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Some Notes on Berger Type Deformed Sasaki Metric in the Cotangent Bundle

Abderrahim Zagane*

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ABSTRACT

In the present paper, we study some notes on Berger type deformed Sasaki metric in the cotangent bundle T^*M over an anti-paraKähler manifold (M, φ, g). We characterize some geodesic properties for this metric. Next we also construct some almost anti-paraHermitian structures on T^*M and search conditions for these structures to be anti-paraKähler and quasi-anti-paraKähler with respect to the Berger type deformed Sasaki metric.

Keywords: *Horizontal lift and vertical lift, cotangent bundle, Berger type deformed Sasaki metric, geodesic, paracomplex structure.*
AMS Subject Classification (2020): *Primary:53C15 ; 53C22 ; 53C55; Secondary: 53C20; 53C07.*

1. Introduction

In this field, one of the first works which deal with the cotangent bundles of a manifold as a Riemannian manifold is that of Patterson, E. M., Walker, A. G. [10] who constructed from an affine symmetric connection on a manifold a Riemannian metric on the cotangent bundle, which they call the Riemann extension of the connection. A generalization of this metric had been given by Sekizawa, M. [15] in his classification of natural transformations of affine connections on manifolds to metrics on their cotangent bundles, obtaining the class of natural Riemann extensions which is a 2-parameter family of metrics, and which had been intensively studied by many authors. On the other hand, inspired by the concept of g-natural metrics on tangent bundles of Riemannian manifolds, Ağca, F. considered another class of metrics on cotangent bundles of Riemannian manifolds, that he callad g-natural metrics [1]. Also, there are studies by other authors Salimov, A. A., Ağca, F. [11], Yano, K., Ishihara, S. [17], Ocak, F. [9], Gezer, A., Altunbas, M. [7] etc...

In a previous work [19] we proposed "Berger Type Deformed Sasaki Metric and Harmonicity on the Cotangent Bundle". In this paper, we give some geodesic properties for the Berger type deformed Sasaki metric, then we establish necessary and sufficient conditions under which a curve be a geodesic with respect to this metric. Secondly we study some almost anti-paraHermitian structures on T^*M and we search conditions for these structures to be anti-paraKähler and quasi-anti-paraKähler. We also characterize some properties of almost anti-paraHermitian structures in the context of almost product Riemannian manifolds are presented.

2. Preliminaries

Let (M^m, g) be an m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi : T^*M \to M$ the natural projection. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)_{i=\overline{1,m}, \bar{i}=m+i}$ on T^*M, where p_i is the component of covector p in each cotangent space T^*_xM, $x \in U$ with respect to the natural coframe dx^i. Let $C^\infty(M)$ (resp. $C^\infty(T^*M)$) be the ring of real-valued C^∞ functions on M(resp. T^*M) and $\Im^r_s(M)$ (resp. $\Im^r_s(T^*M)$) be the module over $C^\infty(M)$ (resp. $C^\infty(T^*M)$) of C^∞ tensor fields of type (r, s). Denote by Γ^k_{ij} the

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*** Corresponding author***

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Christoffel symbols of g and by ∇ the Levi-Civita connection of g. We have two complementary distributions on T^*M, the vertical distribution $VT^*M = Ker(d\pi)$ and the horizontal distribution HT^*M that define a direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M. \tag{2.1}$$

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be a local expressions in $(U, x^i)_{i=\overline{1,m}}$, $U \subset M$ of a vector and covector (1-form) field $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, respectively. Then the complete and horizontal lifts ${}^CX$, ${}^HX \in \Im^1_0(T^*M)$ of $X \in \Im^1_0(M)$ and the vertical lift ${}^V\omega \in \Im^1_0(T^*M)$ of $\omega \in \Im^0_1(M)$ are defined, respectively by

$$\begin{aligned} {}^CX &= X^i \frac{\partial}{\partial x^i} - p_h \frac{\partial^H X}{\partial x^i} \frac{\partial}{\partial x^{\bar{i}}}, & (2.2)\\ {}^HX &= X^i \frac{\partial}{\partial x^i} + p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^{\bar{i}}}, & (2.3)\\ {}^V\omega &= \omega_i \frac{\partial}{\partial x^{\bar{i}}}, & (2.4) \end{aligned}$$

with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\}$, (see [17] for more details).
From (2.3) and (2.4) we see that ${}^H(\frac{\partial}{\partial x^i})$ and ${}^V(dx^i)$ have respectively local expressions of the form

$$\begin{aligned} \tilde{e}_i &= {}^H(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} + p_a \Gamma^a_{hi} \frac{\partial}{\partial x^{\bar{h}}}, & (2.5)\\ \tilde{e}_{\bar{i}} &= {}^V(dx^i) = \frac{\partial}{\partial x^{\bar{i}}}. & (2.6) \end{aligned}$$

The set of vector fields $\{\tilde{e}_i\}$ on $\pi^{-1}(U)$ define a local frame for HT^*M over $\pi^{-1}(U)$ and the set of vector fields $\{\tilde{e}_{\bar{i}}\}$ on $\pi^{-1}(U)$ define a local frame for VT^*M over $\pi^{-1}(U)$. The set $\{\tilde{e}_\alpha\} = \{\tilde{e}_i, \tilde{e}_{\bar{i}}\}$ define a local frame on T^*M, adapted to the direct sum decomposition (2.1). The indices $\alpha, \beta, \ldots = \overline{1, 2m}$ indicate the indices with respect to the adapted frame.
Using (2.3), (2.4) we have,

$$\begin{aligned} {}^HX &= X^i \tilde{e}_i \ , \ \ {}^HX = \begin{pmatrix} X^i \\ 0 \end{pmatrix}, & (2.7)\\ {}^V\omega &= \omega_i \tilde{e}_{\bar{i}} \ , \ \ {}^V\omega = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix}, & (2.8) \end{aligned}$$

with respect to the adapted frame $\{\tilde{e}_\alpha\}_{\alpha=\overline{1,2m}}$, (see [17] for more details).

Lemma 2.1. *[17] Let (M^m, g) be a Riemannian manifold. The bracket operation of vertical and horizontal vector fields on T^*M is given by the formulas*

(1) $[{}^V\omega, {}^V\theta] = 0$,

(2) $[{}^HX, {}^V\theta] = {}^V(\nabla_X \theta)$,

(3) $[{}^HX, {}^HY] = {}^H[X, Y] + {}^V(pR(X, Y))$,

for all vector fields $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, where ∇ and R denotes respectively the Levi-Civita connection and the curvature tensor of (M^m, g).

Let (M^m, g) be a Riemannian manifold, we define to maps

$$\begin{array}{cccc} \sharp: & \Im^0_1(M) & \to & \Im^1_0(M) \\ & \omega & \mapsto & \sharp(\omega) \end{array} \quad and \quad \begin{array}{cccc} \flat: & \Im^1_0(M) & \to & \Im^0_1(M) \\ & X & \mapsto & \flat(X) \end{array}$$

by $g(\sharp(\omega), Y) = \omega(Y)$ and $\flat(X)(Y) = g(X, Y)$ respectively for all $Y \in \Im^1_0(M)$.
Locally for all $X = X^i \frac{\partial}{\partial x^i} \in \Im^1_0(M)$ and $\omega = \omega_i dx^i \in \Im^0_1(M)$, we have

$$\sharp(\omega) = g^{ij} \omega_i \frac{\partial}{\partial x^j} \quad and \quad \flat(X) = g_{ij} X^i dx^j$$

where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).
For each $x \in M$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $T^*_x M$ by, for all $\omega, \theta \in \Im^0_1(M)$

$$g^{-1}(\omega, \theta) = g(\sharp(\omega), \sharp(\theta)) = g^{ij}\omega_i\theta_j.$$

In this case we have $\sharp(\omega) = g^{-1} \circ \omega$ and $\flat(X) = g \circ X$.
In the following, we noted $\sharp(\omega)$ by $\tilde{\omega}$ and $\flat(X)$ by $\tilde{X}$ for all $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$.

