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Differentiable functions in a three-dimensional associative noncommutative algebra

Kuzmenko Tetiana[a], Shpakivskyi Vitalii[b]

[a] Department of Fundamental Scinces, Zhytomyr Millitary Institute, Zhytomyr, Ukraine.

[b] Department of Complex analysis and potential theory, Institute of Mathematics of the NAS of Ukraine, Kyiv, Ukraine.

Abstract

We consider a three-dimensional associative noncommutative algebra $\widetilde{\mathbb{A}}_2$ over the field $\mathbb{C}$ with the basis $\{I_1, I_2, \rho\}$, where I_1, I_2 are idempotents and ρ is nilpotent. The algebra $\widetilde{\mathbb{A}}_2$ contains the algebra of bicomplex numbers $\mathbb{B}(\mathbb{C})$ as a subalgebra. In this paper we consider functions of the form $\Phi(\zeta) = f_1(\xi_1, \xi_2, \xi_3)I_1 + f_2(\xi_1, \xi_2, \xi_3)I_2 + f_3(\xi_1, \xi_2, \xi_3)\rho$ of the variable $\zeta = \xi_1 I_1 + \xi_2 I_2 + \xi_3 \rho$, where ξ_1, ξ_2, ξ_3 are independent complex variables and f_1, f_2, f_3 are holomorphic functions of three complex variables. We construct in an explicit form all functions defined by equalities $d\Phi = d\zeta \cdot \Phi'(\zeta)$ or $d\Phi = \Phi'(\zeta) \cdot d\zeta$. The obtained descriptions we apply to representation of the mentioned class of functions by series. Also we established integral representations of these functions.

Keywords: noncommutative algebra differentiable function Cauchy-Riemann conditions constructive description power series integral representation.
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1. Introduction

The natural task is a generalization of the concept of differentiability and derivative for a function with values in noncommutative algebras. Probably the first attempt to solve this problem was made in a quaternion algebra in the paper [1]. For the quaternionic functions f of the quaternionic variable x the left and right derivative were defined, respectively, by equalities

$$f'_l(x) := \lim_{\Delta x \to 0} (\Delta x)^{-1} \Delta f, \qquad f'_r(x) := \lim_{\Delta x \to 0} \Delta f (\Delta x)^{-1}, \tag{1}$$

Email addresses: kuzmenko.ts15@gmail.com (Kuzmenko Tetiana), shpakivskyi86@gmail.com (Shpakivskyi Vitalii)

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and necessary and sufficient conditions (analogues of the Cauchy–Riemann conditions) of the existence of limits (1) were obtained. In the paper [2] the mentioned analogues of the Cauchy–Riemann conditions were integrated and an explicit form of functions, such that have limits (1), were obtained. Such functions are, respectively, of the form $f(x) = xa + b$ and $f(x) = ax + b$, where a, b are quaternions. This result motivated mathematicians to search anothers definitions of the quaternionic derivative and quaternionic analytic functions (see paper [3]).

Usually, classes of functions satisfying the right (or the left) Dirac equation

$$D_r[f] = \sum_{k=0}^{n} \frac{\partial f}{\partial x_k} e_k = 0 \qquad \left(\text{or} \quad D_l[f] = \sum_{k=0}^{n} e_k \frac{\partial f}{\partial x_k} = 0\right)$$

are considered in noncommutative algebras. The function theory for mentioned classes of functions is constructed in the algebra of quaternions (see monograph [4]), in Clifford algebras [5] and in the other papers. Many generalizations are obtained in this direction. For example, fractional Dirac operator on Clifford algebras are investigated in the papers [6, 7]. In the cited papers showed that the fractional action-like variational approach constructed to model weak dissipative dynamical systems may have interesting features when applied on classical field theory.

In this paper we consider another class of differentiable functions in the special three-dimensional noncommutative algebra. It is known (see, e.g., [8]) that there exists only one three-dimensional associative noncommutative algebra over the field $\mathbb{C}$. It is the algebra $\widetilde{\mathbb{A}}_2$ with the basis $\{I_1, I_2, \rho\}$, whose elements satisfy the following multiplication rules:

$\cdot$	I_1	I_2	ρ
I_1	I_1	0	0
I_2	0	I_2	ρ
ρ	ρ	0	0

. (2)

Note that the subalgebra with the basis $\{I_1, I_2\}$ is the algebra of bicomplex numbers $\mathbb{B}(\mathbb{C})$ or Segre's algebra of commutative quaternions (see, e.g., [9, 10]).

