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Research Article

Scattering analysis of a quantum impulsive boundary value problem with spectral parameter

Yelda Aygar*, Guher Gulcehre Ozbey

Department of Mathematics, Ankara University, 06100 Tandoğan, Ankara, Turkey

Abstract

We are interested in scattering and spectral analysis of an impulsive boundary value problem (IBVP) generated with a q-difference equation with eigenparameter in boundary condition in addition to impulsive conditions. We work on the Jost solution and scattering function of this problem, and by using the scattering solutions, we establish the resolvent operator, continuous spectrum and point spectrum of this problem. Furthermore, we discuss asymptotic behavior of the Jost solution and properties of eigenvalues. Also, we illustrate our results by a detailed example which is the special case of main problem.

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Keywords. q-difference equation, impulsive condition, Jost solution, scattering solution, scattering function, eigenvalue, resolvent operator

1. Introduction

Scattering problems have been a significant research area in mathematical physics. There are many books and papers devoted exclusively to scattering analysis of difference and differential equations or boundary value problems defining with a difference and differential expressions [2, 17–19, 23, 25, 28, 29].

Impulsive cases of such problems have received comparably little attention, although such equations or boundary value problems are of importance in many fields of science such as mathematical modeling, medicine, physics, economics, chemical, engineering, mathematical biology, and other areas of mathematics. Since the theory of impulsive difference equations takes form under favor of the theory of impulsive differential equations, we refer to the monographs [8, 9, 15, 32, 37] for the mathematical theory of such impulsive equations. Note that the impulsive conditions increase the importance of scattering problem. Because scientists formulate the mathematical research by using impulsive equations to understand the daily life. In general, such problems are related to discontinuous material properties. To deal with interior discontinuities, some conditions are imposed on discontinuous points. These points are called impulsive conditions. Impulsive conditions are also called transmission conditions, jump conditions, interface conditions and point conditions in literature [14, 21, 26, 30, 31, 34, 35]. Recently, some researchers have paid more attention to scattering problems of differential and difference equations with impulsive

*Corresponding Author.
Email addresses: yaygar@ankara.edu.tr (Y. Aygar), ozbeyguher@gmail.com (G.G. Ozbey)
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conditions [4,10–13,20]. Differently from these papers, we will consider a quantum impulsive boundary value problem (QIBVP) in this paper and differently from [4], this QIBVP consists spectral parameter in boundary condition. Also, this paper is more comprehensive than [4] in the way of consisting resolvent operator, continuous spectrum and additional properties of eigenvalues. On the other hand, our problem has more applications in the literature since it has eigenparameter-dependent boundary condition. It is well-known that q-calculus can be treated as bridge between mathematics and physics, and it deals with the investigation and applications of quantum derivatives and quantum integrals. It is an interesting topic having interconnections with various problems of mathematical physics and quantum mechanics [3,22,24,38]. Moreover, there are some papers about the spectral analysis of $q-$difference equations in literature [1,5–7,16]. As a result of this, our paper will contribute to literature in a way of different perspective.

In this study, we let $q>1$ and use the notation $q^{\mathbb{N}_0}:=\{q^n : n\in\mathbb{N}_0\}$, where $\mathbb{N}_0$ denotes the set of nonnegative integers. Let us consider the QIBVP consisting of the second order q-difference equation

$$qa\left(t\right)y\left(qt,z\right)+b\left(t\right)y\left(t,z\right)+a\left(\frac{t}{q}\right)y\left(\frac{t}{q},z\right)=\lambda y\left(t,z\right),\ t\in q^{\mathbb{N}}\backslash\left\{q^{m_0-1},q^{m_0},q^{m_0+1}\right\},\tag{1.1}$$

the boundary conditions

$$\left(\xi_0+\xi_1\lambda\right)y\left(q,z\right)+\left(\nu_0+\nu_1\lambda\right)y\left(1,z\right)=0,\qquad \xi_0\nu_1-\xi_1\nu_0\neq 0,\qquad \xi_1\neq\frac{\nu_0}{a\left(1\right)}\tag{1.2}$$

and the impulsive conditions

$$\begin{aligned} y\left(q^{m_0+1},z\right) &= \gamma_1 y\left(q^{m_0-1},z\right) \\ y\left(q^{m_0+2},z\right) &= \gamma_2 y\left(q^{m_0-2},z\right),\quad \gamma_1\gamma_2\neq 0,\quad \gamma_1,\gamma_2\in\mathbb{R}, \end{aligned}\tag{1.3}$$

where $\lambda=2\sqrt{q}\cos z$ is a spectral parameter, $\xi_i,\ \nu_i$ are real numbers for $i=0,1$, $\{a\left(t\right)\}_{t\in q^{\mathbb{N}_0}}$ and $\{b\left(t\right)\}_{t\in q^{\mathbb{N}}}$ are real sequences satisfying the condition

$$\sum_{t\in q^{\mathbb{N}}}\frac{\ln t}{\ln q}\left\{\left|1-a\left(t\right)\right|+\left|b\left(t\right)\right|\right\}<\infty.\tag{1.4}$$

Throughout this work, we will assume that $a\left(t\right)\neq 0$, for all $t\in q^{\mathbb{N}_0}$.

To the best of our observations, there are no results of scattering properties for (1.1)-(1.3) boundary value problem. This paper is organized as follows: In Section 2, we give some auxiliary results and introduce some notations. Section 3 and Section 4 feature the main results of the paper. We get scattering function of QIBVP (1.1)-(1.3) and investigate the properties of this function in Section 3 and we give the resolvent operator, continuous spectrum and discrete spectrum of QIBVP (1.1)-(1.3) in Section 4. Also, we present an asymptotic equation to get the properties of eigenvalues in this section. In Section 5, we are interested in unperturbed form of (1.1)-(1.3). It is a special case of (1.1)-(1.3) and it can be seen as an example of (1.1)-(1.3). This example is provided in order to illustrate our main results. Discussing the properties of Jost solution and scattering function of this unperturbed boundary value problem, we determine the region of eigenvalues and continuous spectrum of unperturbed problem.