Lemma 2.2. *Let (M^{2m}, φ, g) be an almost anti-paraHermitian manifold, we have*

$$\begin{aligned} g^{-1}(\omega, \theta\varphi) &= g(\varphi\tilde{\omega}, \tilde{\theta}), & (2.9)\\ \nabla_X\tilde{\omega} &= \widetilde{\nabla_X\omega}, & (2.10)\\ Xg^{-1}(\omega, \theta) &= g^{-1}(\nabla_X\omega, \theta) + g^{-1}(\omega, \nabla_X\theta), & (2.11)\end{aligned}$$

for all $X \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, where ∇ is the Levi-Civita connection of (M, g).

Proof. (i) Let $\varphi = \varphi^j_h \frac{\partial}{\partial x^j} \otimes d^H X$, $\omega = \omega_k dx^k$ and $\theta = \theta_i dx^i$

$$\begin{aligned} g^{-1}(\omega, \theta\varphi) &= g^{-1}(\omega_k dx^k, \theta_i\varphi^i_h d^H X) = g^{kh}\omega_k\theta_i\varphi^i_h = g^{kh}\tilde{\omega}^t g_{tk}\tilde{\theta}^s g_{is}\varphi^i_h \\ &= \delta^h_t\tilde{\omega}^t\tilde{\theta}^s g_{is}\varphi^i_h = g_{is}\varphi^i_h\tilde{\omega}^h\tilde{\theta}^s = g(\varphi^i_h\tilde{\omega}^h\frac{\partial}{\partial x^i}, \tilde{\theta}^s\frac{\partial}{\partial x^s}) = g(\varphi\tilde{\omega}, \tilde{\theta}).\end{aligned}$$

(ii) For all $Y \in \Im^1_0(M)$

$$\begin{aligned} g(\nabla_X\tilde{\omega}, Y) &= X(g(\tilde{\omega}, Y)) - g(\tilde{\omega}, \nabla_X Y) = X(\omega(Y)) - \omega(\nabla_X Y) \\ &= (\nabla_X\omega)(Y) = g(\widetilde{\nabla_X\omega}, Y),\end{aligned}$$

(iii)

$$\begin{aligned} Xg^{-1}(\omega, \theta) &= Xg(\tilde{\omega}, \tilde{\theta}) = g(\nabla_X\tilde{\omega}, \tilde{\theta}) + g(\tilde{\omega}, \nabla_X\tilde{\theta}) \\ &= g(\widetilde{\nabla_X\omega}, \tilde{\theta}) + g(\tilde{\omega}, \widetilde{\nabla_X\theta}) = g^{-1}(\nabla_X\omega, \theta) + g^{-1}(\omega, \nabla_X\theta).\end{aligned}$$

□

3. Berger type deformed Sasaki metric

An almost product structure φ on a manifold M is a $(1,1)$ tensor field on M such that $\varphi^2 = id_M$, $\varphi \neq \pm id_M$ (id_M is the identity tensor field of type $(1,1)$ on M). The pair (M, φ) is called an almost product manifold.
A linear connection ∇ on (M, φ) such that $\nabla\varphi = 0$ is said to be an almost product connection. There exists an almost product connection on every almost product manifold. [5].
An almost paracomplex manifold is an almost product manifold (M, φ), such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even [4].

Let (M^{2m}, φ) be an almost paracomplex manifold. A Riemannian metric g is said to be an anti-paraHermitian metric (B-metric)[14] if

$$g(\varphi X, \varphi Y) = g(X, Y) \Leftrightarrow g(\varphi X, Y) = g(X, \varphi Y), \tag{3.1}$$

or from (2.9) equivalently

$$g^{-1}(\omega\varphi, \theta\varphi) = g^{-1}(\omega, \theta) \Leftrightarrow g^{-1}(\omega\varphi, \theta) = g^{-1}(\omega, \theta\varphi), \tag{3.2}$$

for all $X, Y \in \Im^1_0(M)$and $\omega, \theta \in \Im^0_1(M)$.

If (M^{2m}, φ) is an almost paracomplex manifold with an anti-paraHermitian metric g, then the triple (M^{2m}, φ, g) is said to be an almost anti-paraHermitian manifold (an almost B-manifold)[14]. Moreover, (M^{2m}, φ, g) is said to be anti-paraKä manifold (B-manifold)[14] if φ is parallel with respect to the Levi-Civita

connection ∇ of g i.e. ($\nabla\varphi = 0$).
A Tachibana operator ϕ_φ applied to the anti-paraHermitian metric (pure metric) g is given by

$$(\phi_\varphi g)(X,Y,Z) = (\varphi X)(g(Y,Z)) - X(g(\varphi Y,Z)) + g((L_Y\varphi)X,Z) + g((L_Z\varphi)X,Y), \quad (3.3)$$

for all $X,Y,Z \in \Im^1_0(M)$ [16].
In an almost anti-paraHermitian manifold, an anti-paraHermitian metric g is called paraholomorphic if

$$(\phi_\varphi g)(X,Y,Z) = 0, \quad (3.4)$$

for all $X,Y,Z \in \Im^1_0(M)$[14].
In [14], Salimov and his collaborators showed that for an almost anti-paraHermitian manifold, the condition $\phi_\varphi g = 0$ is equivalent to $\nabla\varphi = 0$ [12].
It is well known that if (M^{2m},φ,g) is a anti-paraKähler manifold, the Riemannian curvature tensor is pure [14], and for all $Y,Z \in \Im^1_0(M)$.

$$\begin{cases} R(\varphi Y,Z) &= R(Y,\varphi Z) = R(Y,Z)\varphi = \varphi R(Y,Z), \\ R(\varphi Y,\varphi Z) &= R(Y,Z), \end{cases} \quad (3.5)$$

Definition 3.1. [19] Let (M^{2m},φ,g) be an almost anti-paraHermitian manifold and T^*M be its tangent bundle. A fiber-wise Berger type deformation of the Sasaki metric noted $\tilde{g}$ is defined on T^*M by

$$\tilde{g}({}^HX,{}^HY) = g(X,Y) = g(X,Y)\circ\pi, \quad (3.6)$$
$$\tilde{g}({}^HX,{}^V\theta) = 0, \quad (3.7)$$
$$\tilde{g}({}^V\omega,{}^V\theta) = g^{-1}(\omega,\theta) + \delta^2 g^{-1}(\omega,p\varphi)g^{-1}(\theta,p\varphi), \quad (3.8)$$

for all $X,Y \in \Im^1_0(M)$, $\omega,\theta \in \Im^0_1(M)$, where δ is some constant [3].

Lemma 3.1. *[19] Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(T^*M,\tilde{g})$ its cotangent bundle equipped with the Berger type deformed Sasaki metric, we have the following:*

$$\begin{aligned} (1)\ \ {}^HX\tilde{g}({}^V\theta,{}^V\eta) &= \tilde{g}({}^V(\nabla_X\theta),{}^V\eta) + \tilde{g}({}^V\theta,{}^V(\nabla_X\eta)), \\ (2)\ \ {}^V\omega\tilde{g}({}^V\theta,{}^V\eta) &= \delta^2 g^{-1}(\theta,\omega\varphi)g^{-1}(\eta,p\varphi) + \delta^2 g^{-1}(\theta,p\varphi)g^{-1}(\eta,\omega\varphi), \end{aligned}$$

for all $X \in \Im^1_0(M)$ and $\omega,\theta,\eta \in \Im^0_1(M)$.