In the paper [11] locally bounded and Gâteaux-differentiable mappings, defined in the domains of the three-dimensional subspace of the algebra $\mathbb{B}(\mathbb{C})$ and taking values in the algebra $\widetilde{\mathbb{A}}_2$, are considered. Such mappings are described by means of three holomorphic functions of a complex variable.

In paper [12] an algebra with the multiplication table (2) is considered over the field of real numbers $\mathbb{R}$, and the functions $f(x) = f_1(x_1, x_2, x_3)I_1 + f_2(x_1, x_2, x_3)I_2 + f_3(x_1, x_2, x_3)\rho$ of the variable $x = x_1I_1 + x_2I_2 + x_3\rho$ with real x_1, x_2, x_3 and differentiable components f_1, f_2, f_3 are considered in that algebra. All functions with values in the mentioned algebra, for which exists a limit (1), are explicitly described.

In this paper we consider the algebra $\widetilde{\mathbb{A}}_2$ over the field of complex numbers $\mathbb{C}$ and functions of the form

$$\Phi(\zeta) = f_1(\xi_1, \xi_2, \xi_3)I_1 + f_2(\xi_1, \xi_2, \xi_3)I_2 + f_3(\xi_1, \xi_2, \xi_3)\rho \tag{3}$$

of the variable $\zeta = \xi_1 I_1 + \xi_2 I_2 + \xi_3 \rho$, where ξ_1, ξ_2, ξ_3 are independent complex variables and f_1, f_2, f_3 are holomorphic functions of three complex variables. The right- and left-derivatives of the function Φ are defined by equalities

$$d\Phi = d\zeta \cdot \Phi'(\zeta) \tag{4}$$

and

$$d\Phi = \Phi'(\zeta) \cdot d\zeta, \tag{5}$$

respectively. All right- and left-differentiable functions are explicitly constructed. The obtained descriptions applied to the representations of mentioned classes of functions by series. Integral representations of these functions are established too.

2. Right– and left–differentiable functions

Let Ω be a domain in the algebra $\widetilde{\mathbb{A}}_2$. Further a domain $\Omega = \{\zeta = \xi_1 I_1 + \xi_2 I_2 + \xi_3\rho \in \widetilde{\mathbb{A}}_2\}$ we will be identify with the congruent domain in $\mathbb{C}^3$.

A function (3), where $f_k(\xi_1, \xi_2, \xi_3)$, $k = 1, 2, 3$, are holomorphic functions of three complex variables, is called *right–differentiable* (respectively, *left–differentiable*) in a domain Ω, if at every point $\zeta \in \Omega$ there exists the element $\Phi'(\zeta)$ of the algebra $\widetilde{\mathbb{A}}_2$ such that the equality (4) (respectively, (5)) is fulfilled.

Theorem 2.1. *A function $\Phi : \Omega \to \widetilde{\mathbb{A}}_2$ of the form* (3)*, where $f_k : \Omega \to \mathbb{C}^3$ are holomorphic functions, is right–differentiable in the domain $\Omega \subset \widetilde{\mathbb{A}}_2$ if and only if the conditions*

$$\frac{\partial f_1}{\partial \xi_2} = \frac{\partial f_1}{\partial \xi_3} = 0, \qquad \frac{\partial f_2}{\partial \xi_1} = \frac{\partial f_2}{\partial \xi_3} = 0,$$
$$\frac{\partial f_3}{\partial \xi_1} = 0, \qquad \frac{\partial f_1}{\partial \xi_1} = \frac{\partial f_3}{\partial \xi_3} \tag{6}$$

are satisfied.

Proof. Necessity. By virtue of the right–differentiability of the function (3), the following equality is true:

$$d\Phi = df_1 I_1 + df_2 I_2 + df_3\rho =$$
$$= (d\xi_1 I_1 + d\xi_2 I_2 + d\xi_3\rho)\,(AI_1 + BI_2 + C\rho) =$$
$$= Ad\xi_1 I_1 + Bd\xi_2 I_2 + (Cd\xi_2 + Ad\xi_3)\rho,$$

where by $AI_1 + BI_2 + C\rho$ we denote the right-derivative $\Phi'(\zeta)$.

