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ON THE WELL-COVEREDNESS OF SQUARE GRAPHS

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Abstract. The *square* of a graph G is obtained from G by putting an edge between two distinct vertices whenever their distance in G is 2. A graph is *well-covered* if every maximal independent set in the graph is of the same size. In this paper, we investigate the graphs whose squares are well-covered. We first provide a characterization of the trees whose squares are well-covered. Afterwards, we show that a bipartite graph G and its square are well-covered if and only if every component of G is K_1 or $K_{r,r}$ for some $r \geq 1$. Moreover, we obtain a characterization of the graphs whose squares are well-covered in the case $\alpha(G) = \alpha(G^2) + k$ for $k \in \{0, 1\}$.

1. Introduction

A set of vertices in a graph is *independent* if no two vertices in the set are adjacent. If every maximal independent set of vertices has the same cardinality, then the graph is called *well-covered*. These graphs have been introduced by Plummer in [11] and many researches have been done related to them. Most of the research on well-covered graphs appearing in literature has focused on certain subclasses of well-covered graphs such as well-covered line graphs [4], very well covered graphs [6] and well-covered graphs that are 3-regular [3].

The *square* of a graph G, denoted by G^2, is the graph whose vertex set is the same as G, and where two vertices are adjacent in G^2 if and only if their distance is at most 2 in G. Particularly, a graph G is called *square-stable* if it satisfies $\alpha(G) = \alpha(G^2)$ where $\alpha(G)$ denotes the size of a maximum independent set in G. Levit and Mandrescu showed in [8] that every square-stable graph is well-covered, and well-covered trees are exactly the square-stable trees. On the other hand, König–Egerváry square-stable graphs have been studied in [9]. In addition, it has

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been proved in [10] that G^2 is a König–Egerváry graph if and only if G is a square-stable König–Egerváry graph.

In this paper, we study the graphs whose squares are well-covered. We first present some observations for certain graph classes; cycles, paths, P_4-free graphs, and P_5-free graphs. Later, we consider trees, and we define a family $\mathcal{T}$ of trees (see Section 3). Our first result is that the square of a tree is well-covered if and only if the tree is a member of $\mathcal{T}$. We also extend this result to the bipartite graphs that are well-covered. We show that a bipartite graph G and its square are well-covered if and only if every component of G is K_1 or $K_{r,r}$ for some $r \geq 1$. Finally, we consider the graphs satisfying $\alpha(G) = \alpha(G^2) + k$ for $k \in \{0, 1\}$. For the case $k = 0$, we prove that G^2 is well-covered if and only if every component of G is a complete graph. By using this result, we also provide a characterization of the graphs whose squares are well-covered in the case $\alpha(G) = \alpha(G^2) + 1$.

The paper is structured as follows. We start in Section 2 with some definitions and preliminary results on square graphs. In Section 3, we present a characterization of trees whose squares are well-covered, also we extend it to well-covered bipartite graphs. Section 4 is devoted to the square of graphs satisfying $\alpha(G) = \alpha(G^2)+k$ for $k \in \{0, 1\}$. We finish the paper with Section 5 in which we discuss the results that we obtain.

2. Preliminaries

All graphs in this paper are assumed to be simple i.e. finite and undirected, with no loops or multiple edges. We refer to [14] for terminology and notation not defined here. Given a graph $G = (V, E)$ and a subset of vertices S, $G[S]$ denotes the subgraph of G induced by S, and $G - S = G[V - S]$. We denote $G - S$ by $G - v$ when S consists of a single vertex v. For a vertex v, the *open neighbourhood* of v in a subgraph H is denoted by $N_H(v)$ while the *closed neighbourhood* of v is $N_H(v) \cup \{v\}$, denoted by $N_H[v]$. We omit the subscript H whenever there is no ambiguity on H. For a subset $S \subseteq V$, $N_H(S)$ (resp. $N_H[S]$) is the union of the open (resp. closed) neighbourhoods of the vertices in S. We use the notation $[k]$ to denote the set of integers $1, 2, \ldots, k$.

A connected graph with no cycles is called a *tree*. We denote by K_n, C_n and P_n, the complete graph, the cycle and the path on n vertices, respectively. Also, we denote by $K_{r,s}$, the complete bipartite for any $r, s \geq 1$. A star S_k is the complete bipartite graph $K_{1,k}$. The complete bipartite graph $K_{1,3}$ is also known as the *claw*. A subset $S \subset V(G)$ is called a *clique* of G if $G[S]$ is isomorphic to a complete graph. We denote by $d_G(u, v)$ the distance (i.e., the length of the shortest path) between vertices u and v in G.

