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Research Article

Existence of three solutions for Kirchhoff-type three-point boundary value problems

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Abstract

The present paper is an attempt to investigate the multiplicity results of solutions for a three-point boundary value problem of Kirchhoff-type. Indeed, we will use variational methods for smooth functionals, defined on the reflexive Banach spaces in order to achieve the existence of at least three solutions for the equation. Finally, by presenting one example, we will ensure the applicability of our results.

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1. Introduction

This paper attempts to study the existence of at least three weak solutions for the following three-point boundary value problem of Kirchhoff-type

$$\begin{cases} -K(\int_a^b |u'(t)|^2 \mathrm{d}t)u''(t) = \lambda f(t,u(t)) + \mu g(t,u(t)) + h(u(t)), \ t \in (a,b), \\ u(a)=0, \ u(b)=\alpha u(\eta) \end{cases} \qquad (K^{f,g}_{\lambda,\mu})$$

where $K : [0,+\infty[\to \mathbb{R}$ is a continuous function such that there exist two positive numbers m and M with $m \leq K(x) \leq M$ for all $x \geq 0$, $a,b \in \mathbb{R}$ with $a < b$, $\lambda > 0$, $\mu \geq 0$, $f,g : [a,b] \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions, $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|h(\xi_1) - h(\xi_2)| \leq L|\xi_1 - \xi_2|$$

for every $\xi_1, \xi_2 \in \mathbb{R}$ and $h(0) = 0$, $\alpha \in \mathbb{R}$ and $\eta \in (a,b)$.

The theory of multi-point boundary value problems for ordinary differential equations emerges in various areas of applied mathematics and physics, especially in heat conduction [5, 6, 26, 27], the vibration of cables with nonuniform weights [40], and other problems in nonlinear elasticity [47]. The study of multi-point BVPs for linear second-order ordinary differential equations began with Il'in and Moiseev [24]. Afterwards, many researchers studied more general nonlinear multi-point boundary value problems. In recent years, the existence and the multiplicity of solutions for nonlinear multi-point BVPs have been the

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center of attention. Readers may see [8, 10, 12–14, 17, 21–23] and the related references therein.

In recent years, much attention has been paid to the problem of finding solutions for three-point boundary value problems in nonlinear ordinary differential equations. Concerning the review of literature and the most recent findings, readers may see [1, 9, 15, 16, 25, 29, 31–33, 44, 46, 48, 49] and the references therein for more details. For example, Gupta in [15] considered the second-order three-point nonlinear boundary value problem

$$\begin{cases} u'' = f(x, u(x), u'(x)) - e(x), \quad 0 < x < 1, \\ u(0) = 0, \quad u(\eta) = u(1) \end{cases} \tag{1.1}$$

in the case $f(x, u, u') = p_0(x) + p_1(x)u + p_2(x)u'$ whit $p_k : [0, 1] \to \mathbb{R}$ locally integrable. He obtained existence and uniqueness theorems for the boundary value problem (1.1) under natural conditions on f using degree-theoretic arguments. Infante in [25] used the theory of fixed point index for weakly inward A-proper maps to establish the existence of positive solutions of some second-order three-point boundary value problems in which the highest-order derivative occurs nonlinearly. Xu in [48] by employing the fixed point index method, obtained some multiplicity results for positive solutions of some singular semi-positone three-point boundary value problem. Sun in [44] by using a fixed point theorem of cone expansion-compression type due to Krasnosel'skii, established various results on the existence of single and multiple positive solutions for the nonlinear singular third-order three-point boundary value problem

$$\begin{cases} u''(t) - \lambda a(t)F(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(\eta) = u''(1) = 0 \end{cases}$$

with $\lambda > 1$, $\eta \in [\frac{1}{2}, 1)$ where $a(t)$ is a non-negative continuous function defined on $(0, 1)$ and $F : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous. Du et al. in [9] based upon Leray-Schauder degree theory, ensured the existence of at least three solutions for the following problem

$$\begin{cases} u''(t) + \lambda f(t, u(t), u'(t)) = 0, \ 0 < t < 1 \\ u(0) = 0, \ u(1) = \xi u(\eta) \end{cases}$$

where $\xi > 0$, $0 < \eta < 1$ such that $\xi\eta < 1$ and $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Yau in [49] considered the positive solutions of a class of singular third-order three-point boundary value problems by using the Guo-Krasnosel'skii fixed point theorem of cone expansion-compression type. He showed that this class of problems can have n positive solutions provided that the conditions on the nonlinear term on some bounded sets are appropriate. Lin in [29] by using variational method and three-critical-point theorem, studied the existence of at least three solutions for a three-point boundary value problem

$$\begin{cases} u''(t) + \lambda f(t, u) = 0, \ t \in [0, 1], \\ u(0) = 0, \ u(1) = \alpha u(\eta). \end{cases}$$