Then

$$df_1 = Ad\xi_1, \qquad df_2 = Bd\xi_2, \qquad df_3 = Cd\xi_2 + Ad\xi_3.$$

Since the functions f_1, f_2, f_3 are holomorphic, then

$$A = \frac{\partial f_1}{\partial \xi_1}, \quad B = \frac{\partial f_2}{\partial \xi_2}, \quad C = \frac{\partial f_3}{\partial \xi_2}, \quad A = \frac{\partial f_3}{\partial \xi_3}.$$

Taking into account that

$$df_1 = \frac{\partial f_1}{\partial \xi_1}\,d\xi_1 + \frac{\partial f_1}{\partial \xi_2}\,d\xi_2 + \frac{\partial f_1}{\partial \xi_3}\,d\xi_3 = Ad\xi_1,$$

we have

$$\frac{\partial f_1}{\partial \xi_2} = \frac{\partial f_1}{\partial \xi_3} = 0.$$

In the same way we can show that

$$df_2 = \frac{\partial f_2}{\partial \xi_1}\,d\xi_1 + \frac{\partial f_2}{\partial \xi_2}\,d\xi_2 + \frac{\partial f_2}{\partial \xi_3}\,d\xi_3 = Bd\xi_2,$$

so

$$\frac{\partial f_2}{\partial \xi_1} = \frac{\partial f_2}{\partial \xi_3} = 0$$

and

$$df_3 = \frac{\partial f_3}{\partial \xi_1}\,d\xi_1 + \frac{\partial f_3}{\partial \xi_2}\,d\xi_2 + \frac{\partial f_3}{\partial \xi_3}\,d\xi_3 = Cd\xi_2 + Ad\xi_3,$$

therefore

$$\frac{\partial f_3}{\partial \xi_1} = 0, \qquad \frac{\partial f_1}{\partial \xi_1} = \frac{\partial f_3}{\partial \xi_3}.$$

Sufficiency. Using holomorphicity of functions f_1, f_2, f_3, conditions (6), and the multiplication table of the algebra $\widetilde{\mathbb{A}}_2$, we have

$$d\Phi = df_1 I_1 + df_2 I_2 + df_3\rho = \left(\frac{\partial f_1}{\partial \xi_1}\, d\xi_1 + \frac{\partial f_1}{\partial \xi_2}\, d\xi_2 + \frac{\partial f_1}{\partial \xi_3}\, d\xi_3\right) I_1 +$$

$$+\left(\frac{\partial f_2}{\partial \xi_1}\, d\xi_1 + \frac{\partial f_2}{\partial \xi_2}\, d\xi_2 + \frac{\partial f_2}{\partial \xi_3}\, d\xi_3\right) I_2 + \left(\frac{\partial f_3}{\partial \xi_1}\, d\xi_1 + \frac{\partial f_3}{\partial \xi_2}\, d\xi_2 + \frac{\partial f_3}{\partial \xi_3}\, d\xi_3\right)\rho =$$

$$= \frac{\partial f_1}{\partial \xi_1}\, d\xi_1\, I_1 + \frac{\partial f_2}{\partial \xi_2}\, d\xi_2\, I_2 + \left(\frac{\partial f_3}{\partial \xi_2}\, d\xi_2 + \frac{\partial f_1}{\partial \xi_1}\, d\xi_3\right)\rho =$$

$$= (d\xi_1 I_1 + d\xi_2 I_2 + d\xi_3\rho)\left(\frac{\partial f_1}{\partial \xi_1}\, I_1 + \frac{\partial f_2}{\partial \xi_2}\, I_2 + \frac{\partial f_3}{\partial \xi_2}\, \rho\right) =$$

$$= d\zeta \cdot \left(\frac{\partial f_1}{\partial \xi_1}\, I_1 + \frac{\partial f_2}{\partial \xi_2}\, I_2 + \frac{\partial f_3}{\partial \xi_2}\, \rho\right),$$

that means the right–differentiability of the function Φ. □

The following theorem can be proved similarly to the proof of Theorem 2.1.