Theorem 3.1. *[19] Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(T^*M,\tilde{g})$ its cotangent bundle equipped with the Berger type deformed Sasaki metric. The Levi-Civita connection $\widetilde{\nabla}$ of the Berger type deformed Sasaki metric $\tilde{g}$ on T^*M satisfies the following properties:*

$$\begin{aligned} (1)\ \ \widetilde{\nabla}_{{}^HX}{}^HY &= {}^H(\nabla_XY) + \frac{1}{2}{}^V(pR(X,Y)), \\ (2)\ \ \widetilde{\nabla}_{{}^HX}{}^V\theta &= {}^V(\nabla_X\theta) + \frac{1}{2}{}^H(R(\tilde{p},\tilde{\theta})X), \\ (3)\ \ \widetilde{\nabla}_{{}^V\omega}{}^HY &= \frac{1}{2}{}^H(R(\tilde{p},\tilde{\omega})Y), \\ (4)\ \ \widetilde{\nabla}_{{}^V\omega}{}^V\theta &= \frac{\delta^2}{1+\delta^2\alpha}g^{-1}(\omega,\theta\varphi){}^V(p\varphi), \end{aligned}$$

for all $X,Y \in \Im^1_0(M)$, $\omega,\theta \in \Im^0_1(M)$ and $\alpha = g^{-1}(p,p)$, where ∇ and R denotes respectively the Levi-Civita connection and the curvature tensor of (M^{2m},φ,g).

4. Geodesics of Berger-type deformed Sasaki metric

Let (M,g) be a Riemannian manifold and $\gamma: I \to M$ be a curve on M ($I \subset \mathbb{R}$). We define on T^*M the curve $C: I \to T^*M$ by $C(t) = (\gamma(t),\vartheta(t))$, for all $t \in I$ where $\vartheta(t) \in T^*_{\gamma(t)}M$ i.e. $\vartheta(t)$ is a covector field along $\gamma(t)$.

Definition 4.1. Let (M,g) be a Riemannian manifold, $C(t) = (\gamma(t),\vartheta(t))$ be a curve on T^*M and ∇ denote the Levi-Civita connection of (M,g). If $\nabla_{\dot{\gamma}}\vartheta = 0$ the curve $C(t)$ is said to be a horizontal lift of the cure $\gamma(t)$, where $\dot{\gamma}$ the tangent field along $\gamma(t)$.

Lemma 4.1. *[18] Let (M,g) be a Riemannian manifold. If $\omega \in \Im_1^0(M)$ is a covector field on M and $(x,p) \in T^*M$ such that $\omega_x = p$, then we have:*

$$d_x\omega(X_x) = {}^HX_{(x,p)} + {}^V(\nabla_X\omega)_{(x,p)}.$$

for all $X \in \Im_0^1(M)$.

Lemma 4.2. *[18] Let (M,g) be a Riemannian manifold and ∇ denote the Levi-Civita connection of (M,g). If $\gamma(t)$ is a curve on M and $C(t) = (\gamma(t), \vartheta(t))$ is a curve on T^*M, then*

$$\dot{C} = {}^H\dot{\gamma} + {}^V(\nabla_{\dot{\gamma}}\vartheta). \tag{4.1}$$

Theorem 4.1. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(T^*M,\tilde{g})$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric. If $\widetilde{\nabla}$ denote the Levi-Civita connection of $(T^*M,\tilde{g})$ and $C(t) = (\gamma(t),\vartheta(t))$ is curve on T^*M such that $\vartheta(t)$ is a covector field along $\gamma(t)$, then*

$$\widetilde{\nabla}_{\dot{C}}\dot{C} = {}^H\Big[\nabla_{\dot{\gamma}}\dot{\gamma} + R(\tilde{\vartheta}, \widetilde{\nabla_{\dot{\gamma}}\vartheta})\dot{\gamma}\Big] + {}^V\Big[\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\vartheta + \frac{\delta^2}{1+\delta^2\alpha}g^{-1}(\nabla_{\dot{\gamma}}\vartheta,(\nabla_{\dot{\gamma}}\vartheta)\varphi)\vartheta\varphi\Big] \tag{4.2}$$

where ∇ is the Levi-Civita connection of (M^{2m},φ,g).

Proof. Using Lemma 4.1 we obtain

$$\begin{aligned}
\widetilde{\nabla}_{\dot{C}}\dot{C} &= \widetilde{\nabla}_{[{}^H\dot{\gamma} + {}^V(\nabla_{\dot{\gamma}}\vartheta)]}[{}^H\dot{\gamma} + {}^V(\nabla_{\dot{\gamma}}\vartheta)] \\
&= \widetilde{\nabla}_{{}^H\dot{\gamma}}{}^H\dot{\gamma} + \widetilde{\nabla}_{{}^H\dot{\gamma}}{}^V(\nabla_{\dot{\gamma}}\vartheta) + \widetilde{\nabla}_{{}^V(\nabla_{\dot{\gamma}}\vartheta)}{}^H\dot{\gamma} + \widetilde{\nabla}_{{}^V(\nabla_{\dot{\gamma}}\vartheta)}{}^V(\nabla_{\dot{\gamma}}\vartheta) \\
&= {}^H(\nabla_{\dot{\gamma}}\dot{\gamma}) + \frac{1}{2}{}^V(\vartheta R(\dot{\gamma},\dot{\gamma})) + {}^V(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\vartheta) + \frac{1}{2}{}^H(R(\tilde{\vartheta},\widetilde{\nabla_{\dot{\gamma}}\vartheta})\dot{\gamma}) \\
&\quad + \frac{1}{2}{}^H(R(\tilde{\vartheta},\widetilde{\nabla_{\dot{\gamma}}\vartheta})\dot{\gamma}) + \frac{\delta^2}{1+\delta^2\alpha}g^{-1}(\nabla_{\dot{\gamma}}\vartheta,(\nabla_{\dot{\gamma}}\vartheta)\varphi)^V(\vartheta\varphi) \\
&= {}^H(\nabla_{\dot{\gamma}}\dot{\gamma}) + {}^H(R(\tilde{\vartheta},\widetilde{\nabla_{\dot{\gamma}}\vartheta})\dot{\gamma}) + {}^V(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\vartheta) + \frac{\delta^2}{1+\delta^2\alpha}g^{-1}(\nabla_{\dot{\gamma}}\vartheta,(\nabla_{\dot{\gamma}}\vartheta)\varphi)^V(\vartheta\varphi) \\
&= {}^H\Big[\nabla_{\dot{\gamma}}\dot{\gamma} + R(\tilde{\vartheta},\widetilde{\nabla_{\dot{\gamma}}\vartheta})\dot{\gamma}\Big] + {}^V\Big[\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\vartheta + \frac{\delta^2}{1+\delta^2\alpha}g^{-1}(\nabla_{\dot{\gamma}}\vartheta,(\nabla_{\dot{\gamma}}\vartheta)\varphi)\vartheta\varphi\Big].
\end{aligned}$$

□

Theorem 4.2. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, T^*M its cotangent bundle equipped with the Berger-type deformed Sasaki metric and $C(t) = (\gamma(t),\vartheta(t))$ be a curve on T^*M such that $\vartheta(t)$ is a covector field along $\gamma(t)$, then $C(t)$ is a geodesic on T^*M if and only if*

$$\begin{cases} \nabla_{\dot{\gamma}}\dot{\gamma} = -R(\tilde{\vartheta},\widetilde{\nabla_{\dot{\gamma}}\vartheta})\dot{\gamma}, \\ \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\vartheta = -\dfrac{\delta^2}{1+\delta^2\alpha}g^{-1}(\nabla_{\dot{\gamma}}\vartheta,(\nabla_{\dot{\gamma}}\vartheta)\varphi)\vartheta\varphi. \end{cases} \tag{4.3}$$

where ∇ is the Levi-Civita connection of (M^{2m},φ,g).