2. Preliminaries

We now give some definitions and preliminary results. Denote the Hilbert space $\ell_2\left(q^{\mathbb{N}_0}\right)$ consisting of complex-valued functions with the inner product

$$\langle f,g\rangle_q:=\sum_{t\in q^{\mathbb{N}_0}}\mu\left(t\right)f\left(t\right)\overline{g\left(t\right)},\qquad f,g:q^{\mathbb{N}_0}\to\mathbb{C}$$

and the norm

$$\|f\|_q := \left(\sum_{t\in q^{\mathbb{N}_0}} \mu(t)\,|f(t)|^2\right)^{\frac{1}{2}}, \qquad f: q^{\mathbb{N}_0} \to \mathbb{C},$$

where $\mu(t) := (q-1)\,t$ is the graininess function for all $t \in q^{\mathbb{N}_0}$.

Let us define two semi-strips

$$D_0 := \left\{ z \in \mathbb{C} : \mathrm{Im} z > 0, -\frac{\pi}{2} \le \mathrm{Re} z \le \frac{3\pi}{2} \right\}$$

and

$$D := D_0 \cup \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right].$$

Throughout the remainder of the paper, we suppose that $P(t,z)$ and $Q(t,z)$ are the fundamental solutions of (1.1) for $z \in D$, $\lambda = 2\sqrt{q}\cos z$ and $t \in q^{\mathbb{N}_0}$, satisfying the initial conditions

$$P(1,z) = 0 \qquad P(q,z) = 1$$

$$Q(1,z) = \frac{1}{a(1)} \qquad Q(q,z) = 0,$$

respectively. For each $t \in q^{\mathbb{N}_0}$, $P(t,z)$ is polynomial of degree $(n-1)$ and is called a polynomial of the first kind, $Q(t,z)$ is polynomial of degree $(n-2)$ and is known as a polynomial of the second kind.

Definition 2.1. The Wronskian of two solutions $y = \{y(t,\lambda)\}$ and $u = \{u(t,\lambda)\}$ of (1.1) is defined by

$$W[y,u](t) = \mu(t)\,a(t)\,\{y(t,\lambda)\,u(qt,\lambda) - y(qt,\lambda)\,u(t,\lambda)\}$$

for $t \in q^{\mathbb{N}_0}$.

It can be easily shown that the Wronskian is independent from the value of t and $W[P,Q] = (1-q)$ for all $z \in \mathbb{C}$. Note that, we can write the other solution ψ of (1.1) as a linear combination of fundamental solutions. We can introduce this solution as

$$\psi(t,z) = -(\nu_0 + \lambda\nu_1)\,P(t,z) + a(1)\,(\xi_0 + \lambda\xi_1)\,Q(t,z), \quad t \in \left\{1, q, q^2, \ldots, q^{m_0-1}\right\}. \tag{2.1}$$

On the other hand, to introduce the Jost solution of (1.1)-(1.3), we need the bounded solution $e(t,z)$ of (1.1) which satisfies the condition

$$\lim_{t\to\infty} e(t,z)\, e^{-i\frac{\ln t}{\ln q}z} \sqrt{\mu(t)} = 1, \qquad z \in D \tag{2.2}$$

for $\lambda = 2\sqrt{q}\cos z$ [1]. It is represented by

$$e(t,z) = \rho(t)\,\frac{e^{i\frac{\ln t}{\ln q}z}}{\sqrt{\mu(t)}} \left(1 + \sum_{r\in q^{\mathbb{N}}} A(t,r)\, e^{i\frac{\ln r}{\ln q}z}\right)$$

$t \in \{q^{m_0+1}, q^{m_0+2}, \ldots\}$ in [1], where $\rho(t)$ and $A(t,r)$ are given in terms of the sequences $\{a(t)\}$ and $\{b(t)\}$ as

$$\rho(t) := \prod_{s\in q^{\mathbb{N}}}^{\infty} [a(s)]^{-1},$$

$$A(t,q) := -\frac{1}{\sqrt{q}} \sum_{s\in[qt,\infty)\cap q^{\mathbb{N}}} b(s),$$

$$A\left(t,q^{2}\right):=\sum_{s\in[qt,\infty)\cap q^{\mathbb{N}}}\left\{1-a^{2}\left(s\right)+\frac{1}{q}b\left(s\right)\sum_{p\in[qs,\infty)\cap q^{\mathbb{N}}}b\left(p\right)\right\},$$

$$A\left(t,q^{2}r\right):=A\left(qt,r\right)+\sum_{s\in[qt,\infty)\cap q^{\mathbb{N}}}\left\{\left(1-a^{2}\left(s\right)\right)A\left(qs,r\right)-\frac{b\left(s\right)}{\sqrt{q}}A\left(s,qr\right)\right\}$$

for $r\in q^{\mathbb{N}}$. It is clear from [1] that $e\left(t,z\right)$ is asymptotically equal to the solution

$$\widetilde{e}\left(t,z\right)=\frac{e^{i\frac{\ln t}{\ln q}z}}{\sqrt{\mu\left(t\right)}}$$

of the equation $qy\left(qt,z\right)+y\left(\frac{t}{q},z\right)=\lambda y\left(t,z\right)$ for $t\in q^{\mathbb{N}}$ and $\lambda=2\sqrt{q}\cos z$. Hereafter, by using (2.2) and Definition 2.1, we can write

$$W\left[e\left(t,z\right),e\left(t,-z\right)\right]=-\frac{2i}{\sqrt{q}}\sin z \tag{2.3}$$

for $t\in q^{\mathbb{N}_{m_0+1}}:=\{q^{m_0+1},q^{m_0+2},\ldots\}$ and $z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$. It follows from (2.3) that $\{e\left(t,z\right)\}_{t\in q^{\mathbb{N}_{m_0+1}}}$ and $\{e\left(t,-z\right)\}_{t\in q^{\mathbb{N}_{m_0+1}}}$ are fundamental system of solutions of (1.1)-(1.3) for $z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\setminus\{0,\pi\}$.