Theorem 2.2. *A function $\Phi : \Omega \to \widetilde{\mathbb{A}}_2$ of the form* (3)*, where $f_k : \Omega \to \mathbb{C}^3$ are holomorphic functions, is left–differentiable in the domain $\Omega \subset \widetilde{\mathbb{A}}_2$ if and only if the conditions*

$$\frac{\partial f_1}{\partial \xi_2} = \frac{\partial f_1}{\partial \xi_3} = 0, \qquad \frac{\partial f_2}{\partial \xi_1} = \frac{\partial f_2}{\partial \xi_3} = 0,$$
$$\frac{\partial f_3}{\partial \xi_2} = 0, \quad \frac{\partial f_2}{\partial \xi_2} = \frac{\partial f_3}{\partial \xi_3} \tag{7}$$

are satisfied.

3. Constructive description of differentiable functions

The following conditions are consequence of equalities (6) and holomorphicity of the functions f_1, f_2, f_3:

$$f_1(\xi_1, \xi_2, \xi_3) = f_1(\xi_1), \quad f_2(\xi_1, \xi_2, \xi_3) = f_2(\xi_2), \quad f_3(\xi_1, \xi_2, \xi_3) = f_3(\xi_2, \xi_3).$$

By virtue the last equality of conditions (6) we have:

$$f_3(\xi_2, \xi_3) = \xi_3\, \frac{\partial f_1}{\partial \xi_1} + \varphi(\xi_2) = \xi_3 f_1'(\xi_1) + \varphi(\xi_2).$$

Since $\frac{\partial f_3}{\partial \xi_1} = 0$, then $f_1(\xi_1) = \alpha\xi_1 + \beta$, where α, β are arbitrary complex numbers.

Therefore, every right-differentiable function can be expressed in the form

$$\Phi(\zeta) = (\alpha\xi_1 + \beta)\, I_1 + f_2(\xi_2) I_2 + (f_3(\xi_2) + \alpha\xi_3)\,\rho, \quad \alpha, \beta \in \mathbb{C}. \tag{8}$$

Herewith

$$\Phi'(\zeta) = \alpha I_1 + f_2'(\xi_2) I_2 + f_3'(\xi_2)\rho.$$

Similarly, every left-differentiable function can be expressed in the form

$$\Psi(\zeta) = f_1(\xi_1) I_1 + (\alpha\xi_2 + \beta)\, I_2 + (f_3(\xi_1) + \alpha\xi_3)\,\rho, \quad \alpha, \beta \in \mathbb{C}, \tag{9}$$

and

$$\Psi'(\zeta) = f_1'(\xi_1)I_1 + \alpha I_2 + f_3'(\xi_1)\rho.$$

Let

$$\Omega_1 := \{\xi_1 \in \mathbb{C} \text{ such that } \exists(\xi_2, \xi_3) \in \mathbb{C}^2 : (\xi_1, \xi_2, \xi_3) \in \Omega\};$$
$$\Omega_2 := \{\xi_2 \in \mathbb{C} \text{ such that } \exists(\xi_1, \xi_3) \in \mathbb{C}^2 : (\xi_1, \xi_2, \xi_3) \in \Omega\}.$$

The following statement follows immediately from the equalities (8) and (9), due to their right-hand side are right– and left–differentiable functions in the domain

$$\Pi_2 := \{\zeta = \xi_1 I_1 + \xi_2 I_2 + \xi_3\rho \in \widetilde{\mathbb{A}}_2 : \xi_2 \in \Omega_2\} \equiv \mathbb{C} \times \Omega_2 \times \mathbb{C}, \tag{10}$$

$$\Pi_1 := \{\zeta = \xi_1 I_1 + \xi_2 I_2 + \xi_3\rho \in \widetilde{\mathbb{A}}_2 : \xi_1 \in \Omega_1\} \equiv \Omega_1 \times \mathbb{C} \times \mathbb{C}. \tag{11}$$

respectively.

Theorem 3.1. *Every right–differentiable function $\Phi : \Omega \to \widetilde{\mathbb{A}}_2$ can be extended to the right–differentiable function in the domain Π_2.*

Theorem 3.2. *Every left–differentiable function $\Psi : \Omega \to \widetilde{\mathbb{A}}_2$ can be extended to the left–differentiable function in the domain Π_1.*

It is easy to see that, in general, the product of two right–differentiable functions is not right–differentiable function, and the product of two left–differentiable functions is not left–differentiable function.