We say that G is F-free if no induced subgraph of G is isomorphic to F. The degree of a vertex x, the maximum and the minimum degrees of a graph G are denoted by $d_G(x)$, $\Delta(G)$ and $\delta(G)$, respectively. A *leaf* is a vertex with degree one while an *isolated* vertex is a vertex with degree zero. An edge of a graph is said to be *pendant* if one of its vertices is a leaf vertex. If a vertex is adjacent to every other vertex in G, then it is called a *full vertex.* In a graph G, a vertex v is called *simplicial* if its neighbourhood $N_G(v)$ induces a complete graph in G.

A *matching* is a set of edges of G having pairwise no common endvertex. A *perfect matching* of a graph is a matching in which every vertex of the graph is incident to exactly one edge of the matching.

We start with some known results and observations on the well-coveredness of square graphs.

Theorem 1. *[2] In a graph G, an independent set S is maximum if and only if every independent set disjoint from S can be matched into S.*

Observation 1. *The following properties can be easily obtained.*

(*i*) *The only paths whose squares are well-covered are P_1, P_2, P_3, P_6.*
(*ii*) *The only cycles whose squares are well-covered are $C_3, C_4, \ldots, C_8, C_{10}$.*

Since the square of a P_4-free graph is a complete graph, the following holds.

Observation 2. *The squares of P_4-free graphs are well-covered.*

For a graph G, a subset $S \subset V(G)$ is called a *dominating set* of G if any vertex which is not in S is adjacent to a vertex in S. A set S of vertices is said to *dominate* another set T if every vertex in T is adjacent to at least one vertex in S.

Theorem 2. *[1] Every connected P_5-free graph has either a dominating clique or a dominating P_3.*

By using Theorem 2, we shall show that the P_5-free graphs whose squares are well-covered are complete graphs.

Proposition 1. *Let G be a P_5-free graph. Then, G^2 is well-covered if and only if G^2 is a complete graph.*

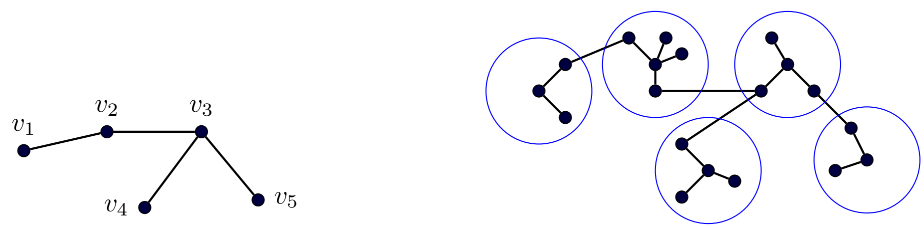
Proof. The sufficiency is clear since complete graphs are well-covered. Thus, we suppose that G^2 is well-covered. Since G is P_5-free, each pair of vertices in G is at distance at most 3. By Theorem 2, we deduce that G has a vertex v which is at distance at most 2 from each vertex of G, and so v is a full vertex in G^2. It follows that G^2 is a complete graph since G^2 is well-covered. □

3. The Square of Bipartite Graphs

In this section, we first consider the square of trees and provide a characterization of those which are well-covered. Later, we extend this result to the bipartite graphs that are well-covered.

For a tree T, we define a class $\mathcal{C}(T)$ of trees as follows. Any member of the class $\mathcal{C}(T)$ is a tree obtained from T by replacing each vertex v with a star S_k for $k \geq 2$, and where if two vertices $u, v \in V(T)$ are adjacent, then we add precisely one edge between two leaf vertices of the corresponding stars so that each star has a pendant edge in the resulting graph. If a graph G is in $\mathcal{C}(T)$, we denote it by $G \cong T(S_{k_1}, S_{k_2}, \ldots, S_{k_n})$ for some stars $\{S_{k_i}\}_{i\in[n]}$ and the tree T with $n = |T|$. For instance, $P_6 = P_2(S_2, S_2)$, $S_k = P_1(S_k)$, and the graph G depicted in Figure 1(b) is $G = T(S_2, S_4, S_3, S_3, S_2)$ for the tree T depicted in Figure 1(a).