Now, we define the following solution of (1.1)-(1.3) by using $P\left(t,z\right),Q\left(t,z\right)$ and $e\left(t,z\right)$ for $z\in D$ as

$$E\left(t,z\right):=\begin{cases}\alpha\left(z\right)P\left(t,z\right)+\beta\left(z\right)Q\left(t,z\right) & ,\quad t\in\{1,q,q^{2},\ldots,q^{m_0-1}\}\\ \qquad e\left(t,z\right) & ,\quad t\in q^{\mathbb{N}_{m_0+1}},\end{cases} \tag{2.4}$$

where α and β are z-dependent coefficients. By using impulsive conditions (1.3) for $E\left(t,z\right)$, we write

$$\begin{aligned}E\left(q^{m_0+1},z\right)&=\gamma_1E\left(q^{m_0-1},z\right)\\ E\left(q^{m_0+2},z\right)&=\gamma_2E\left(q^{m_0-2},z\right)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\gamma_1}e\left(q^{m_0+1},z\right)&=\alpha\left(z\right)P\left(q^{m_0-1},z\right)+\beta\left(z\right)Q\left(q^{m_0-1},z\right)\\ \frac{1}{\gamma_2}e\left(q^{m_0+2},z\right)&=\alpha\left(z\right)P\left(q^{m_0-2},z\right)+\beta\left(z\right)Q\left(q^{m_0-2},z\right).\end{aligned} \tag{2.5}$$

By using the definition of Wronskian and equation (2.5), we obtain the coefficients $\alpha\left(z\right)$ and $\beta\left(z\right)$ for $z\in D$ as

$$\alpha\left(z\right)=\frac{q^{m_0-2}a\left(q^{m_0-2}\right)}{\gamma_1\gamma_2}\left\{\gamma_2\alpha_1\left(z\right)-\gamma_1\alpha_2\left(z\right)\right\} \tag{2.6}$$

$$\beta\left(z\right)=-\frac{q^{m_0-2}a\left(q^{m_0-2}\right)}{\gamma_1\gamma_2}\left\{\gamma_2\beta_1\left(z\right)-\gamma_1\beta_2\left(z\right)\right\}, \tag{2.7}$$

where

$$\begin{aligned}\alpha_1\left(z\right)&:=e\left(q^{m_0+1},z\right)Q\left(q^{m_0-2},z\right)\\ \alpha_2\left(z\right)&:=e\left(q^{m_0+2},z\right)Q\left(q^{m_0-1},z\right)\\ \beta_1\left(z\right)&:=e\left(q^{m_0+1},z\right)P\left(q^{m_0-2},z\right)\end{aligned}$$

and

$$\beta_2(z) := e\left(q^{m_0+2}, z\right) P\left(q^{m_0-1}, z\right).$$

The function $E(z) := \{E(t,z)\}$ is called Jost solution of the QIBVP (1.1)-(1.3).

Lemma 2.2. *The coefficients $\alpha(z)$ and $\beta(z)$ satisfy the following equations for $z \in D$:*

$$\alpha(-z) = \overline{\alpha(z)} \quad \text{and} \quad \beta(-z) = \overline{\beta(z)}.$$

Proof. It is known that $\overline{P(t,z)} = P(t,-z)$ and $\overline{Q(t,z)} = Q(t,-z)$ for $z \in D$. It follows from that $\alpha(-z) = \overline{\alpha(z)}$ and $\beta(-z) = \overline{\beta(z)}$ for $z \in D$ by using (2.6) and (2.7). It completes the proof. □

Now, we will consider the following solution $F(z) = \{F(t,z)\}$ of QIBVP (1.1)-(1.3) for $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$

$$F(t,z) := \begin{cases} \psi(t,z) & , \quad t \in \{1, q, q^2, \ldots, q^{m_0-1}\} \\ c(z)\, e(t,z) + d(z)\, e(t,-z) & , \quad t \in q^{\mathbb{N}_{m_0+1}}. \end{cases} \tag{2.8}$$

To get the coefficients $c(z)$ and $d(z)$, we will use same way as finding $\alpha(z)$ and $\beta(z)$. By using (1.3) and (2.3), we obtain

$$c(z) = -\frac{(q-1)\, q^{m_0+\frac{3}{2}} a\left(q^{m_0+1}\right)}{2i \sin z} \{\gamma_1 c_1(z) - \gamma_2 c_2(z)\} \tag{2.9}$$

and

$$d(z) = \frac{(q-1)\, q^{m_0+\frac{3}{2}} a\left(q^{m_0+1}\right)}{2i \sin z} \{\gamma_1 d_1(z) - \gamma_2 d_2(z)\}, \tag{2.10}$$

where

$$\begin{aligned} c_1(z) &:= \psi\left(q^{m_0-1}, z\right) e\left(q^{m_0+2}, -z\right) \\ c_2(z) &:= \psi\left(q^{m_0-2}, z\right) e\left(q^{m_0+1}, -z\right) \\ d_1(z) &:= \psi\left(q^{m_0-1}, z\right) e\left(q^{m_0+2}, z\right) \end{aligned}$$

and

$$d_2(z) := \psi\left(q^{m_0-2}, z\right) e\left(q^{m_0+1}, z\right).$$

Corollary 2.3. *The coefficients $c(z)$ and $d(z)$ satisfy the following relationship for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$*

$$d(z) = c(-z) = \overline{c(z)}.$$

Lemma 2.4. *For all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, we get*

$$W[E(t,z), F(t,z)] := \begin{cases} \dfrac{a\left(q^{m_0-2}\right)}{a\left(q^{m_0+1}\right)} \dfrac{2i \sin z}{\gamma_1 \gamma_2} \dfrac{d(z)}{q^{\frac{7}{2}}} & , \quad t \in \{1, q, q^2, \ldots, q^{m_0-1}\} \\ -\dfrac{2i \sin z}{\sqrt{q}} d(z) & , \quad t \in q^{\mathbb{N}_{m_0+1}}. \end{cases}$$