In view of Theorems 3.1 and 3.2 further we will be consider right– and left–differentiable functions defined, respectively, in the domains Π_k , $k = 1, 2$, of the form (10) and (11).

4. Application of constructive description to representation of differentiable functions by series

Using the representation (8) of right–differentiable function $\Phi : \Pi_2 \to \widetilde{\mathbb{A}}_2$ by means of holomorphic functions of a complex variable, we show that the function Φ in the domain $B_\varepsilon(\zeta_0) := \mathbb{C} \times \{\xi_2 \in \mathbb{C} : |\xi_2 - \xi_{20}| < \varepsilon\} \times \mathbb{C}$ is expressed into the power series, where $\zeta_0 := \xi_{10}I_1 + \xi_{20}I_2 + \xi_{30}\rho$.

Theorem 4.1. *Let a function $\Phi : \Pi_2 \to \widetilde{\mathbb{A}}_2$ is right–differentiable in a domain $\Pi_2 = \mathbb{C} \times D_2 \times \mathbb{C}$. Then at every point $\zeta = \xi_1 I_1 + \xi_2 I_2 + \xi_3\rho$ of the domain $B_\varepsilon(\zeta_0) \subset \Pi_2$ the function Φ is expressed as the sum of the absolutely convergent power series*

$$\Phi(\zeta) = \sum_{n=0}^{\infty} (\zeta - \zeta_0)^n \, p_n \,, \tag{12}$$

where

$$\begin{gathered} p_0 = (\alpha\zeta_0 + \beta)I_1 + b_0 I_2 + c_0\rho, \qquad p_1 = \alpha I_1 + b_1 I_2 + c_1\rho, \\ p_k = b_k I_2 + c_k\rho, \qquad k = 2, 3, \ldots, \end{gathered} \tag{13}$$

and b_n, c_n are coefficients of the Taylor series of the functions f_2, f_3 from the equality (8) for $\zeta \in \Pi_2$:

$$f_2(\xi_2) = \sum_{n=0}^{\infty} b_n(\xi_2 - \xi_{20})^n, \qquad f_3(\xi_2) = \sum_{n=0}^{\infty} c_n(\xi_2 - \xi_{20})^n. \tag{14}$$

Proof. Since in the equality (8) the functions $\alpha\xi_1 + \beta$ and $\alpha\xi_3$ are holomorphic in the whole complex plane, then their series are absolutely convergent in the whole complex plane, and since the functions f_2, f_3 are holomorphic in the domain $\{\xi_2 \in \mathbb{C} : |\xi_2 - \xi_{20}| < \varepsilon\}$, then the series (14) are absolutely convergent in the corresponding domain. Then the equality (8) takes the form

$$\Phi(\zeta) = (\alpha\xi_1 + \beta)I_1 + \sum_{n=0}^{\infty} b_n(\xi_2 - \xi_{20})^n I_2 + \sum_{n=0}^{\infty} c_n(\xi_2 - \xi_{20})^n \rho + \alpha\xi_3\rho =$$

$$= \alpha(\xi_1 I_1 + \xi_3\rho) + \beta I_1 + \sum_{n=0}^{\infty} b_n(\xi_2 - \xi_{20})^n I_2 + \sum_{n=0}^{\infty} c_n(\xi_2 - \xi_{20})^n \rho$$

for all $\zeta \in \Pi_2$. Now, taking into account the relation

$$(\zeta - \zeta_0)^n I_2 = (\xi_2 - \xi_{20})^n I_2, \quad \zeta I_1 = \xi_1 I_1 + \xi_3\rho$$

for all $\zeta \in \widetilde{\mathbb{A}}_2$ and $n = 0, 1, \ldots,$ we have

$$\Phi(\zeta) = \alpha\zeta I_1 + \beta I_1 + \sum_{n=0}^{\infty} b_n(\zeta - \zeta_0)^n I_2 + \sum_{n=0}^{\infty} c_n(\zeta - \zeta_0)^n \rho =$$

$$= (\alpha\zeta_0 + \beta)I_1 + \alpha(\zeta - \zeta_0)I_1 + \sum_{n=0}^{\infty} b_n(\zeta - \zeta_0)^n I_2 + \sum_{n=0}^{\infty} c_n(\zeta - \zeta_0)^n \rho =$$

$$= \sum_{n=0}^{\infty}(\zeta - \zeta_0)^n p_n\,,$$

where coefficients are defined by the equality (13) and series (12) is absolutely convergent in the domain Π_2. □

A similar statement is true for the left–differentiable function $\Psi(\zeta)$ in the domain $Q_\varepsilon(\zeta_0) := \{\xi_1 \in \mathbb{C} : |\xi_1 - \xi_{10}| < \varepsilon\} \times \mathbb{C} \times \mathbb{C}$.