Let $\mathcal{T}$ stand for the family of all trees that belong to a class $\mathcal{C}(T)$ for some tree T.



(a) T (b) G

Figure 1. A tree T and a member G of $\mathcal{C}(T)$.

Notice that if $G \cong T(S_{k_1}, S_{k_2}, \ldots, S_{k_n})$ for some stars $\{S_{k_i}\}_{i\in[n]}$ and a tree T with $|T| = n$, then we have $\alpha(G^2) = |T| = n$ by taking the centres of all stars, where the equality holds because each star corresponds to a clique in the square of G.

Proposition 2. *[7] If G is a well-covered graph and I is an independent set of vertices in G, then $G - N_G[I]$ must also be well-covered. In particular, $\alpha(G) = \alpha(G - N_G[S]) + |S|$.*

A vertex v of a graph G is called *shedding* if for every independent set S in $G - N_G[v]$, there is a vertex $u \in N_G(v)$ so that $S \cup \{u\}$ is independent. In other words, v is a shedding vertex if there is no independent set $I \subset V(G - N_G[v])$ which dominates $N_G(v)$.

Lemma 1. *For a tree T, the square of every graph in $\mathcal{C}(T)$ is well-covered.*

Proof. Given a tree T with n vertices, suppose that $G \in \mathcal{C}(T)$ and $G \cong T(S_{k_1}, S_{k_2}, \ldots, S_{k_n})$ for some star $\{S_{k_i}\}_{i\in[n]}$. Let H_i be the subgraph induced by the vertices of S_{k_i} in G for $i \in [n]$. By the definition of G, each H_i has at least one vertex that is a leaf in G. Thus the center of each star S_{k_i} is a shedding vertex in G, also in G^2. This implies that each maximal independent set of G^2 contains a vertex of H_i for each $i \in [n]$. Moreover, any maximal independent set of G^2 cannot contain two vertices of H_i for $i \in [n]$ since each H_i induces a clique in G^2. Hence G^2 is a well-covered graph. □

We next give a complete characterization of the trees whose squares are well-covered.

Theorem 3. *The square of a tree is well-covered if and only if the tree belongs to* $\mathcal{T}$.

Proof. The sufficiency has been proved in Lemma 1. So we assume that G is a tree, and G^2 is well-covered. The claim follows when G has at most three vertices, so let $|G| \geq 4$. Consider a leaf vertex v_1 in G, let w_1 be its unique neighbour. Set $G = G_1$, we similarly pick a leaf vertex v_i in $G_i = G_{i-1} - N_{G_{i-1}}[w_{i-1}]$ for $2 \leq i \leq p$ so that $G_p - N_{G_p}[w_p]$ is an edgeless graph where w_i is the unique neighbour of v_i in the graph G_i for each $i \in [p]$. Obviously, each $N_G[w_i]$ induces a star S_{k_i} in G with $k_i = d_{G_i}(w_i)$ for $i \in [p]$. We write $S = \{v_1, v_2 \ldots, v_p\}$, $T = \{w_1, w_2, \ldots, w_p\}$ and $H = G_p - N_{G_p}[w_p]$. Clearly, S is an independent set in G^2, also H is an edgeless graph. On the other hand, $V(H)$ does not need to be an independent set in G^2 while $V(H)$ is an independent set in G. That is, some pair of vertices in $V(H)$ may have a common neighbour in $G - V(H)$. For $u, v, w \in V(H)$, if each pair of u, v, w has a common neighbour c_j in G for $j \in [3]$, then $c_1 = c_2 = c_3$ since G has no cycle. Also, in such a case, we deduce that the vertices u, v, w induce a clique in G^2. Therefore, $V(H)$ induces a graph in G^2 whose each component is a clique or an isolated vertex.

