}$-module which is not a total-$\mathcal{A}$-module. Therefore, by applying cite[Corollary 2.4]BEK and Theorem 4.4, we get $T = R \ltimes M$ is a $\mathcal{CA}$-ring which is not a total-$\mathcal{A}$-ring.

Proposition 4.7. *Let R be a ring and M a free R-module. Then $R \ltimes M$ is a $\mathcal{CA}$-ring if and only if R is so.*

Proof. Assume that $R \ltimes M$ is a $\mathcal{CA}$-ring. By Theorem 4.4, $R \oplus M =: N$ is a $\mathcal{CA}$-module. Since M is a free R-module, then $N = R^{(\Lambda)}$ is a free R-module for some set Λ. Hence, by Corollary 4.3, R is a $\mathcal{CA}$-ring. Conversely, assume that R is a $\mathcal{CA}$-ring. Then, by Corollary 4.3, M is a $\mathcal{CA}$-module as M is a free R-module. Also, note that $\mathrm{Z}(R) = \mathrm{Z}_R(M) = \mathrm{Z}_R(R \oplus M)$. Hence, by Theorem 4.4, $R \oplus M$ is a $\mathcal{CA}$-module. It follows, by Theorem 4.4, that $R \ltimes M$ is a $\mathcal{CA}$-ring, as desired. □

Proposition 4.8. *Let R be a ring and $N \subseteq M$ be R-modules such that M is an essential extension of N. Then M is a $\mathcal{CA}$-module if and only if N is so.*

Proof. Note that, by cite[Theorem 2.2 (4)]AC3, $\mathrm{Z}_R(N) = \mathrm{Z}_R(M)$. Then, by Theorem 4.1, N is a $\mathcal{CA}$-module implies that M is so. Conversely, assume that M is a $\mathcal{CA}$-module. Let $I \subseteq \mathrm{Z}_R(N) = \mathrm{Z}_R(M)$ be a countably generated ideal of R. Then, since M is a $\mathcal{CA}$-module, we get $\mathrm{ann}_M(I) \neq 0$. As M is an essential extension of N, then $\mathrm{ann}_M(I) capN \neq 0$. Thus there exists $0 \neq x \in N$ such that $Ix = 0$ so that $\mathrm{ann}_N(I) \neq 0$. It follows that N is a $\mathcal{CA}$-module establishing the desired equivalence. □

5. $\mathcal{CA}$-property and direct products of modules

This section goal is to study the behavior of the $\mathcal{CA}$-property with respect to direct products of rings. In fact, it is known that the direct product $\prod_i R_i$ of rings $(R_i)_{i\in\Lambda}$ is an $\mathcal{A}$-ring if and only if so is each ring R_i cite[Proposition 1.3]HKLR. As to $\mathcal{SA}$-property, it is proved that, if the cardinal $|\Lambda|$ of the set Λ is ≥ 2, then $\prod_i R_i$ is never an $\mathcal{SA}$-ring. We aim next at characterizing when $\prod_i R_i$ is a $\mathcal{CA}$-ring.

Theorem 5.1. *Let $(R_i)_{i\in\Lambda}$ be a family of rings and $(M_i)_{i\in\Lambda}$ a family of modules such that M_i is an R_i-module for each $i \in \Lambda$. Then the following assertions are equivalent:*

(1) $\prod_{i\in\Lambda} M_i$ *is a $\mathcal{CA}$-module over* $\prod_{\Lambda} R_i$*;*
(2) Λ *is a finite set and M_i is a $\mathcal{CA}$-module over R_i for each $i \in \Lambda$.*

We need the following lemma.

Lemma 5.2. *Let $(R_i)_{i\in\Lambda}$ be a family of rings and $(M_i)_{i\in\Lambda}$ a family of modules such that M_i is an R_i-module for each $i \in \Lambda$. If Λ is an infinite set, then $\prod_{i\in\Lambda} M_i$ is never a $\mathcal{CA}$-module over $\prod_{\Lambda} R_i$.*

Proof. Let $R := \prod_{i\in\Lambda} R_i$ and $M := \prod_{i\in\Lambda} M_i$. Assume that Λ is infinite. Let

$$\Gamma = \{i_0, i_1, cdots, i_n, cdots\} \subseteq \Lambda$$

be an infinite countable subset of Λ. Let $x_n = (a_i)_{i\in\Lambda} \in R$ such that $a_k = 1$ for $k = i_n$ or $k \in \Lambda \setminus \Gamma$, and $a_j = 0$ otherwise. Let $J = (x_n)_{n\in\mathbb{N}}$ be the countably generated ideal of R generated by the x_n. It is easily seen that J is a proper ideal of R and that $J \subseteq \mathrm{Z}_R(M)$. Let $b = (b_i)_{i\in\Lambda}$ be an element of M such that $Jb = (0)$. Then $x_n b = 0$ for each positive integer n. Hence, by the construction of the x_n, we get $b_i = 0$ for each $i \in \Lambda$ so that $b = 0$. Therefore $\mathrm{ann}_M(J) = (0)$. It follows that M is not a $\mathcal{CA}$-module over R, as desired. □