Proof. Using the definition of Wronskian for $t \in q^{\mathbb{N}_{m_0+1}}$, we write

$$\begin{aligned} W[E,F] =& \mu(t)\, a(t)\, \{E(t,z)\, F(qt,z) - E(qt,z)\, F(t,z)\} \\ =& \mu\left(q^{m_0+1}\right) a\left(q^{m_0+1}\right) \left\{E\left(q^{m_0+1},z\right) F\left(q^{m_0+2},z\right) - E\left(q^{m_0+2},z\right) F\left(q^{m_0+1},z\right)\right\} \\ =& \mu\left(q^{m_0+1}\right) a\left(q^{m_0+1}\right) \left\{ \begin{array}{c} e\left(q^{m_0+1},z\right)\left[c(z)\, e\left(q^{m_0+2},z\right) + d(z)\, e\left(q^{m_0+2},-z\right)\right] \\ -e\left(q^{m_0+2},z\right)\left[c(z)\, e\left(q^{m_0+1},z\right) + d(z)\, e\left(q^{m_0+1},-z\right)\right] \end{array} \right\}. \end{aligned}$$

By using

$$W\left[e\left(q^{m_0+1},z\right), e\left(q^{m_0+1},-z\right)\right] = -\frac{2i}{\sqrt{q}} \sin z,$$

we get

$$\begin{aligned} W[E(t,z), F(t,z)] &= \mu\left(q^{m_0+1}\right) a\left(q^{m_0+1}\right) d(z) \left(-\frac{2i\sin z}{\sqrt{q}\mu\left(q^{m_0+1}\right) a\left(q^{m_0+1}\right)}\right) \\ &= -\frac{2i\sin z}{\sqrt{q}} d(z). \end{aligned}$$

Similarly, if we apply the definitions $E(t,z)$, $\alpha(z)$, $\beta(z)$, $F(t,z)$ and $d(z)$ given in (2.4), (2.6), (2.7), (2.8) and (2.10), respectively, we find

$$W[E(t,z), F(t,z)] = \frac{a\left(q^{m_0-2}\right)}{a\left(q^{m_0+1}\right)} \frac{2i\sin z}{\gamma_1\gamma_2} \frac{d(z)}{q^{\frac{7}{2}}}$$

for $t = 1, q, q^2, \ldots, q^{m_0-1}$. This completes the proof. □

3. Jost solution and scattering solution

Now, we define the Jost function J of QIBVP (1.1)-(1.3) by applying the boundary conditions (1.2) to the Jost solution $E(t,z)$ of (1.1)-(1.3) and we write

$$\begin{aligned} J(z) &:= (\xi_0 + \lambda\xi_1)\, E(q,z) + (\nu_0 + \lambda\nu_1)\, E(1,z) \\ &= \alpha(z)(\xi_0 + \lambda\xi_1) + \frac{\beta(z)}{a(1)}(\nu_0 + \lambda\nu_1). \end{aligned} \tag{3.1}$$

It is evident that

$$J(-z) = \alpha(-z)(\xi_0 + \lambda\xi_1) + \frac{\beta(-z)}{a(1)}(\nu_0 + \lambda\nu_1).$$

Furthermore, the function J is analytic in $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ and continuous in $\overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \mathrm{Im} z \geq 0\}$. Similarly to the Sturm-Liouville equation, the function J is called the Jost function of QIBVP (1.1)-(1.3).

Lemma 3.1. *For all* $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, *the Jost function J can be written as a multiple of coefficient $d(z)$*

$$J(z) = -\frac{a\left(q^{m_0-2}\right)}{a\left(q^{m_0+1}\right)} \frac{2i\sin z}{a(1)} \frac{q^{-\frac{7}{2}}}{\gamma_1\gamma_2(q-1)} d(z).$$

Proof. It follows from (2.10) that

$$d(z) = \frac{(q-1)\, q^{m_0+\frac{3}{2}} a\left(q^{m_0+1}\right)}{2i\sin z} \left\{ \begin{array}{c} \gamma_1\psi\left(q^{m_0-1},z\right) e\left(q^{m_0+2},z\right) \\ -\gamma_2\psi\left(q^{m_0-2},z\right) e\left(q^{m_0+1},z\right) \end{array} \right\}$$

If we use (2.1) in last equation, we get

$$d(z)=\frac{(q-1)\,q^{m_0+\frac{3}{2}}a\left(q^{m_0+1}\right)}{2i\sin z}\left\{\begin{array}{c}-\left(\nu_0+\lambda\nu_1\right)\dfrac{\beta(z)\,\gamma_1\gamma_2}{q^{m_0-2}a\left(q^{m_0-2}\right)}\\ \\ -a(1)\left(\xi_0+\lambda\xi_1\right)\dfrac{\alpha(z)\,\gamma_1\gamma_2}{q^{m_0-2}a\left(q^{m_0-2}\right)}\end{array}\right\}$$

$$=-\frac{(q-1)\,q^{m_0+\frac{3}{2}}a\left(q^{m_0+1}\right)\gamma_1\gamma_2}{2i\sin zq^{m_0-2}a\left(q^{m_0-2}\right)}\left\{\frac{(\nu_0+\lambda\nu_1)}{a(1)}\beta(z)+\left(\xi_0+\lambda\xi_1\right)\alpha(z)\right\},$$

this equation gives

$$d(z)=-\frac{(q-1)\,q^{\frac{7}{2}}a\left(q^{m_0+1}\right)a(1)\,\gamma_1\gamma_2}{2i\sin za\left(q^{m_0-2}\right)}J(z)$$

from (3.1). It completes the proof of Lemma 3.1. □

Theorem 3.2. *The coefficient $d(z)$ is not zero for all $z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\setminus\{0,\pi\}$.*