Let R be a maximal subset of $V(H)$ containing no pair having a common neighbour in G. Thus, $S \cup R$ is a maximal independent set in G^2, and it follows that $\alpha(G^2) = |R| + p$ since G^2 is well-covered. In particular, at most $|R|$ vertices of H can be contained in any maximal independent set of G^2 since $V(H)$ induces a graph in G^2 whose each component is a clique or an isolated vertex. On the other hand, let H_i be the subgraph induced by the vertices of $N_{G_i}[w_i]$ in G for $i \in [p]$. Obviously, for each $i \in [p]$, the graph H_i is a star of size at least 2, and so $V(H_i)$ induces a complete graph in G^2. Therefore, any maximal independent set of G^2 contains at most one vertex from each $V(H_i)$ for $i \in [p]$. If there exists a maximal independent set L of G^2, and $\ell \in [p]$ such that L contains no vertex in $V(H_\ell)$, then we deduce that $|L| < |S \cup R|$, since any maximal independent set of G^2 contains at most $|R|$ vertices of H and contains at most one vertex from each $V(H_i)$ for $i \in [p]$. However, this contradicts that G^2 is well-covered. Hence any maximal independent set of G^2 contains exactly one vertex from each H_i for $i \in [p]$.

We now claim that T is an independent set in G^2. Indeed, if there exist $w_i, w_j \in T$ with $i < j$ having a common neighbour z in G, then z is adjacent to all vertices of $V(H_i) \cup V(H_j)$ in G^2. Extending of z into a maximal independent set in G^2 gives a set that does not contain any vertex from H_j. This is a contradiction with the fact that any maximal independent set of G^2 contains exactly one vertex from each H_i for $i \in [p]$. Thus, T is an independent set in G^2. Moreover, T is a dominating set in G^2 since every vertex of H is adjacent to some vertices of $N_{G_j}(w_j)$ for $w_j \in T$ in G. In this manner, T is a maximal independent set in G^2. Since G^2 is well-covered and $|S| = |T|$, we infer that H has no vertex, i.e., $V(H) = \emptyset$. Hence $\alpha(G^2) = |S| = |T| = p$.

Next, we claim that each $w_j \in T$ for $j \in [p]$ is a shedding vertex in G. Assume for a contradiction that there is a vertex $w_j \in T$ such that all vertices of $N_G(w_j)$ are dominated by an independent set $A \subset V(G - N_G[w_j])$ in G. Suppose that A is a minimal set with respect to this property. Then, the path between any pair of vertices in A is of size 5 in G since the graph G is a tree. This implies that A is an independent set in G^2 as well. Clearly, A dominates w_j in G^2 due to $d_{G^2}(a, w_j) = 2$ for every $a \in A$. The extension of A into a maximal independent set in G^2 gives a set that does not contain any vertex in $V(H_j)$. This is a contradiction with the fact that any maximal independent set of G^2 contains exactly one vertex from each H_i for $i \in [p]$. We then conclude that each w_j for $j \in [p]$ is a shedding vertex in G.

Finally we claim that for each $i \in [p]$, H_i consists of at least three vertices. Assume to the contrary that there exists $i \in [p]$ such that $N_{G_i}[w_i]$ induces a K_2 in G, so $N_{G_i}[w_i] = \{w_i, v_i\}$. Note that w_i cannot be adjacent to a leaf vertex of a star H_k for $k \in [p] \setminus \{i\}$ since T is an independent set in G^2. It then follows from the connectivity of G that there exists a star H_j induced by $N_{G_j}[w_j]$ for $j \in [p] \setminus \{i\}$ such that v_i is adjacent to a leaf vertex z of the star H_j. This implies that z is adjacent to all vertices of $V(H_i) \cup V(H_j)$ in G^2. Hence $(T - \{w_i, w_j\}) \cup \{z\}$ is a maximal independent set in G^2, a contradiction.

Consequently, we have a tree L such that $G = L(S_{k_1}, \dots, S_{k_p})$ where $S_{k_i} = H_i$ for $i \in [p]$, and L is the graph on the vertex set $T = \{w_1, w_2, \dots, w_p\}$ such that two vertices are adjacent in L if they are at distance 3 in G. Observe that the centres of two stars S_{k_i} and S_{k_j} have no common neighbour in G since T is an independent set in G^2. Also, each S_{k_i} has a pendant edge in G since L is connected and each w_i is a shedding vertex in G. Hence, G belongs to $\mathcal{C}(L)$. □

We now turn our attention to the square of bipartite graphs. Let us first give a useful result on well-covered bipartite graphs as follows.