Proof. The statement is a direct consequence of Theorem 4.1 and definition of geodesic. □

Corollary 4.1. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(T^*M,\tilde{g})$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric. The curve $C(t) = (\gamma(t),\widetilde{\dot{\gamma}(t)})$ is a geodesic on T^*M if and only if $\gamma(t)$ is a geodesic on M.*

Proof. $\dot{\gamma}$ is the tangent field along $\gamma(t)$, i.e. $\dot{\gamma}(t) \in TM$, then $\widetilde{\dot{\gamma}}(t) = \widetilde{\dot{\gamma}(t)} \in T^*M$. From (2.10), we have $\nabla_{\dot{\gamma}}\widetilde{\dot{\gamma}} = \widetilde{\nabla_{\dot{\gamma}}\dot{\gamma}}$ and $\gamma(t)$ is a geodesic on M equivalent to $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. Using Theorem 4.2 we deduce the result. □

Corollary 4.2. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(T^*M,\tilde{g})$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric. Then the horizontal lift $C(t) = (\gamma(t),\vartheta(t))$ of the curve $\gamma(t)$ is a geodesic on T^*M if and only if $\gamma(t)$ is a geodesic on M.*

Proof. Let $C(t) = (\gamma(t), \vartheta(t))$ be a horizontal lift of the curve $\gamma(t)$, equivalent to $\nabla_{\dot{\gamma}}\vartheta = 0$ and $\gamma(t)$ is a geodesic on M equivalent to $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. Using Theorem 4.2 we deduce the result. □

Remark 4.1. If $\gamma(t)$ is a geodesic on M, locally we have:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0 \quad \Leftrightarrow \quad \frac{d^2\gamma^h}{dt^2} + \sum_{i,j=1}^{2m} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^h = 0, \quad h = \overline{1, 2m}.$$

If $C(t) = (\gamma(t), \vartheta(t))$ horizontal lift of the curve $\gamma(t)$, locally we have:

$$\nabla_{\dot{\gamma}}\vartheta = 0 \quad \Leftrightarrow \quad \frac{d\vartheta_h}{dt} - \sum_{i,j=1}^{2m} \Gamma_{jh}^i \frac{d\gamma^j}{dt}\vartheta_i = 0, \quad h = \overline{1, 2m}.$$

Remark 4.2. Using Corollary 4.1, Corollary 4.2 and Remark 4.1 we can construct an infinity of examples of geodesics on $(T^*M, \tilde{g})$.

Example 4.1. Let $(\mathbb{R}^2, \varphi, g)$ be an anti-paraKähler manifold such that

$$g = e^{2x}dx^2 + e^{2y}dy^2.$$

and

$$\varphi\frac{\partial}{\partial x} = \frac{e^x}{e^y}\frac{\partial}{\partial y} \quad , \quad \varphi\frac{\partial}{\partial y} = \frac{e^y}{e^x}\frac{\partial}{\partial x}$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{11}^1 = \Gamma_{22}^2 = 1,$$

The geodesics $\gamma(t) = (\gamma^1(t), \gamma^2(t))$ such that $\gamma(0) = (a, b) \in \mathbb{R}^2$ and $\dot{\gamma}(0) = (v, w) \in \mathbb{R}^2$ satisfy the system of equations,

$$\frac{d^2\gamma^k}{dt^2} + \sum_{i,j=1}^{2} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^k = 0 \Leftrightarrow \begin{cases} \dfrac{d^2\gamma^1}{dt^2} + (\dfrac{d\gamma^1}{dt})^2 = 0 \\ \dfrac{d^2\gamma^2}{dt^2} + (\dfrac{d\gamma^2}{dt})^2 = 0 \end{cases}$$

Hence $\dot{\gamma}(t) = \dfrac{v}{1+vt}\dfrac{\partial}{\partial x} + \dfrac{w}{1+wt}\dfrac{\partial}{\partial y}$ and $\gamma(t) = (a + \ln(1+vt), b + \ln(1+wt))$.

1) $\widetilde{\dot{\gamma}(t)} = \sum_{i,j=1}^{2} g_{ij}\dot{\gamma}^j(t)dx^i = e^{2a+2\ln(1+vt)}\dfrac{v}{1+vt}dx + e^{2b+2\ln(1+wt)}\dfrac{w}{1+wt}dy,$

From Corollary 4.1, the curve $C(t) = (\gamma(t), \widetilde{\dot{\gamma}(t)})$ is a geodesic on T^*M.

2) If $C(t) = (\gamma(t), \vartheta(t))$ is horizontal lift of the curve $\gamma(t)$ i.e. $\nabla_{\dot{\gamma}}\vartheta = 0$ then,

$$\frac{d\vartheta_h}{dt} - \sum_{i,j=1}^{2m} \Gamma_{jh}^i \frac{d\gamma^j}{dt}\vartheta_i = 0 \Leftrightarrow \begin{cases} \dfrac{d\vartheta_1}{dt} - \dfrac{d\gamma^1}{dt}\vartheta_1 = 0 \\ \dfrac{d\vartheta_2}{dt} - \dfrac{d\gamma^2}{dt}\vartheta_2 = 0 \end{cases}$$

Hence $\vartheta(t) = k_1e^{a+\ln(1+vt)}dx + k_2e^{b+\ln(1+wt)}dy$, where $k_1, k_2 \in \mathbb{R}$.

From Corollary 4.2, the curve $C(t) = (\gamma(t), \vartheta(t))$ is a geodesic on T^*M.

Corollary 4.3. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, $(T^*M, \tilde{g})$ its cotangent bundle equipped with the Berger-type deformed Sasaki metric and $\gamma(t)$ be a geodesic on M. If $C(t) = (\gamma(t), \vartheta(t))$ is a geodesic on T^*M, then M^{2m} is a flat.*

5. Some almost anti-paraHermitian Structures.

Lemma 5.1. *Let (M^{2m},φ,g) be an almost anti-paraHermitian manifold and define a tensor field $J_\varphi\in\Im^1_1(T^*M)$ by*

$$\begin{cases} J_\varphi{}^HX &= -{}^H(\varphi X)\\ J_\varphi{}^V\omega &= {}^V\omega\end{cases}\tag{5.1}$$

for all $X\in\Im^1_0(M)$ and $\omega\in\Im^0_1(M)$. Then the couple (T^*M,J_φ) is an almost paracomplex manifold .

Proof. By virtue of (5.1), we have

$$\begin{cases} J^2_\varphi{}^HX = J_\varphi(J_\varphi{}^HX) = J_\varphi(-{}^H(\varphi X)) = {}^H(\varphi(\varphi X)) = {}^H(\varphi^2X),\\[2ex] J^2_\varphi{}^V\omega = J_\varphi(J_\varphi{}^V\omega) = J_\varphi{}^V\omega = {}^V\omega,\end{cases}$$

for any $X\in\Im^1_0(M)$ and $\omega\in\Im^0_1(M)$. Hence $\varphi^2=id_M$ then $J^2_\varphi=id_{T^*M}$. □

Theorem 5.1. *Let (M^{2m},φ,g) be an almost anti-paraHermitian manifold, $(T^*M,\tilde g)$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric and the almost paracomplex structure J_φ defined by (5.1). The triple $(T^*M,J_\varphi,\tilde g)$ is an almost anti-paraHermitian manifold.*

Proof. For all $X,Y\in\Im^1_0(M)$ and $\omega,\theta\in\Im^0_1(M)$, from (5.1) we have

$$\begin{aligned} (i)\ \tilde g(J_\varphi{}^HX,{}^HY) &= \tilde g(-{}^H(\varphi X),{}^HY) = -g(\varphi X,Y) = -g(X,\varphi Y) = \tilde g({}^HX,-{}^H(\varphi Y)) = \tilde g({}^HX,J_\varphi{}^HY),\\ (ii)\ \tilde g(J_\varphi{}^HX,{}^V\theta) &= \tilde g(-{}^H(\varphi X),{}^V\theta) = 0 = \tilde g({}^HX,{}^V\theta) = \tilde g({}^HX,J_\varphi{}^V\theta),\\ (iii)\ \tilde g(J_\varphi{}^V\omega,{}^V\theta) &= \tilde g({}^V\omega,{}^V\theta) = \tilde g({}^V\omega,J_\varphi{}^V\theta).\end{aligned}$$