Proof of Theorem 5.1. 1) $\Rightarrow$ 2) Assume that $M := \prod_{i\in\Lambda} M_i$ is a $\mathcal{CA}$-module over R. Then, by Lemma 5.2, Λ is a finite set. Let $j \in \Lambda$ and let $I \subseteq \mathrm{Z}_{R_j}(M_j)$ be a countably generated ideal of R_j. Let $K = \prod_{i\in\Lambda} K_i$ such that $K_i = R$ for all $i \neq j$ and $K_j = I$. Then K is a countably generated ideal of R such that $K \subseteq \mathrm{Z}_R(M)$. Therefore, there exists a nonzero element $a = (a_i)_{i\in\Lambda}$ of M such that $Ka = 0$. Then $Ra_i = 0$ for all $i \neq j$. Hence $a_i = 0$ for all $i \neq j$ so that $a_j \neq 0$. It follows that $Ia_j = 0$ and $0 \neq a_j \in M_j$. Consequently, M_j is a $\mathcal{CA}$-module over R_j.
2) $\Rightarrow$ 1) Assume that Λ is a finite set and that M_i is a $\mathcal{CA}$-module for each $i \in \Lambda$. Then it suffices to handle the case where $R = R_1 \times R_2$ and $M = M_1 \times M_2$ with M_1 is a $\mathcal{CA}$-module over R_1 and M_2 is a $\mathcal{CA}$-module over R_2. Let $K \subseteq \mathrm{Z}_R(M)$ be a countably generated ideal of R. Then $K = I_1 \times I_2$ such that I_1 is a countably generated ideal of R_1 and I_2 is a countably generated ideal of R_2 with either $I_1 \subseteq \mathrm{Z}_{R_1}(M_1)$ or $I_2 \subseteq \mathrm{Z}_{R_2}(M_2)$. Suppose that $I_1 \subseteq \mathrm{Z}_{R_1}(M_1)$. Therefore $\mathrm{ann}_{M_1}(I) \neq 0$ as M_1 is a $\mathcal{CA}$-module over R_1. Let $a \in M_1 \setminus \{0\}$ such that $Ia = 0$. Then $K(a, 0) = (0, 0)$ and $(0, 0) \neq (a, 0) \in M$. Hence $\mathrm{ann}_M(K) \neq (0)$. It follows that M is a $\mathcal{CA}$-module over R completing the proof of the theorem.

□

We next list some consequences of Theorem 5.1.

Corollary 5.3. *Let $(R_i)_{i\in\Lambda}$ be a family of rings. Then the following assertions are equivalent:*

(1) $\prod_{i\in\Lambda} R_i$ *is a* $\mathcal{CA}$*-ring;*
(2) Λ *is a finite set and* R_i *is a* $\mathcal{CA}$*-ring for each* $i \in \Lambda$.

We deduce the following result on finite direct products of modules.

Corollary 5.4. *Let* $n \geq 2$ *be an integer. Let* $R_1, R_2, cdots, R_n$ *be rings and* $M_1, M_2, cdots, M_n$ *be modules such that* M_i *is an* R_i*-module for each* $i = 1, 2, cdots, n$. *Then* $M_1 \times M_2 \times cdots \times M_n$ *is a* $\mathcal{CA}$*-module over* $R_1 \times R_2 \times cdots \times R_n$ *if and only if* M_i *is a* $\mathcal{CA}$*-module over* R_i *for each* $i = 1, 2, cdots, n$.

We give the following particular version of Corollary 5.3. In particular, it sheds light on why the direct sum of two $\mathcal{CA}$-modules over a ring R is not a $\mathcal{CA}$-module over R, in general.

Corollary 5.5. *Let* R *be a ring and* M, N *be* R*-modules. Then* $M \oplus N$ *is a* $\mathcal{CA}$*-module over* $R \oplus R$ *if and only if* M *and* N *are* $\mathcal{CA}$*-modules over* R.

We close by the following result which represents the ring theoretic version of Corollary 5.3.

Corollary 5.6. *Let* $R_1, R_2, cdots, R_n$ *be rings with* $n \geq 2$ *an integer. Then* $R_1 \times R_2 \times cdots \times R_n$ *is a* $\mathcal{CA}$*-ring if and only if* R_i *is a* $\mathcal{CA}$*-ring for each* $i = 1, 2, cdots, n$.

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