Theorem 4. *[12,13] Let G be a connected bipartite graph. Then G is well-covered if and only if G has a perfect matching M such that for every edge $uv \in M$, $N_G[\{u, v\}]$ induces a complete bipartite graph.*

Lemma 2. *Let G be a connected bipartite graph with at least 2 vertices. If G and G^2 are well-covered, then $G = K_{r,r}$ for $r \geq 1$.*

Proof. Suppose that G is a connected bipartite graph with a bipartition I_1 and I_2 where $|I_i| \geq 1$ for $i \in \{1, 2\}$. Assume that G and G^2 are well-covered. By Theorem 4, G has a perfect matching M, and clearly $|I_1| = |I_2| = r$ for $r \in \mathbb{N}$. Let $I_1 = \{x_1, x_2, \dots, x_r\}$, $I_2 = \{y_1, y_2, \dots, y_r\}$, and $M = \{x_1y_1, x_2y_2, \dots, x_ry_r\}$. It follows from Theorem 4 that for every edge $x_iy_i \in M$, $N_G[\{x_i, y_i\}]$ induces a complete bipartite graph.

Assume for a contradiction that $G \neq K_{r,r}$. Then, there exist $i, j \in [r]$ with $i \neq j$ such that $x_iy_j \notin E(G)$ and $x_jy_i \in E(G)$ since G is connected. We may assume, without loss of generality, that $i = 1$ and $j = 2$. Recall that for every edge $x_ky_k \in M$, $N_G[\{x_k, y_k\}]$ induces a complete bipartite graph. Therefore, every vertex in $N_G(x_1)$ is adjacent to each vertex of $N_G(y_1)$. Similarly, every vertex in $N_G(y_2)$ is adjacent to each vertex of $N_G(x_2)$. This implies that $N_G(x_1)$ is complete to $N_G(y_1)$, also $N_G(y_2)$ is complete to $N_G(x_2)$. Thus, $N_G(x_1) \subseteq N_G(x_2)$ and $N_G(y_2) \subseteq N_G(y_1)$. Consequently, y_1 is adjacent to all vertices of $N_G[\{x_1, y_2\}]$ in the graph G^2.

Consider the graph $G^2 - N_{G^2}[y_1]$, clearly it is a well-covered graph by Proposition 2. Since y_1 is adjacent to all vertices of $N_G[\{x_1, y_2\}]$ in the graph G^2, we deduce that none of x_1, y_2 is adjacent to a vertex of the graph $G^2 - N_{G^2}[y_1]$, i.e., they are isolated vertices in $G^2 - N_{G^2}[y_1]$. Then for a maximal independent set S in $G^2 - N_{G^2}[y_1]$, we observe that $S \cup \{y_1\}$ and $S \cup \{x_1, y_2\}$ are two maximal independent sets in G^2 with different sizes, contradicting to the well-coveredness of G^2. Hence, $G = K_{r,r}$ for $r \geq 1$. □

The following is an immediate consequence of Lemma 2 together with the fact that the square of $K_{r,r}$ for $r \in \mathbb{N}$ is a complete graph.

Theorem 5. *A bipartite graph G and its square are well-covered if and only if every component of G is either K_1 or $K_{r,r}$ for some $r \geq 1$*

It was shown in [12] that a tree with at least two vertices is well-covered if and only if it has a perfect matching consisting of pendant edges. Then we have the following by Theorems 3 and 5.

Corollary 1. *A tree T and its square are well-covered if and only if T is either K_1 or K_2.*

4. Square-Stable and Well-Covered Graphs

Recall that a graph G is square-stable if it satisfies $\alpha(G) = \alpha(G^2)$. It was shown in [8] that square-stable graphs are well-covered. However, the square of a square-stable graph does not need to be well-covered; e.g., the square of P_4 is not well-covered.

In this section, we investigate the squares of square-stable graphs as well as the squares of graphs satisfying $\alpha(G) = \alpha(G^2) + 1$.

Theorem 6. *[8] Any square-stable graph is well-covered.*

In what follows, we state our first result in this section, which is the characterization of the square-stable graphs whose squares are well-covered.