Since g anti-paraHermitian metric, then $\tilde g$ anti-paraHermitian metric. □

Proposition 5.1. *Let (M^{2m},φ,g) be an almost anti-paraHermitian manifold, $(T^*M,\tilde g)$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric and the almost paracomplex structure J_φ defined by (5.1), then we get*

$$\begin{aligned} 1.\quad (\phi_{J_\varphi}\tilde g)({}^HX,{}^HY,{}^HZ) &= -(\phi_\varphi g)(X,Y,Z),\\ 2.\quad (\phi_{J_\varphi}\tilde g)({}^V\omega,{}^HY,{}^HZ) &= 0,\\ 3.\quad (\phi_{J_\varphi}\tilde g)({}^HX,{}^V\theta,{}^HZ) &= \tilde g\big((pR(\varphi X+X,Z))^V,{}^V\theta\big),\\ 4.\quad (\phi_{J_\varphi}\tilde g)({}^HX,{}^HY,{}^V\eta) &= \tilde g\big((pR(\varphi X+X,Y))^V,{}^V\eta\big),\\ 5.\quad (\phi_{J_\varphi}\tilde g)({}^V\omega,{}^V\theta,{}^HZ) &= 0,\\ 6.\quad (\phi_{J_\varphi}\tilde g)({}^V\omega,{}^HY,{}^V\eta) &= 0,\\ 7.\quad (\phi_{J_\varphi}\tilde g)({}^HX,{}^V\theta,{}^V\eta) &= 0,\\ 8.\quad (\phi_{J_\varphi}\tilde g)({}^V\omega,{}^V\theta,{}^V\eta) &= 0,\end{aligned}$$

for all $X,Y,Z\in\Im^1_0(M)$ and $\omega,\theta,\eta\in\Im^0_1(M)$, where R denote the curvature tensor of (M,g).

Proof. We calculate Tachibana operator ϕ_{J_φ} applied to the pure metric $\tilde g$. This operator is characterized by (3.3), from Lemma 3.1 we have

$$\begin{aligned} 1.\ (\phi_{J_\varphi}\tilde g)({}^HX,{}^HY,{}^HZ) &= (J_\varphi{}^HX)\tilde g({}^HY,{}^HZ) - {}^HX\tilde g(J_\varphi{}^HY,{}^HZ) + \tilde g\big((L_{{}^HY}J_\varphi)^HX,{}^HZ\big) + \tilde g\big({}^HY,(L_{{}^HZ}J_\varphi)^HX\big)\\ &= -{}^H(\varphi X)\tilde g({}^HY,{}^HZ) + {}^HX\tilde g({}^H(\varphi Y),{}^HZ) + \tilde g\big(L_{{}^HY}J_\varphi{}^HX - J_\varphi(L_{{}^HY}{}^HX),{}^HZ\big)\\ &\quad +\tilde g\big({}^HY,L_{{}^HZ}J_\varphi{}^HX - J_\varphi(L_{{}^HZ}{}^HX)\big)\\ &= -(\varphi X)g(Y,Z) + Xg(\varphi Y,Z) - \tilde g\big([{}^HY,{}^H(\varphi X)],{}^HZ\big) + \tilde g\big([{}^HY,{}^HX],{}^H(\varphi Z)\big)\\ &\quad -\tilde g\big({}^HY,[{}^HZ,{}^H(\varphi X)]\big) + \tilde g\big({}^H(\varphi Y),[{}^HZ,{}^HX]\big)\\ &= -(\varphi X)g(Y,Z) + Xg(\varphi Y,Z) - g\big([Y,\varphi X],Z\big) + g\big([Y,X],\varphi Z\big) - g\big(Y,[Z,\varphi X]\big)\\ &\quad +g\big(\varphi Y,[Z,X]\big)\\ &= -(\varphi X)g(Y,Z) + Xg(\varphi Y,Z) - g\big(L_Y(\varphi X),Z\big) + g\big(\varphi L_YX,Z\big) - g\big(Y,L_Z(\varphi X)\big)\\ &\quad +g\big(Y,\varphi L_ZX\big)\\ &= -(\phi_\varphi g)(X,Y,Z).\end{aligned}$$

$$\begin{aligned}
2.\ (\phi_{J_\varphi}\tilde{g})({}^V\omega, {}^HY, {}^HZ) &= (J_\varphi{}^V\omega)\tilde{g}({}^HY, {}^HZ) - {}^V\omega\tilde{g}(J_\varphi{}^HY, {}^HZ)) + \tilde{g}\big((L_{{}^HY}J_\varphi)^V\omega, {}^HZ\big) + \tilde{g}\big({}^HY, (L_{{}^HZ}J_\varphi)^V\omega\big) \\
&= \tilde{g}\big([{}^HY, {}^V\omega], {}^HZ\big) + \tilde{g}\big([{}^HY, {}^V\omega], {}^H(\varphi Z)\big) + \tilde{g}\big({}^HY, [{}^HZ, {}^V\omega]\big) + \tilde{g}\big({}^H(\varphi Y), [{}^HZ, {}^V\omega]\big) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
3.\ (\phi_{J_\varphi}\tilde{g})({}^HX, {}^V\theta, {}^HZ) &= (J_\varphi{}^HX)\tilde{g}({}^V\theta, {}^HZ) - {}^HX\tilde{g}(J_\varphi{}^V\theta, {}^HZ) + \tilde{g}\big((L_{{}^V\theta}J_\varphi)^HX, {}^HZ\big) + \tilde{g}\big({}^V\theta, (L_{{}^HZ}J_\varphi)^HX\big) \\
&= -\tilde{g}\big([{}^V\theta, {}^H(\varphi X)], {}^HZ\big) + \tilde{g}\big([{}^V\theta, {}^HX], {}^H(\varphi Z)\big) - \tilde{g}\big({}^V\theta, [{}^HZ, {}^H(\varphi X)]\big) - \tilde{g}\big({}^V\theta, [{}^HZ, {}^HX]\big) \\
&= -\tilde{g}\big({}^V\theta, {}^V(pR(Z, \varphi X))\big) - \tilde{g}\big({}^V\theta, {}^V(pR(Z, X))\big) \\
&= \tilde{g}\big({}^V(pR(\varphi X + X, Z)), {}^V\theta\big).
\end{aligned}$$

$$\begin{aligned}
4.\ (\phi_{J_\varphi}\tilde{g})({}^HX, {}^HY, {}^V\eta) &= (J_\varphi{}^HX)\tilde{g}({}^HY, {}^V\eta) - {}^HX\tilde{g}(J_\varphi{}^HY, {}^V\eta) + \tilde{g}\big((L_{{}^HY}J_\varphi)^HX, {}^V\eta\big) + \tilde{g}\big({}^HY, (L_{{}^V\eta}J_\varphi)^HX\big) \\
&= -\tilde{g}\big([{}^HY, {}^H(\varphi X)], {}^V\eta\big) - \tilde{g}\big([{}^HY, {}^HX], {}^V\eta\big) - \tilde{g}\big({}^HY, [{}^V\eta, {}^H(\varphi X)]\big) + \tilde{g}\big({}^H(\varphi Y), [{}^V\eta, {}^HX]\big) \\
&= -\tilde{g}\big({}^V(pR(Y, \varphi X)), {}^V\eta\big) - \tilde{g}\big({}^V(pR(Y, X)), {}^V\eta\big) \\
&= \tilde{g}\big((pR(\varphi X + X, Y))^V, {}^V\eta\big).
\end{aligned}$$

The other formulas are obtained by a similar calculation. □

Theorem 5.2. *Let (M^{2m}, φ, g) be a anti-paraKähler manifold, $(T^*M, \tilde{g})$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric and the almost paracomplex structure J_φ defined by (5.1). The triple $(T^*M, J_\varphi, \tilde{g})$ is a anti-paraKähler manifold if and only if M^{2m} is flat.*