Theorem 4.2. *Let a function $\Psi : \Pi_1 \to \widetilde{\mathbb{A}}_2$ is left–differentiable in a domain $\Pi_1 = D_1 \times \mathbb{C} \times \mathbb{C}$. Then at every point $\zeta = \xi_1 I_1 + \xi_2 I_2 + \xi_3\rho$ of the domain $Q_\varepsilon(\zeta_0) \subset \Pi_1$ the function Ψ is expressed as the sum of the absolutely convergent power series*

$$\Psi(\zeta) = \sum_{n=0}^{\infty} q_n(\zeta - \zeta_0)^n, \tag{15}$$

where

$$q_0 = a_0 I_2 + (\alpha\zeta_0 + \beta)I_2 + c_0\rho, \qquad q_1 = a_1 I_1 + \alpha I_2 + c_1\rho,$$
$$q_k = a_k I_1 + c_k\rho, \qquad k = 2, 3, \ldots, \tag{16}$$

and a_n, c_n are coefficients of the Taylor series of the functions f_1, f_3 from the equality (9)for $\zeta \in \Pi_1$:

$$f_1(\xi_1) = \sum_{n=0}^{\infty} a_n(\xi_1 - \xi_{10})^n, \qquad f_3(\xi_1) = \sum_{n=0}^{\infty} c_n(\xi_1 - \xi_{10})^n. \tag{17}$$

Here we note the paper [18] in which a new Taylor–type series expansion based on the Appell polynomials is presented, and such series expansion is applied to the description of the hypercomplex derivative.

5. Application of constructive description to the construction of the integral representations of differentiable functions

Integral representations of the differentiable in some sense, functions are important for applications, especially for solving boundary value problems. For example, in the papers [13, 14] integral representation of monogenic functions in the two-dimensional biharmonic algebra were applied to solving boundary value problems for two-dimensional biharmonic equation. In the monograph [15] integral representations of axial-symmetric potential and Stokes flow functions were obtained and such representations were applied to the solution of Dirichlet boundary problem for the mentioned potentials. Furthermore, in the paper [16] integral representations of monogenic functions in a commutative algebra were applied to the construction of exact solutions of the linear homogeneous PDEs with constant coefficients.

In this section we establish integral representations of the right– and left–differentiable functions in the algebra $\widetilde{\mathbb{A}}_2$.

It is easy to show that the invertse element to the element $\zeta = \xi_1 I_1 + \xi_2 I_2 + \xi_3\rho$ is of the form

$$\zeta^{-1} = \frac{1}{\xi_1}\, I_1 + \frac{1}{\xi_2}\, I_2 - \frac{\xi_3}{\xi_1\xi_2}\, \rho, \tag{18}$$

where $\xi_1 \neq 0$ and $\xi_2 \neq 0$.

Let $\tau := t_1 I_1 + t_2 I_2$, where $t_1, t_2 \in \mathbb{C}$ and $d\tau := dt_1 dt_2$. The equality

$$(\tau - \zeta)^{-1} = \frac{1}{t_1 - \xi_1}\, I_1 + \frac{1}{t_2 - \xi_2}\, I_2 - \frac{\xi_3}{(t_1 - \xi_1)(t_2 - \xi_2)}\, \rho \tag{19}$$

follows from the equality (18).

Let D_k be an arbitrary domain on the plane t_k for $k = 1, 2$. In addition, let a function $\Phi : \Pi_2 \to \widetilde{\mathbb{A}}_2$ is right–differentiable in the domain $\Pi_2 = \mathbb{C} \times D_2 \times \mathbb{C}$ of the form (10), and a function $\Psi : \Pi_1 \to \widetilde{\mathbb{A}}_2$ is a left–differentiable in the domain $\Pi_1 = D_1 \times \mathbb{C} \times \mathbb{C}$ of the form (11). Denote by Γ_k an arbitrary closed Jordan piece-smooth curve in $\overline{D}_k$, which surround the points ξ_k, $k = 1, 2$.