Proof. Suppose that there exists a point z_0 in $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\setminus\{0,\pi\}$ such that $d(z_0)=0$. It follows from that $c(z_0)=d(z_0)=0$ as a result of Corollary 2.2. It gives $F(t,z_0)=0$ for all $t\in q^{\mathbb{N}_0}$, but this gives a contradiction. Because F is not a trivial solution of QIBVP (1.1)-(1.3). So, the assumption is not true, i.e., $d(z)\neq 0$ for all $z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\setminus\{0,\pi\}$. □

Definition 3.3. The function

$$S(z):=\frac{\overline{J(z)}}{J(z)},\quad z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\setminus\{0,\pi\}$$

is called the scattering function of QIBVP (1.1)-(1.3).

It is obvious from Definition 3.1 and Lemma 3.1 that scattering function can be also given with the help of coefficient $d(z)$ as

$$S(z)=\frac{J(-z)}{J(z)}=-\frac{d(-z)}{d(z)} \tag{3.2}$$

for all $z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\setminus\{0,\pi\}$.

Theorem 3.4. *For all $z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\setminus\{0,\pi\}$, the function $S(z)$ has the following properties:*

i) $S(-z)=S^{-1}(z)=\overline{S(z)}$
ii) $|S(z)|=1$.

Proof. i) By using (3.1) and (3.2), we get

$$S(-z)=\frac{J(z)}{J(-z)}$$

and

$$\overline{S(z)}=\frac{\overline{J(-z)}}{\overline{J(z)}}=\frac{J(z)}{J(-z)}=S^{-1}(z)$$

for all $z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\setminus\{0,\pi\}$.

ii) Since $|S(z)|^2 = \overline{S(z)}S(z)$, the equation (3.2) gives us

$$|S(z)| = \frac{J(z)}{J(-z)}\frac{J(-z)}{J(z)} = 1$$

for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$. □

4. Resolvent operator, continuous and discrete spectrum of QIBVP

In the following, we will define another solution $G(z) := \{G(t,z)\}$ of (1.1)-(1.3) for all $z \in D$ to get the resolvent operator of QIBVP (1.1)-(1.3).

$$G(t,z) := \begin{cases} \psi(t,z) & , \quad t \in \{1, q, q^2, \ldots, q^{m_0-1}\} \\ r(z)\, e(t,z) + k(z)\, \widehat{e}(t,z) & , \quad t \in q^{\mathbb{N}_{m_0+1}}, \end{cases} \tag{4.1}$$

where $\widehat{e}(t,z)$ denotes the unbounded solution of (1.1) for $t \in q^{\mathbb{N}_{m_0+1}}$ and satisfies the condition

$$\lim_{t\to\infty} \widehat{e}(t,z)\, e^{i\frac{\ln t}{\ln q}z}\sqrt{\mu(t)} = 1, \quad z \in D.$$

It can be easily shown that

$$W[e(t,z), \widehat{e}(t,z)] = -\frac{2i}{\sqrt{q}}\sin z$$

for $t \in q^{\mathbb{N}_{m_0+1}}$ and $z \in D$, and $e(t,z)$, $\widehat{e}(t,z)$ are independent solutions for $z \in D \setminus \{0, \pi\}$. Similar to previous solutions, we find the coefficients $r(z)$ and $k(z)$ for $z \in D \setminus \{0, \pi\}$ uniquely, as

$$r(z) = -\frac{(q-1)\, q^{m_0+\frac{3}{2}} a(q^{m_0+1})}{2i \sin z}\{\gamma_1 r_1(z) - \gamma_2 r_2(z)\} \tag{4.2}$$

$$k(z) = \frac{(q-1)\, q^{m_0+\frac{3}{2}} a(q^{m_0+1})}{2i \sin z}\{\gamma_1 k_1(z) - \gamma_2 k_2(z)\}, \tag{4.3}$$

where

$$\begin{aligned} r_1(z) &:= \psi\left(q^{m_0-1}, z\right)\widehat{e}\left(q^{m_0+2}, z\right) \\ r_2(z) &:= \psi\left(q^{m_0-2}, z\right)\widehat{e}\left(q^{m_0+1}, z\right) \\ k_1(z) &:= \psi\left(q^{m_0-1}, z\right) e\left(q^{m_0+2}, z\right) \\ k_2(z) &:= \psi\left(q^{m_0-2}, z\right) e\left(q^{m_0+1}, z\right). \end{aligned}$$

The solution G is called the unbounded solution of (1.1)-(1.3), and for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, the coefficient $k(z)$ has the following relation between the coefficients of the solution F

$$k(z) = d(z) = \overline{c(z)}.$$

Moreover, using (2.4), (4.1) and Definition 2.1, we obtain

$$W[E(t,z), G(t,z)] := \begin{cases} \frac{a(q^{m_0-2})}{a(q^{m_0+1})}\frac{2i\sin z}{\gamma_1\gamma_2}\frac{d(z)}{q^{\frac{7}{2}}} & , \quad t \in \{1, q, q^2, \ldots, q^{m_0-1}\} \\ -\frac{2i\sin z}{\sqrt{q}} d(z) & , \quad t \in q^{\mathbb{N}_{m_0+1}} \end{cases} \tag{4.4}$$

for all $z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\}$, samely as Lemma 2.3.