Proof. For all $\overline{X}, \overline{Y}, \overline{Z} \in \Im^1_0(T^*M)$ such as $\overline{X} = {}^HX, {}^V\omega$, $\overline{Y} = {}^HY, {}^V\theta$ and $\overline{Z} = {}^HZ, {}^V\eta$
Since (M^{2m}, φ, g) is a anti-paraKähler manifold equivalent to $(\phi_\varphi g)(X, Y, Z) = 0$

$$\begin{aligned}
(\phi_{J_\varphi}\tilde{g}))(\overline{X}, \overline{Y}, \overline{Z}) = 0 \quad &\Leftrightarrow \quad \begin{cases} \tilde{g}\big((pR(\varphi X + X, Z))^V, {}^V\theta\big) & = 0 \\ \tilde{g}\big((pR(\varphi X + X, Y))^V, {}^V\eta\big) & = 0 \end{cases} \\
&\Leftrightarrow \quad \begin{cases} pR(\varphi X + X, Z) & = 0 \\ pR(\varphi X + X, Y) & = 0 \end{cases}
\end{aligned}$$

Since $\varphi \neq \pm id_M$ then $\varphi X + X \neq 0$ Thus we get

$$(\phi_{J_\varphi}\tilde{g}))(\overline{X}, \overline{Y}, \overline{Z}) = 0 \quad \Leftrightarrow \quad R = 0.$$

□

Now we study a quasi-anti-paraKähler manifold. Let (M^{2m}, φ, g) be a non-integrable almost anti-paraHermitian manifold, if

$$\underset{X,Y,Z}{\sigma} g((\nabla_X\varphi)Y, Z) = 0.$$

for all $X, Y, Z \in \Im^1_0(M)$, where σ is the cyclic sum by three arguments, then the triple (M^{2m}, φ, g) is a quasi-anti-para-Kähler manifold [6, 8]. It is well known that

$$\underset{X,Y,Z}{\sigma} g((\nabla_X\varphi)Y, Z) = 0$$

is equivalent to

$$(\phi_\varphi g)(X, Y, Z) + (\phi_\varphi g)(Y, Z, X) + (\phi_\varphi g)(Z, X, Y) = 0,$$

which was proven in [13].

Theorem 5.3. *Let (M^{2m}, φ, g) be a anti-paraKähler manifold, $(T^*M, \tilde{g})$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric and the almost paracomplex structure J_φ defined by (5.1). The triple $(T^*M, J_\varphi, \tilde{g})$ is a quasi-anti-paraKähler manifold.*

Proof. We put, for all $\overline{X},\overline{Y},\overline{Z}\in\Im^1_0(T^*M)$,

$$A(\overline{X},\overline{Y},\overline{Z})=(\phi_{J_\varphi}\tilde{g})(\overline{X},\overline{Y},\overline{Z})+(\phi_{J_\varphi}\tilde{g})(\overline{Y},\overline{Z},\overline{X})+(\phi_{J_\varphi}\tilde{g})(\overline{Z},\overline{X},\overline{Y})$$

By virtue of Proposition 5.1 and using (3.5) we have

$$\begin{array}{rcl}
1.\ A({}^HX,{}^HY,{}^HZ) &=& 0,\\
2.\ A({}^V\omega,{}^HY,{}^HZ) &=& \tilde{g}\big((pR(\varphi Y+Y,Z))^V,{}^V\omega\big)+\tilde{g}\big((pR(\varphi Z+Z,Y))^V,{}^V\omega\big)\\
&=& \tilde{g}\big((pR(\varphi Y,Z)-pR(Y,\varphi Z))^V,{}^V\omega\big)=0\\
3.\ A({}^V\omega,{}^V\theta,{}^HZ) &=& 0.\\
4.\ A({}^V\omega,{}^V\theta,{}^V\eta) &=& 0.
\end{array}$$

then $(T^*M,J_\varphi,\tilde{g})$ is a quasi-anti-paraKähler manifold. □

We consider another almost para-complex structure J on T^*M defined by

$$\begin{cases} J^HX &= -{}^HX\\ J\omega^V &= \omega^V\end{cases} \tag{5.2}$$

for all $X\in\Im^1_0(M)$ and $\omega\in\Im^0_1(M)$ [4]. From direct computations, it is easy to see that the triple $(T^*M,J,\tilde{g})$ is an almost anti-paraHermitian manifold. which give the following results.

Proposition 5.2. *Let (M^{2m},φ,g) be an almost anti-paraHermitian manifold, $(T^*M,\tilde{g})$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric and the almost paracomplex structure J defined by (5.2), then we get*

$$\begin{array}{rrcl}
1. & (\phi_J\tilde{g})({}^HX,{}^HY,Z^H) &=& 0,\\
2. & (\phi_J\tilde{g})(\omega^V,{}^HY,Z^H) &=& 0,\\
3. & (\phi_J\tilde{g})({}^HX,\theta^V,Z^H) &=& 2\tilde{g}\big((pR(X,Z))^V,\theta^V\big),\\
4. & (\phi_J\tilde{g})({}^HX,{}^HY,\eta^V) &=& 2\tilde{g}\big((pR(X,Y))^V,\eta^V\big),\\
5. & (\phi_J\tilde{g})(\omega^V,\theta^V,Z^H) &=& 0,\\
6. & (\phi_J\tilde{g})(\omega^V,{}^HY,\eta^V) &=& 0,\\
7. & (\phi_J\tilde{g})({}^HX,\theta^V,\eta^V) &=& 0,\\
8. & (\phi_J\tilde{g})(\omega^V,\theta^V,\eta^V) &=& 0,
\end{array}$$

for all $X,Y,Z\in\Im^1_0(M)$ and $\omega,\theta,\eta\in\Im^0_1(M)$, where R denote the curvature tensor of (M,g).

Theorem 5.4. *Let (M^{2m},φ,g) be a anti-paraKähler manifold, $(T^*M,\tilde{g})$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric and the almost paracomplex structure J defined by (5.2). The triple $(T^*M,J,\tilde{g})$ is a anti-paraKähler manifold if and only if M^{2m} is flat.*

Now we define a tensor field S of type $(1,2)$ and linear connection $\widehat{\nabla}$ on T^*M by,

$$S(\overline{X},\overline{Y})=\frac{1}{2}\big[(\widetilde{\nabla}_{J_\varphi\overline{Y}}J_\varphi)\overline{X}+J_\varphi\big((\widetilde{\nabla}_{\overline{Y}}J_\varphi)\overline{X}\big)-J_\varphi\big((\widetilde{\nabla}_{\overline{X}}J_\varphi)\overline{Y}\big)\big]. \tag{5.3}$$

$$\widehat{\nabla}_{\overline{X}}\overline{Y}=\widetilde{\nabla}_{\overline{X}}\overline{Y}-S(\overline{X},\overline{Y}). \tag{5.4}$$

for all $\overline{X},\overline{Y}\in\Im^1_0(T^*M)$, where $\widetilde{\nabla}$ is the Levi-Civita connection of $(T^*M,\tilde{g})$ given by Theorem 3.1. $\widehat{\nabla}$ is an almost product connection on T^*M (see [5, p.151] for more details).

Lemma 5.2. *Let (M^{2m},φ,g) be a anti-paraKähler manifold, $(T^*M,\tilde{g})$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric and the almost product structure J_φ defined by (5.1). Then tensor field S is as follows,*

$$\begin{array}{rrcl}
(1) & S({}^HX,{}^HY) &=& \frac{1}{4}{}^V(pR(X,\varphi Y+Y)),\\
(2) & S({}^HX,{}^V\theta) &=& \frac{1}{4}{}^H(R(\tilde{p},\tilde{\theta})(\varphi X+X)),\\
(3) & S({}^V\omega,{}^HY) &=& -\frac{1}{2}{}^H(R(\tilde{p},\tilde{\omega})(\varphi Y+Y)),\\
(4) & S({}^V\omega,{}^V\theta) &=& 0,
\end{array}$$

for all $X,Y\in\Im^1_0(M)$ and $\omega,\theta\in\Im^0_1(M)$, *Proof.* (1) Using (5.1), (5.3) and (3.5) we have