Now we shall construct integral representation of right– and left–differentiable functions.

Theorem 5.1. *Every right–differentiable function $\Phi : \Pi_2 \to \widetilde{\mathbb{A}}_2$ in a domain $\Pi_2 = \mathbb{C} \times D_2 \times \mathbb{C}$ is expressed by formula*

$$\Phi(\zeta) = \frac{1}{(2\pi i)^2} \int\limits_{\Gamma_1\times\Gamma_2} \left(\frac{\alpha\, t_1 + \beta}{t_2 - \xi_2}\, I_1 - \alpha I_2 + \frac{f_3(t_2)}{t_2 - \xi_2}\, \rho \right) (\tau - \zeta)^{-1} d\tau +$$

$$+ \frac{1}{(2\pi i)^2} \int\limits_{\Gamma_1\times\Gamma_2} \frac{f_2(t_2)}{t_1 - \xi_1} (\tau - \zeta)^{-1} d\tau\, I_2\,, \tag{20}$$

where α, β are some complex numbers and f_2, f_3 are some holomorphic functions in the domain D_2.

Proof. Let us show that the representation (20) by equivalent transformations is reduced to representation (8). Using the equality (19) and the multiplication table of the algebra, the equality (20) takes the form

$$\Phi(\zeta) = \frac{1}{(2\pi i)^2} \int\limits_{\Gamma_1\times\Gamma_2} \frac{(\alpha\, t_1 + \beta) I_1}{(t_1 - \xi_1)(t_2 - \xi_2)}\, d\tau - \frac{1}{(2\pi i)^2} \int\limits_{\Gamma_1\times\Gamma_2} \frac{\alpha\, I_2}{t_2 - \xi_2}\, d\tau +$$

$$+ \frac{1}{(2\pi i)^2} \int\limits_{\Gamma_1\times\Gamma_2} \frac{\alpha\, \xi_3\, \rho}{(t_1 - \xi_1)(t_2 - \xi_2)}\, d\tau + \frac{1}{(2\pi i)^2} \int\limits_{\Gamma_1\times\Gamma_2} \frac{f_3(t_2)\, \rho}{(t_1 - \xi_1)(t_2 - \xi_2)}\, d\tau +$$

$$+ \frac{1}{(2\pi i)^2} \int\limits_{\Gamma_1\times\Gamma_2} \frac{f_2(t_2)\, I_2}{(t_1 - \xi_1)(t_2 - \xi_2)}\, d\tau =: \sum_{k=1}^{5} J_k\,.$$

Now, applying the Cauchy formula to the functions of several complex variables (see, e.g., [17, p.45]), we have

$$J_1 = (\alpha\, \xi_1 + \beta) I_1,\ J_2 = 0,\ J_3 = \alpha\, \xi_3 \rho,\ J_4 = f_3(\xi_2)\rho,\ J_5 = f_2(\xi_2) I_2\,.$$

Thus, the function Φ is of the form (8). □

The following theorem can be proved similarly.

Theorem 5.2. *Every left–differentiable function $\Psi : \Pi_1 \to \widetilde{\mathbb{A}}_2$ in a domain $\Pi_1 = D_1 \times \mathbb{C} \times \mathbb{C}$ is expressed by formula*

$$\Psi(\zeta) = \frac{1}{(2\pi i)^2} \int\limits_{\Gamma_1 \times \Gamma_2} \left(\frac{f_1(t_1)}{t_2 - \xi_2} I_1 - \alpha I_2 + \frac{f_3(t_1)}{t_2 - \xi_2} \rho \right) (\tau - \zeta)^{-1} d\tau +$$

$$+ \frac{1}{(2\pi i)^2} \int\limits_{\Gamma_1 \times \Gamma_2} \frac{\alpha t_2 + \beta}{t_1 - \xi_1} (\tau - \zeta)^{-1} d\tau \, I_2 \, ,$$

where α, β are some complex numbers and f_1, f_3 are some holomorphic functions in the domain D_1.

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