Theorem 7. *Let G be a square-stable graph. Then, G^2 is well-covered if and only if every component of G is a complete graph.*

Proof. The sufficiency is clear since any complete graph is well-covered. Thus, we suppose that G^2 is well-covered. Note that every component of a well-covered graph is also well-covered. In addition, every component of a square-stable graph is also square-stable. Let H be a component of G. Then, H is square-stable, and H^2 is well-covered, so $\alpha(H) = \alpha(H^2) = k$. By contradiction suppose that H is not a complete graph. Then, there exist $u, v, w \in V(H)$ such that u, v are non-adjacent and $u, v \in N_H(w)$. Consider the graph $H^2 - N_{H^2}[w]$, it is clearly well-covered by Proposition 2, and $\alpha(H^2 - N_{H^2}[w]) = k - 1$. Also, if S is an independent set in $H^2 - N_{H^2}[w]$, then S is independent in $H - N_{H^2}[w]$ as well. Thus, $\alpha(H - N_{H^2}[w]) \geq k - 1$. Notice that $H - N_{H^2}[w]$ has neither u, v nor their neighbours in H. However, $S \cup \{u, v\}$ induces an independent set in H, and so $\alpha(H) \geq |S \cup \{u, v\}| = k + 1$, contradicting that $\alpha(H) = \alpha(H^2) = k$. Hence, H is a complete graph. □

A graph G is called *almost well-covered*, which is introduced in [5], if any maximal independent set is of size $\alpha(G)$ or $\alpha(G) - 1$.

Unlike square-stable graphs, we now consider the graphs satisfying $\alpha(G) = \alpha(G^2) + 1$.

Proposition 3. *If G is a graph with $\alpha(G) = \alpha(G^2) + 1$, then G is either well-covered or almost well-covered.*

Proof. Assume for a contradiction that G has a maximal independent set T such that $|T| \leq \alpha(G) - 2$. Let $\alpha(G) = k$, we pick a maximum independent set S in G^2, and so $|S| = k - 1$. Obviously, S is an independent set in G as well. Also, $d_G(u, v) \geq 3$ for every $u, v \in S$.

First we assume that S is maximal in G. Then, every vertex of $T - S$ has a neighbour in $S - T$. Notice that a vertex of $T - S$ cannot have more than one neighbour in $S - T$ since $d_G(u, v) \geq 3$ for every $u, v \in S$. We also note that $|S| \geq |T| + 1$, and let $R := N_G(T - S) \cap S$. It then follows that $T \cup (S - R)$ is a maximal independent set including T in the graph G with $|T \cup (S - R)| \geq \alpha(G) - 1$, contradicting to the maximality of T.

We now assume that S is not maximal in G. Then, there exists a vertex $u \in V(G) - S$ such that $S \cup \{u\}$ is an independent set in G. In fact, $S \cup \{u\}$ is a maximum independent set in G since $\alpha(G) = \alpha(G^2) + 1$. We write $S' := S \cup \{u\}$, and clearly $|S'| = \alpha(G)$. Similarly as before, consider T and S', if $u \in T \cap S'$, then it turns out the previous case. Thus, we further assume that $u \notin T$, i.e., $u \in S' - T$. Let $R = N_G(T - S') \cap S'$. It then follows that $T - S'$ has at most $|T - S'| + 1$ neighbours in S' since $d_G(x, y) \geq 3$ for every $x, y \in S' - u$. In particular $|R| \leq |T - S'| + 1$. We therefore deduce that $T \cup (S' - R)$ is a maximal independent set including T in the graph G with $|T \cup (S' - R)| \geq \alpha(G) - 1$, contradicting to the maximality of T. Consequently, G has no maximal independent set of size at most $\alpha(G) - 2$. This completes the proof. □

We next deal with the graphs satisfying $\alpha(G) = \alpha(G^2) + 1$ under the assumption that G^2 is well-covered. We manage to characterize those graphs in Theorem 8 with a series of lemmas.

Lemma 3. *Let G be a graph with $\alpha(G) = \alpha(G^2) + 1$. If G^2 is well-covered, then G is claw-free.*

Proof. Let G^2 be a well-covered graph. Assume for a contradiction that G contains a claw. Let $\{x, y, z\}$ be an independent set in G, and suppose that w is adjacent to all vertices of $\{x, y, z\}$ in the graph G. Pick a maximal independent set S containing w in G^2, clearly S is also maximum in G^2 due to the well-coveredness of G^2. Thus $\alpha(G^2) = |S|$ and $\alpha(G) = |S|+1$. On the other hand, $G^2 - N_{G^2}[w]$ is well-covered by Proposition 2, and $\alpha(G^2 - N_{G^2}[w]) = |S| - 1$. Notice that $S - w$ is an independent set in $G - N_{G^2}[w]$. It then follows that $(S - w) \cup \{x, y, x\}$ is independent set in G. However, this contradicts that $\alpha(G) = \alpha(G^2) + 1$. Hence, G is claw-free. □