$$\begin{aligned}
S({}^{H}X,{}^{H}Y) &= \frac{1}{2}\big[(\widetilde{\nabla}_{J_\varphi {}^{H}Y}J_\varphi)^{H}X + J_\varphi\big((\widetilde{\nabla}_{{}^{H}Y}J_\varphi)^{H}X\big) - J_\varphi\big((\widetilde{\nabla}_{{}^{H}X}J_\varphi)^{H}Y\big)\big]\\
&= \frac{1}{2}\big[\widetilde{\nabla}_{{}^{H}(\varphi Y)}{}^{H}(\varphi X) + J_\varphi(\widetilde{\nabla}_{{}^{H}(\varphi Y)}{}^{H}X) - J_\varphi\big(\widetilde{\nabla}_{{}^{H}Y}{}^{H}(\varphi X)\big) - \widetilde{\nabla}_{{}^{H}Y}{}^{H}X + J_\varphi\big(\widetilde{\nabla}_{{}^{H}X}{}^{H}(\varphi Y)\big)\\
&\quad +\widetilde{\nabla}_{{}^{H}X}{}^{H}Y\big]\\
&= \frac{1}{2}\big[(\nabla_{\varphi Y}\varphi X)^{H} + \frac{1}{2}{}^{V}(pR(\varphi Y,\varphi X)) - (\varphi\nabla_{\varphi Y}X)^{H} + \frac{1}{2}{}^{V}(pR(\varphi Y,X)) + (\varphi\nabla_{Y}\varphi X)^{H}\\
&\quad -\frac{1}{2}{}^{V}(pR(Y,\varphi X)) - (\nabla_{Y}X)^{H} - \frac{1}{2}{}^{V}(pR(Y,X)) - (\varphi\nabla_{X}\varphi Y)^{H} + \frac{1}{2}{}^{V}(pR(X,\varphi Y))\\
&\quad +(\nabla_{X}Y)^{H} + \frac{1}{2}{}^{V}(pR(X,Y))\big]\\
&= \frac{1}{2}\big[\frac{1}{2}{}^{V}(pR(X,\varphi Y)) + \frac{1}{2}{}^{V}(pR(X,Y))\big]\\
&= \frac{1}{4}{}^{V}(pR(X,\varphi Y + Y)).
\end{aligned}$$

(2) By a similar calculation to (1), we have

$$\begin{aligned}
S({}^{H}X,{}^{V}\theta) &= \frac{1}{2}\big[(\widetilde{\nabla}_{J_\varphi {}^{V}\theta}J_\varphi)^{H}X + J_\varphi\big((\widetilde{\nabla}_{{}^{V}\theta}J_\varphi)^{H}X\big) - J_\varphi\big((\widetilde{\nabla}_{{}^{H}X}J_\varphi)^{V}\theta\big)\big]\\
&= \frac{1}{2}\big[-\widetilde{\nabla}_{{}^{V}\theta}{}^{H}(\varphi X) - J_\varphi(\widetilde{\nabla}_{{}^{V}\theta}{}^{H}X) - J_\varphi\big(\widetilde{\nabla}_{{}^{V}\theta}{}^{H}(\varphi X)\big) - \widetilde{\nabla}_{{}^{V}\theta}{}^{H}X - J_\varphi\big(\widetilde{\nabla}_{{}^{H}X}{}^{V}\theta\big) + \widetilde{\nabla}_{{}^{H}X}{}^{V}\theta\big]\\
&= \frac{1}{2}\big[-\frac{1}{2}{}^{H}(R(\tilde{p},\tilde{\theta})\varphi X) + \frac{1}{2}{}^{H}(\varphi R(\tilde{p},\tilde{\theta})X) + \frac{1}{2}{}^{H}(\varphi R(\tilde{p},\tilde{\theta})\varphi X) - \frac{1}{2}{}^{H}(R(\tilde{p},\tilde{\theta})X)\\
&\quad -(\nabla_{X}\theta)^{V} + \frac{1}{2}{}^{H}(\varphi R(\tilde{p},\tilde{\theta})X) + (\nabla_{X}\theta)^{V} + \frac{1}{2}{}^{H}(R(\tilde{p},\tilde{\theta})X)\big]\\
&= \frac{1}{2}\big[\frac{1}{2}{}^{H}(R(\tilde{p},\tilde{\theta})X) + \frac{1}{2}{}^{H}(R(\tilde{p},\tilde{\theta})\varphi X)\big]\\
&= \frac{1}{4}{}^{H}(R(\tilde{p},\tilde{\theta})(\varphi X + X)).
\end{aligned}$$

The other formulas are obtained by a similar calculation. □

Theorem 5.5. *Let (M^{2m},φ,g) be a anti-paraKähler manifold, $(T^*M,\tilde{g})$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric and the almost product structure J_φ defined by (5.1). Then the almost product connection $\widehat{\nabla}$ defined by (5.4) is as follows,*

$$\begin{aligned}
(1)\ \widehat{\nabla}_{{}^{H}X}{}^{H}Y &= (\nabla_{X}Y)^{H} + \frac{1}{4}{}^{V}(pR(X,Y-\varphi Y)),\\
(2)\ \widehat{\nabla}_{{}^{H}X}{}^{V}\theta &= (\nabla_{X}\theta)^{V} + \frac{1}{4}{}^{H}(R(\tilde{p},\tilde{\theta})(X-\varphi X)),\\
(3)\ \widehat{\nabla}_{{}^{V}\omega}{}^{H}Y &= \frac{1}{2}{}^{H}(R(\tilde{p},\tilde{\theta})(2Y+\varphi Y)),\\
(4)\ \widehat{\nabla}_{{}^{V}\omega}{}^{V}\theta &= \frac{\delta^{2}}{1+\delta^{2}\alpha}g^{-1}(\omega,\theta\varphi)(p\varphi)^{V},
\end{aligned}$$

for all $X,Y\in\Im^1_0(M)$ and $\omega,\theta\in\Im^0_1(M)$.

Proof. The proof of Theorem 5.5 follows directly from Theorem 3.1, Lemma 5.2 and formula (5.4). □

Lemma 5.3. *Let (M^{2m},φ,g) be a anti-paraKähler manifold, $(T^*M,\tilde{g})$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric and the almost product structure J_φ defined by (5.1) and $\widehat{T}$ denote the torsion tensor of $\widehat{\nabla}$,* *then we have:*

$$\begin{array}{rcl}(1)\quad \widehat{T}({}^HX,{}^HY) &=& \frac{1}{2}{}^V(pR(\varphi X+X,Y)),\\ (2)\quad \widehat{T}({}^HX,{}^V\theta) &=& \frac{3}{4}{}^H(R(\tilde{p},\tilde{\theta})(\varphi X+X)),\\ (3)\quad \widehat{T}({}^V\omega,{}^HY) &=& -\frac{3}{4}{}^H(R(\tilde{p},\tilde{\omega})(\varphi Y+Y)),\\ (4)\quad \widehat{T}({}^V\omega,{}^V\theta) &=& 0,\end{array}$$

for all $X,Y\in\Im_0^1(M)$ and $\omega,\theta\in\Im_1^0(M)$.

Proof. The proof of Lemma 5.3 follows directly from Lemma 5.2 and formula

$$\widehat{T}(\overline{X},\overline{Y}) = \widehat{\nabla}_{\overline{X}}\overline{Y}-\widehat{\nabla}_{\overline{Y}}\overline{X}-[\overline{X},\overline{Y}]=S(\overline{Y},\overline{X})-S(\overline{X},\overline{Y})$$

for all $\overline{X},\overline{Y}\in\Im_0^1(T^*M)$. □

Theorem 5.6. *Let (M^{2m},φ,g) be a anti-paraKähler manifold, $(T^*M,\tilde{g})$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric and $\widehat{\nabla}$ the almost product connection defined by (5.4), then $\widehat{\nabla}$ is symmetric if and only if M^{2m} is flat.*

Proof. Since $\varphi\neq\pm id_M$ then $\varphi X+X\neq 0$ Thus we get $R=0$. □

Let (M^{2m},φ,g) be an almost anti-paraHermitian manifold. We define a tensor field $F\in\Im_1^1(T^*M)$ by,

$$\begin{cases} F\,{}^HX &= -{}^HX+\eta g(X,\widetilde{p\varphi})^H(\widetilde{p\varphi})\\ F\,{}^V\omega &= {}^V\omega+\mu g^{-1}(\omega,p\varphi)^V(p\varphi)\end{cases}\tag{5.5}$$

for all $X\in\Im_0^1(M)$ and $\omega\in\Im_1^0(M)$, where $\eta,\mu:[0,+\infty[\to[0,+\infty[$ are positive smooth functions.
If $\eta=\mu=0$, then F is the almost paracomplex structure defined by (5.2).
In the following, we consider $\eta\neq 0$ and $\mu\neq 0$.