Theorem 4.1. *Assume (1.4). Then the resolvent operator of QIBVP (1.1)-(1.3) is given by*

$$(Rh)(t) := \sum_{r \in q^{\mathbb{N}}} R(t,r)\, h(r,z), \quad h \in \ell_2\left(q^{\mathbb{N}}\right),$$

where

$$R_{t,z}(z) := \begin{cases} -\dfrac{\mu\left(\frac{r}{q}\right) G(r,z)\, E(t,z)}{W[E,G]} & , \quad r = q^k \\[2ex] -\dfrac{\mu\left(\frac{r}{q}\right) G(t,z)\, E(r,z)}{W[E,G]} & , \quad r = tq^k \end{cases}$$

is the Green function of (1.1)-(1.3) for $r \neq q^{m_0}$.

Proof. It is necessary to solve the equation

$$qa(t)\, y(qt,z) + b(t)\, y(t,z) + a\left(\frac{t}{q}\right) y\left(\frac{t}{q}, z\right) - \lambda y(t,z) = h(t,z) \tag{4.5}$$

to get the Green function of QIBVP (1.1)-(1.3). Since $E(t,z)$ and $G(t,z)$ are fundamental solutions of QIBVP (1.1)-(1.3), we can write the general solution of $g = \{g(t,z)\}$ of (4.5) as

$$g(t,z) = m(t)\, E(t,z) + n(t)\, G(t,z), \tag{4.6}$$

where $m(t), n(t)$ are coefficients and are different from zero. Using the method of variation of parameters, we obtain $m(t)$ and $n(t)$ by

$$m(t) = -\sum_{r \in q^{\mathbb{N}}} \frac{h(r,z)\, G(r,z)\, \mu\left(\frac{r}{q}\right)}{W[E,G]}, \quad r \neq q^{m_0} \tag{4.7}$$

$$n(t) = -\sum_{r \in [qt,\infty)q^{\mathbb{N}}} \frac{h(r,z)\, E(r,z)\, \mu\left(\frac{r}{q}\right)}{W[E,G]}, \quad r \neq q^{m_0}. \tag{4.8}$$

It follows from (4.6), (4.7) and (4.8) that the Green function of (1.1)-(1.3) is $R_{t,z}(z)$ given in Theorem 4.1 and it is easy to write the resolvent operator of QIBVP (1.1)-(1.3) given in Theorem 4.1 by using this Green function. □

Now, we can define the discrete spectrum, i.e., the set of eigenvalues of QIBVP (1.1)-(1.3) by using Theorem 4.1 and the definition of eigenvalues [36]. If we denote the set of eigenvalues of (1.1)-(1.3) by σ_d, we can write

$$\sigma_d := \{\lambda \in \mathbb{C} : \lambda = 2\sqrt{q}\cos z, z \in D_0, d(z) = 0\}.$$

Theorem 4.2. *Under the condition (1.4), $d(z)$ satisfies the following asymptotic equation for $z \in D$*

$$d(z) = e^{4iz}\left[A + o(1)\right], \quad |z| \to \infty, \quad A \neq 0.$$

Proof. As we know, $P(t,z)$ is polynomial of degree $(n-1)$., and $Q(t,z)$ is polynomial of degree $(n-2)$. with respect to λ. By considering this and using (1.2), (2.1), (2.10), we find

$$\lim_{|z|\to\infty} \left\{ \psi(t,z)\, e^{i\frac{\ln t}{\ln q} z} \right\} = -\frac{\nu_1}{q^{n-\frac{3}{2}} a(q)\, a(q^2) \ldots a(q^{n-1})}, \quad t \in q^{\mathbb{N}_0} \tag{4.9}$$

and

$$\lim_{|z|\to\infty}\left\{e(t,z)\,e^{-i\frac{\ln t}{\ln q}z}\right\}=\frac{\rho(t)}{\sqrt{\mu(t)}},\quad t\in q^{\mathbb{N}_0}\tag{4.10}$$

where

$$\rho(t):=\prod_{s\in[t,\infty)\cap q^{\mathbb{N}}}^{\infty}[a(s)]^{-1}.$$

It follows from (2.10), (4.9) and (4.10) that

$$d(z)=\frac{(q-1)\,q^{m_0+\frac{3}{2}}a\left(q^{m_0+1}\right)}{2i\sin z}\left\{\begin{array}{l}\gamma_1\psi\left(q^{m_0-1},z\right)e^{i\frac{\ln q^{m_0-1}}{\ln q}z}e\left(q^{m_0+2},z\right)e^{-i\frac{\ln q^{m_0+2}}{\ln q}z}e^{3iz}\\ -\gamma_2\psi\left(q^{m_0-2},z\right)e^{i\frac{\ln q^{m_0-2}}{\ln q}z}e\left(q^{m_0+1},z\right)e^{-i\frac{\ln q^{m_0+1}}{\ln q}z}e^{3iz}\end{array}\right\}$$

and if we write last equation in limit form, we find

$$\lim_{|z|\to\infty}\left\{d(z)\,e^{-4iz}\right\}=-\frac{A_1\gamma_1\nu_1\rho\left(q^{m_0+2}\right)}{q^{m_0-\frac{5}{2}}a(q)\,a(q^2)\ldots a\left(q^{m_0-2}\right)\sqrt{\mu\left(q^{m_0+2}\right)}}\lim_{|z|\to\infty}\frac{1}{e^{2iz}-1}$$

$$+\frac{A_1\gamma_2\nu_1\rho\left(q^{m_0+1}\right)}{q^{m_0-\frac{7}{2}}a(q)\,a(q^2)\ldots a\left(q^{m_0-3}\right)\sqrt{\mu\left(q^{m_0+1}\right)}}\lim_{|z|\to\infty}\frac{1}{e^{2iz}-1},$$

where $A_1:=(q-1)\,q^{m_0+\frac{3}{2}}a\left(q^{m_0+1}\right)$. Last equation gives

$$\lim_{|z|\to\infty}\left\{d(z)\,e^{-4iz}\right\}=-A\lim_{|z|\to\infty}\frac{1}{e^{2iz}-1},\tag{4.11}$$

where

$$A:=\frac{(q-1)\,q^5\gamma_1\nu_1a\left(q^{m_0+1}\right)\rho\left(q^{m_0+1}\right)}{\sqrt{\mu\left(q^{m_0+1}\right)}a(q)\,a(q^2)\ldots a\left(q^{m_0-3}\right)}\left\{\frac{a\left(q^{m_0+1}\right)}{q^{\frac{3}{2}}a\left(q^{m_0-2}\right)}-\frac{\gamma_2}{\gamma_1}\right\}.$$