Lemma 4. *Let G be a graph with $\alpha(G) = \alpha(G^2) + 1$. Suppose that G^2 is well-covered. If v is a non-simplicial vertex in G, then every component of $G - N_{G^2}[v]$ is a complete graph.*

Proof. Suppose that v is a non-simplicial vertex in G, and let $\alpha(G^2) = k$. Then v has two non-adjacent neighbours x, y in G. Consider a maximum independent set S containing v in G^2, obviously it is an independent set in G as well. Also $\alpha(G) = |S| + 1 = k + 1$ since $\alpha(G) = \alpha(G^2) + 1$. Note that $S' = (S - v) \cup \{x, y\}$ is an independent set in G. Since $\alpha(G) = \alpha(G^2) + 1$, we then deduce that S' is a maximum independent set in G. Consider now the graph $H = G - N_{G^2}[v]$, if there exists an independent set T larger than $S - v$ in H, then $T \cup \{x, y\}$ would be an independent set in G of size at least $|S|+2 = \alpha(G)+1$, a contradiction. Therefore, $S - v$ is a maximum independent set in H, and so $\alpha(H) = |S| - 1 = k - 1$.

Now we claim that H^2 is a well-covered graph. Obviously, $S-v$ is an independent set in H^2, so $\alpha(H^2) \geq k - 1$. If H^2 has a maximal independent set T larger than $S - v$, then T would be an independent set in H as well, which contradicts the fact that $S-v$ is a maximum independent set in H. Hence, $\alpha(H^2) = |S|-1 = k-1$. This implies that H is square-stable, and so H is well-covered by Theorem 6. It remains to show that H^2 has no maximal independent set smaller than $S-v$. Assume to the contrary that H^2 has a maximal independent set T with $|T| \leq k - 2$. Obviously, T is not independent set in $G^2 - N_{G^2}[v]$ since otherwise $T \cup \{v\}$ would be a maximal independent set in G^2, contradicting that G^2 is well-covered with $\alpha(G^2) = k = |S|$. This implies that some vertices of T are adjacent in $G^2 - N_{G^2}[v]$ while they are non-adjacent in H^2. Then there exists $p, q \in T$ and a vertex $z \in N_{G^2}(v)$ which is at distance 2 from v in G such that z is a common neighbour of p and q (see Figure 2). However, we observe that w, z, p, q induce a claw in G where w is a common neighbour of v and z in G, a contradiction by Lemma 3. Consequently, H is square-stable, and H^2 is well-covered. Hence, the result follows from Theorem 7. □

We now ready to prove our second main result in this section.

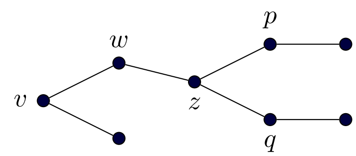


FIGURE 2. An illustration of the vertices v, w, z, p, q in the graph G.

Theorem 8. *Let G be a graph with $\alpha(G) = \alpha(G^2) + 1$, and $k \in \mathbb{N}$. Then, G^2 is well-covered if and only if for every non-simplicial vertex v, the graph $G - N_{G^2}[v]$ consists of k complete graphs such that any pair of such components has no common neighbour in G.*

Proof. The sufficiency follows from Lemmas 3 and 4. So we assume that for every non-simplicial vertex v, the graph $G - N_{G^2}[v]$ consists of k complete graphs such that any two such components have no common neighbour in G. Let $C_1, C_2, \ldots, C_k$ be the components of $G - N_{G^2}[v]$ for a non-simplicial vertex v in G where each C_i is a complete graph. Let x, y be two non-adjacent neighbours of v in G, and let u_i be a vertex of C_i for $i \in [k]$. Note that $d_G(u_i, u_j) \geq 3$ for any pair $u_i, u_j \in I$ with $i \neq j$ since no pair of the components $C_1, C_2, \ldots, C_k$ has a common neighbour in G. Consider the set $\{x, y, u_1, u_2, \ldots, u_k\}$, it is an independent set G. Thus, $\alpha(G) \geq k + 2$. On the other hand, any maximal independent set containing v in G^2 can have at most one vertex from each C_i. Thus, such a maximal independent set has at most $k + 1$ vertices, and so $\alpha(G^2) \leq k + 1$. By combining $\alpha(G) \geq k + 2$ and $\alpha(G^2) \leq k + 1$ together with the fact that $\alpha(G) = \alpha(G^2) + 1$, we deduce that $\alpha(G) = k + 2 = \alpha(G^2) + 1$. This also implies that $\{x, y, u_1, u_2, \ldots, u_k\}$ is a maximum independent set in G.