Remark 5.1.

$$\begin{cases} F\,{}^H(\widetilde{p\varphi}) &= (-1+\eta\alpha)^H(\widetilde{p\varphi})\\ F\,{}^V(p\varphi) &= (1+\mu\alpha)^V(p\varphi)\end{cases}\tag{5.6}$$

where $\alpha=g^{-1}(p,p)$.

Lemma 5.4. *Let (M^{2m},φ,g) be an almost anti-paraHermitian manifold and $(T^*M,\tilde{g})$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric. Then the endomorphisme F defined by (5.5) is an almost paracomplex structure if and only if $\eta=\frac{2}{\alpha}$ and $\mu=\frac{-2}{\alpha}$. i.e.*

$$\begin{cases} F\,{}^HX &= -{}^HX+\frac{2}{\alpha}g(X,\widetilde{p\varphi})^H(\widetilde{p\varphi})\\ F\,{}^V\omega &= {}^V\omega-\frac{2}{\alpha}g^{-1}(\omega,p\varphi)^V(p\varphi)\end{cases}\tag{5.7}$$

for all $X\in\Im_0^1(M)$, $\omega\in\Im_1^0(M)$ and $\alpha=g^{-1}(p,p)$.

Proof. 1) Let $X\in\Im_0^1(M)$, $\omega\in\Im_1^0(M)$,

$$\begin{array}{rcl}F^2({}^HX) &=& F(-{}^HX+\eta g(X,\widetilde{p\varphi})^H(\widetilde{p\varphi}))\\ &=& {}^HX-2\eta g(X,\widetilde{p\varphi})^H(\widetilde{p\varphi})+\eta^2\alpha g(X,\widetilde{p\varphi})^H(\widetilde{p\varphi})\\ &=& {}^HX+\eta(-2+\eta\alpha)g(X,\widetilde{p\varphi})^H(\widetilde{p\varphi}).\end{array}\tag{5.8}$$

$$\begin{array}{rcl}F^2({}^V\omega) &=& F({}^V\omega+\mu g^{-1}(\omega,p\varphi)^V(p\varphi))\\ &=& {}^V\omega+2\mu g^{-1}(\omega,p\varphi)^V(p\varphi)+\mu^2\alpha g^{-1}(\omega,p\varphi)^V(p\varphi)\\ &=& {}^V\omega+\mu(2+\mu\alpha)g(\omega,p\varphi)^V(p\varphi).\end{array}\tag{5.9}$$

From (5.7) and (5.8), then $F^2 = Id_{T^*M}$ equivalent to $\eta = \frac{2}{\alpha}$ and $\mu = \frac{-2}{\alpha}$.
2) Let $\{E_i\}_{i=\overline{1,2m}}$ and $\{\omega^i\}_{i=\overline{1,2m}}$ be a local orthonormal frame, coframe on M respectively. then

$$\begin{aligned} T_{(x,p)}T^*M^- &= \{\overline{Z} \in T_{(x,p)}T^*M, F\overline{Z} = -\overline{Z}\} = Span(A_i), \\ T_{(x,p)}T^*M^+ &= \{\overline{Z} \in T_{(x,p)}T^*M, F\overline{Z} = \overline{Z}\} = Span(B_i) \end{aligned}$$

where $A_i = {}^He_i - \frac{1}{\alpha}g(e_i, \widetilde{p\varphi})^H(\widetilde{p\varphi})$, $B_i = {}^V\omega^i - \frac{1}{\alpha}g^{-1}(\omega^i, p\varphi)^V(p\varphi)$.
Indeed

$$\begin{aligned} F(A_i) &= F({}^He_i - \frac{1}{\alpha}g(e_i, \widetilde{p\varphi})^H(\widetilde{p\varphi})) = F^He_i - \frac{1}{\alpha}g(e_i, \widetilde{p\varphi})F^H(\widetilde{p\varphi}) \\ &= -{}^He_i + \frac{2}{\alpha}g(e_i, \widetilde{p\varphi})^H(\widetilde{p\varphi}) - \frac{1}{\alpha}g(e_i, u)^H(\widetilde{p\varphi}) = -{}^He_i + \frac{1}{\alpha}g(e_i, \widetilde{p\varphi})^H(\widetilde{p\varphi})) = -A_i. \\ F(B_i) &= F({}^V\omega^i - \frac{1}{\alpha}g^{-1}(\omega^i, p\varphi)^V(p\varphi)) = F^V\omega^i - \frac{1}{\alpha}g^{-1}(\omega^i, p\varphi)F^V(p\varphi) \\ &= {}^V\omega^i - \frac{2}{\alpha}g^{-1}(\omega^i, p\varphi)^V(p\varphi) + \frac{1}{\alpha}g^{-1}(\omega^i, p\varphi)^V(p\varphi) = {}^V\omega^i - \frac{1}{\alpha}g^{-1}(\omega^i, p\varphi)^V(p\varphi) = B_i. \end{aligned}$$

□

Theorem 5.7. *Let (M^{2m}, φ, g) be an almost anti-paraHermitian manifold, $(T^*M, \tilde{g})$ be its cotangent bundle equipped with the Berger-type deformed Sasaki metric and the almost paracomplex structure F defined by (5.7). The triple $(T^*M, J_\varphi, \tilde{g})$ is an almost anti-paraHermitian manifold.*

Proof.

$$\begin{aligned} (1)\ \tilde{g}(F^HX, {}^HY) &= \tilde{g}(-{}^HX + \frac{2}{\alpha}g(X, \widetilde{p\varphi})^H(\widetilde{p\varphi}), {}^HY) = -\tilde{g}({}^HX, {}^HY) + \frac{2}{\alpha}g(X, \widetilde{p\varphi})g(Y, \widetilde{p\varphi}) \\ &= \tilde{g}({}^HX, -{}^HY + \frac{2}{\alpha}g(Y, \widetilde{p\varphi})^H(\widetilde{p\varphi})) = \tilde{g}({}^HX, F^HY). \\ (2)\ \tilde{g}(F^V\omega, {}^V\theta) &= \tilde{g}({}^V\omega - \frac{2}{\alpha}g^{-1}(\omega, p\varphi)^V(p\varphi), {}^V\theta) = \tilde{g}({}^V\omega, {}^V\theta) - \frac{2}{\alpha}g^{-1}(\omega, p\varphi)g^{-1}(\theta, p\varphi)(1 + \delta^2\alpha) \\ &= \tilde{g}({}^V\omega, {}^V\theta - \frac{2}{\alpha}g^{-1}(\theta, p\varphi)^V(p\varphi)) = \tilde{g}({}^V\omega, F^V\theta). \\ (3)\ \tilde{g}(F^HX, {}^V\theta) &= \tilde{g}(-{}^HX + \frac{2}{\alpha}g(X, \widetilde{p\varphi})^H(\widetilde{p\varphi}), {}^V\theta) = 0 = \tilde{g}({}^HX, {}^V\theta - \frac{2}{\alpha}g^{-1}(\theta, p\varphi)^V(p\varphi)) = \tilde{g}({}^HX, F^V\theta). \end{aligned}$$

□

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Affiliations

ABDERRAHIM ZAGANE
ADDRESS: University Ahmed Zabana of Relizane, Dept. of mathematics, 48000, Relizane-Algeria.
E-MAIL: zaganeabr2018@gmail.com
ORCID ID:0000-0001-9339-3787