(4.11) gives us, $\lim_{|z|\to\infty}\left\{d(z)\,e^{-4iz}\right\}=A$ for all $z\in D$ and it completes the proof of Theorem 4.2. □

We can say that the set of eigenvalues of QIBVP (1.1)-(1.3) is bounded under the assumption (1.4) by using Theorem 4.2. If we denote the continuous spectrum of QIBVP (1.1)-(1.3) by σ_c, we present the following theorem.

Theorem 4.3. *Assume (1.4). Then the continuous spectrum of the operator L generated by QIBVP (1.1)-(1.3) is $[-2\sqrt{q},2\sqrt{q}]$, i.e., $\sigma_c(L)=[-2\sqrt{q},2\sqrt{q}]$.*

Proof. Let L_1 and L_2 denote q-difference operators generated in $\ell_2\left(q^{\mathbb{N}}\right)$ by the following q-difference expressions

$$(\ell_1y):=y\left(\frac{t}{q},z\right)+qy(qt,z),\quad t\neq q^{m_0-1},q^{m_0+1}$$

and for $t\neq q^{m_0-1},q^{m_0},q^{m_0+1}$

$$(\ell_2y):=\left(a\left(\frac{t}{q}\right)-1\right)y\left(\frac{t}{q},z\right)+b(t)\,y(t,z)+q\,(a(t)-1)\,y(qt,z),$$

respectively, with the boundary condition (1.2). It is evident that $L=L_1+L_2$ and L_2 is a compact operator in $\ell_2\left(q^{\mathbb{N}}\right)$ under the assumption (1.4) (see [33]). We can also write the operator L_1 by the sum of two operators L_3 and L_4, i.e; $L_1=L_3+L_4$, where L_3 is a self-adjoint operator with $\sigma_c(L_3)=[-2\sqrt{q},2\sqrt{q}]$ and defined by the $q-$difference expression ℓ_1 and the boundary condition $y(0) = 0$. On the other hand, L_4 is a finite dimensional operator in $\ell_2\left(q^{\mathbb{N}}\right)$. Since L_4 is a finite dimensional operator in $\ell_2\left(q^{\mathbb{N}}\right)$, it is also a compact operator. It gives that the sum of two compact operators $L_2 + L_4$ is a compact operator, too. It follows from that $L = L_3 + L_4 + L_2$ and by using the Weyl Theorem (see [27]) of a compact perturbation, we get

$$\sigma_c\left(L_3\right) = \sigma_c\left(L\right) = \left[-2\sqrt{q}, 2\sqrt{q}\right].$$

□

Note that from the definition of eigenvalues of QIBVP (1.1)-(1.3), we write

$$\sigma_d \subset (-\infty, -2\sqrt{q}) \cup (2\sqrt{q}, \infty).$$

Theorem 4.4. *Assume (1.4). Then the operator L has a finite number of real eigenvalues.*

Proof. As we know, $\{a\left(t\right)\}_{t\in q^{\mathbb{N}_0}}$ and $\{b\left(t\right)\}_{t\in q^{\mathbb{N}}}$ are real sequences. It follows from that the operator L is selfadjoint. Since the operator L is selfadjoint, its eigenvalues are real. To complete the proof of Theorem 4.4, we have to show that the function d has finitely many zeros in D_0. By using formula $\sigma_d \subset (-\infty, -2\sqrt{q}) \cup (2\sqrt{q}, \infty)$ of (1.1)-(1.3), we obtain that the limit points of the set of all eigenvalues of (1.1)-(1.3) or of L could not be different from $\mp 2\sqrt{q}$, $\mp\infty$. Since $\lambda = 2\sqrt{q}\cos z$, the limit points of the set of all eigenvalues of L could not be $\mp\infty$ for $z \in D_0$. On the other hand for $z = 0$ and $z = \pi$, the limit points of the set of all eigenvalues could be $\mp 2\sqrt{q}$. But from operator theory and Theorem 4.3, the eigenvalues of selfadjoint operators are not the elements of its continuous spectrum. Because of this reason, we also cannot consider $z = 0$ and $z = \pi$ as zeros of the function d, i.e., the set of all eigenvalues of the operator L has not any limit points. This result with Bolzano-Weierstrass Theorem gives that the set of zeros of the function d in D_0 is finite. □