It only remains to show that G^2 has no maximal independent set of size less than $k + 1$. Assume to the contrary that there exists such a maximal independent set I of size r in G^2 with $r \leq k$. Clearly, $d_G(u, v) \geq 3$ for each pair $u, v \in I$. If a vertex $v \in I$ is a non-simplicial vertex, then $G - N_{G^2}[v]$ consists of k complete graphs. Also each vertex in $I - v$ belongs to a component of $G - N_{G^2}[v]$. Since $|I - v| = r - 1 < k$, the set I does not contain any vertex of some component of $G - N_{G^2}[v]$, a contradiction with the maximality of I in G^2. We further suppose that all vertices of I are simplicial in G. Clearly I is an independent set in G. However, it is not maximal in G by Proposition 3, since $\alpha(G) = k + 2 \geq |S| + 2$. Then, there exists an independent set $T \subset V(G) - I$ such that $I \cup T$ is a maximal independent set in G. Let u be a vertex in T. Recall that I is a maximal independent set in G^2, and so u is at distance 2 from some vertices of I. It follows that there exist $w \in I$ and $z \in V(G) - (I \cup T)$ such that z is a common neighbour of u and w in G, and so z is a non-simplicial vertex in G. By assumption, the graph $G - N_{G^2}[z]$ consists of k complete graphs such that any two of such components have no common neighbour in G. Similarly as before, let $D_1, D_2, \ldots, D_k$ be the components of $G - N_{G^2}[z]$ where each D_i is a complete graph. Let u_i be a vertex of D_i for $i \in [k]$.

Notice that every vertex in $I - w$ is at distance at least 2 from z in G since I is a maximal independent set in G^2. By the same reason, for a vertex $x \in (I - w) \cap N_{G^2}[z]$, we have $d_G(x, s) \geq 3$ for every $s \in I$. We then deduce that $x \notin N_G[z]$. Also, $x \notin N_G(u)$ since $S \cup T$ is an independent set in G. Thus, if there exists a vertex $x \in (I - w) \cap N_{G^2}[z]$ such that x is not adjacent to any D_i, then this gives a contradiction since the set $\{u, w, x, u_1, u_2, \ldots, u_k\}$ would be an independent set in G of size $\alpha(G) + 1$. Thus, every vertex $x \in (I - w) \cap N_{G^2}[z]$ is adjacent to a D_i. However, if there is such a vertex x, then x would have two non-adjacent neighbours; one is from $N_G(z)$, and the other is from a C_i for $i \in [k]$, it contradicts that all vertices of I are simplicial in G. Consequently, we deduce that $(I - w) \cap N_{G^2}[z] = \emptyset$. Thus every vertex of $I - w$ comes from the D_i's. But, this again contradicts that I is maximal in G since $|I - w| = r - 1 < k$. Hence, G^2 is well-covered. □

5. Conclusion

In this paper, we studied the graphs whose squares are well-covered. After we introduced some basic observations on those graphs, we exhibited an infinite family $\mathcal{T}$ of trees. We provided a characterization of the trees whose square well-covered which is based on the family $\mathcal{T}$. Also, we extended this result into bipartite graphs that are well-covered.

In the second part, we were interested in the graphs satisfying $\alpha(G) = \alpha(G^2) + k$ for $k \in \{0, 1\}$ where the case $\alpha(G) = \alpha(G^2)$ is also known as the square-stable graphs. Levit and Mandrescu showed in [8] that every square-stable graph is well-covered, and well-covered trees are exactly the square-stable trees. By using this result, we first proved that for the case $k = 0$, G^2 is well-covered if and only if every component of G is a complete graph. Moreover, the graphs for the case $k = 1$ have been characterized. In fact, we showed that if G^2 is well-covered, and v is a non-simplicial vertex in G, then every component of $G - N_{G^2}[v]$ is a complete graph. We conjecture that indeed $G - N_{G^2}[v]$ consists of a unique complete graph. That is, we believe that $\alpha(G^2) = 2$ when G^2 is a connected well-covered graph with $\alpha(G) = \alpha(G^2) + 1$.

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