5. Example

We conclude the paper by defining an unperturbed q-difference equation with impulsive and boundary condition, as a special case of (1.1)-(1.3). This special case introduces our example and it illustrates our theoretical findings. We will discuss our main results on this example. Let us consider the q-discrete unperturbed impulsive problem

$$\begin{aligned} qy\left(qt,z\right) + y\left(\frac{t}{q},z\right) &= 2\sqrt{q}\cos z y\left(t,z\right), \quad t \in q^{\mathbb{N}} \backslash \left\{q^2, q^3, q^4\right\} \\ \left(\xi_0 + \xi_1\lambda\right) y\left(q,z\right) + \left(\nu_0 + \nu_1\lambda\right) y\left(1,z\right) &= 0 \\ y\left(q^4,z\right) &= \gamma_1 y\left(q^2,z\right) \\ y\left(q^5,z\right) &= \gamma_2 y\left(q,z\right), \end{aligned} \tag{5.1}$$

where $\xi_0, \xi_1, \nu_0, \nu_1, \gamma_1, \gamma_2 \in \mathbb{R}$ and $\gamma_1\gamma_2 \neq 0$. It is evident that in problem (1.1)-(1.3), we suppose $a\left(t\right) \equiv 1$, $b\left(t\right) \equiv 0$ for all $t \in q^{\mathbb{N}_0}$ and $m_0 = 3$ for the problem (5.1). Then the solution $e\left(t,z\right)$ turns into

$$e\left(t,z\right) = \frac{e^{i\frac{\ln t}{\ln q}z}}{\sqrt{\mu\left(t\right)}}$$

and the fundamental solutions $P\left(t,z\right)$ and $Q\left(t,z\right)$ of (1.1)-(1.3) have the following values for $t = 1, q, q^2$

$$P\left(1,z\right) = 0 \qquad P\left(q,z\right) = 1 \qquad P\left(q^2,z\right) = \frac{\lambda}{q}$$

$$Q(1,z)=\frac{1}{a(1)} \qquad Q(q,z)=0 \qquad Q(q^2,z)=-\frac{1}{qa(1)}.$$

Thus by using (2.4) and (2.10), we find $d(z)$ and Jost solution of this problem

$$d(z)=\frac{(q-1)\,q^{\frac{9}{2}}a(q^4)}{2i\sin z}\left\{\gamma_1\psi\left(q^2,z\right)e\left(q^5,z\right)-\gamma_2\psi(q,z)\,e\left(q^4,z\right)\right\} \tag{5.2}$$

$$E(t,z)=\begin{cases}\alpha(z)\,P(t,z)+\beta(z)\,Q(t,z) & , \quad t\in\{1,q,q^2\}\\[2ex] \dfrac{e^{i\frac{\ln t}{\ln q}z}}{\sqrt{\mu(t)}} & , \quad t\in\{q^4,q^5,\ldots\},\end{cases}$$

respectively. By using equation (5.2), we find the scattering function of (5.1)

$$S(z)=e^{-10iz}\left\{\frac{\gamma_1\psi(q^2,z)\sqrt{q}-\gamma_2\psi(q,z)\,e^{iz}}{\gamma_1\psi(q^2,z)\sqrt{q}-\gamma_2\psi(q,z)\,e^{-iz}}\right\}.$$

Moreover, continuous spectrum of the problem (5.1) is $\left[-2\sqrt{q},2\sqrt{q}\right]$ from Theorem 4.3. To get the eigenvalues of the problem (5.1), it is necessary to find the zeros of $d(z)$ for $z\in D_0$. Because from the definition of eigenvalues, we write

$$\sigma_d=\{\lambda=2\sqrt{q}\cos z : d(z)=0, z\in D_0\} \tag{5.3}$$

for this problem, where $d(z)$ is defined by (5.2). From the values of $P(t,z)$, $Q(t,z)$ for $t=q,q^2$, we obtain

$$\psi(q,z)=-(\nu_0+\lambda\nu_1)$$

$$\psi\left(q^2,z\right)=-\frac{(\nu_0+\lambda\nu_1)\,\lambda}{q}-\frac{(\xi_0+\lambda\xi_1)}{q}.$$

It follows from that last equation and (5.2) that

$$d(z)=\frac{(q-1)\,q^{\frac{9}{2}}a(q^4)}{2i\sin z}\begin{bmatrix}\gamma_1\left\{-\dfrac{(\nu_0+\lambda\nu_1)\,\lambda}{q}-\dfrac{(\xi_0+\lambda\xi_1)}{q}\right\}\dfrac{e^{5iz}}{\sqrt{\mu(q^5)}}\\[3ex] -\gamma_2\left\{-(\nu_0+\lambda\nu_1)\right\}\dfrac{e^{4iz}}{\sqrt{\mu(q^4)}}\end{bmatrix}. \tag{5.4}$$

Equation (5.4) implies that $d(z)=0$ if and only if

$$\frac{\gamma_2}{\gamma_1}=\frac{\lambda e^{iz}}{q^{3/2}}+\frac{(\xi_0+\lambda\xi_1)}{\nu_0+\lambda\nu_1}\frac{e^{iz}}{q^{3/2}}. \tag{5.5}$$

For the simplicity on calculations, if we choose $\xi_1=\nu_0=1$ and $\xi_0=\nu_1=0$ in (5.5), we find

$$e^{2iz}=\frac{q\gamma_2}{2\gamma_1}-1.$$

Let $\gamma_2=(2/q)B\gamma_1$, $B\in\mathbb{R}$. By using last equation, we get $e^{2iz}=B-1$. It gives us

$$2iz_k=\ln|B-1|+iArg(B-1)+2ik\pi,\ \ k\in\mathbb{Z}$$

i.e.,

$$z_k=-\frac{i}{2}\ln|B-1|+\frac{1}{2}Arg(B-1)+k\pi,\ \ k\in\mathbb{Z}. \tag{5.6}$$

It is clear from (5.3) and (5.6) that the boundary value problem (5.1) has eigenvalues if and only if $\ln|B-1|<0$. It implies that $-1<B-1<1$. Consequently, the necessary condition for the QIBVP (5.1) to have an eigenvalue is that $0<B<2$. These eigenvalues are real and lie on $(-\infty,-2\sqrt{q})\cup(2\sqrt{q},\infty)$. Also, for $k=0$, we obtain $\lambda_0=2\sqrt{q}$, for

$k = 1$, we obtain $\lambda_1 = -2\sqrt{q}$. Since $\lambda = \mp 2\sqrt{q}$ are continuous spectrum, they are not an eigenvalue of (5.1).

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