Kirchhoff's model [28] as a extension of the classical D'Alembert's wave equation for free vibrations of elastic strings takes into account the changes in length of the string produced by transverse vibrations. It received great attention only after Lions [30] proposed an abstract framework for the problem. For some results on solvability of Kirchhoff type problems, we refer the reader to the papers [18, 19, 35–37, 41, 43]. For example, in [19] based on a three critical point theorem, the existence of an interval of positive real parameters λ for which the boundary-value problem of Kirchhoff-type

$$\begin{cases} -K(\int_a^b |u'(x)|^2 \mathrm{d}x)u'' = \lambda f(x, u), \ t \in (a, b) \\ u(a) = u(b) = 0 \end{cases}$$

where $K : [0, +\infty[\to \mathbb{R}$ is a continuous function, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\lambda > 0$, was discussed. In [37] Molica Bisci and Rădulescu, applying mountain pass results studied the existence of solutions to nonlocal equations involving the p-Laplacian. More precisely, they proved the existence of at least one nontrivial weak solution, and under additional assumptions, the existence of infinitely many weak solutions. In [36], they also by using an abstract linking theorem for smooth functionals established a multiplicity result on the existence of weak solutions for a nonlocal Neumann problem driven by a nonhomogeneous elliptic differential operator. The existence and the multiplicity of stationary higher order problems of the Kirchhoff type (in n-dimensional domains, $n \geq 1$) has also been analyzed in several papers recently through such variational methods as the symmetric mountain pass theorem in [7] and a three critical point theorem in [4]. Furthermore, in [2, 3] some evolutionary higher order Kirchhoff problems were discussed, chiefly emphasizing the qualitative properties of the solutions.

We refer to [11, 34, 38, 39] for related non-local problems concerning the variational analysis of solutions of some classes of boundary value problems.

In the present paper, employing the three critical points theorem obtained in [42], which we will recall in the next section (Theorem 2.2), we establish the existence of at least three weak solutions for three-point boundary value problem of Kirchhoff-type $(K^{f,g}_{\lambda,\mu})$ for appropriate values of the parameters λ and μ belonging to real intervals. An example is presented in order to illustrate our main result.

Here, we state a special case of our main result.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function and let $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant L satisfying $0 < L < 4$ and $h(0) = 0$. Assume that*

$$\max\left\{\limsup_{u\to 0}\frac{\int_0^u f(t)\mathrm{d}t}{|u|^2}, \limsup_{|u|\to\infty}\frac{\int_0^u f(t)\mathrm{d}t}{|u|^2}\right\} \leq 0$$

and

$$\sup_{u\in E_\eta}\frac{\int_0^1\int_0^{u(t)} f(s)\mathrm{d}s\mathrm{d}t}{\int_0^1 |u'(t)|^2\mathrm{d}t - 2\int_0^1\int_0^{u(s)} h(\xi)\mathrm{d}\xi\mathrm{d}s} > 0$$

where $E_\eta := \{u \in \mathrm{W}^{1,2}(0,1) : \ u(0) = 0, u(1) = u(\eta)\}$ with $0 < \eta < 1$. Then, for each compact interval $[c, d] \subset (\lambda^\star, \infty)$ where

$$\lambda^\star = \inf\left\{\frac{\int_0^1 |u'(t)|^2\mathrm{d}t - 2\int_0^1\int_0^{u(s)} h(\xi)\mathrm{d}\xi\mathrm{d}s}{2\int_0^1\int_0^{u(t)} f(s)\mathrm{d}s\mathrm{d}t} : \ u \in E_\eta, \int_0^1\int_0^{u(t)} f(s)\mathrm{d}s\mathrm{d}t > 0\right\},$$

there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and for every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$ there exists $\gamma > 0$ such that, for each $\mu \in [0, \gamma]$, the problem

$$\begin{cases} -u''(t) = \ \lambda f(u(t)) + \mu g(u(t)) + h(u(t)), \ t \in (0,1), \\ u(0) = 0, \ u(1) = u(\eta) \end{cases}$$

has at least three non-negative weak solutions $u_i \in E_\eta$ for $i = 1, 2, 3$ such that

$$\int_0^1 |u'(t)|^2\mathrm{d}t < R^2$$

for $i = 1, 2, 3$.

2. Preliminaries

Our main tool is Theorem 2.2 which has been obtained by Ricceri ([42, Theorem 2]). It is as follows:

If X is a real Banach space, denoted by $\mathcal{W}_X$ the class of all functionals $\Phi : X \to \mathbb{R}$ possessing the following property:
If $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to\infty} \Phi(u_n) \le \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u.

Remark 2.1. If X is uniformly convex and $g : [0, \infty) \to \mathbb{R}$ is a continuous and strictly increasing function, then, by a classical result, the functional $u \to g(\|u\|)$ belongs to the class $\mathcal{W}_X$.

Theorem 2.2. *Let X be a separable and reflexive real Banach space; let $\Phi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1-functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a C^1-functional with compact derivative. Assume that Φ has a strict local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$. Finally, setting*

$$\rho = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\},$$

$$\sigma = \sup_{u\in\Phi^{-1}(0,\infty)} \frac{J(u)}{\Phi(u)},$$

assume that $\rho < \sigma$. Then for each compact interval $[c, d] \subset (\frac{1}{\sigma}, \frac{1}{\rho})$ (with the conventions $\frac{1}{0} = \infty, \frac{1}{\infty} = 0$), there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and every C^1-functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\gamma > 0$ such that, for each $\mu \in [0, \gamma]$,

$$\Phi'(u) = \lambda J'(u) + \mu\Psi'(u)$$

has at least three solutions in X whose norms are less than R.

We refer the reader to the paper [20,45] in which Theorem 2.2 was successfully employed to ensure the existence of at least three solutions for impulsive perturbed elastic beam fourth-order equations of Kirchhoff-type.

Here and in the sequel, we take

$$E = \mathrm{W}_1^{1,2}(a, b) := \{u \in \mathrm{W}^{1,2}(a, b) : \ u(a) = 0, u(b) = \alpha u(\eta)\}.$$

endowed with the norm

$$\|u\| := \Big(\int_a^b |u'(t)|^2 \mathrm{d}t\Big)^{\frac{1}{2}}.$$

Theorem 2.3 ([29, Theorem 3.2])**.** *E is a separable and reflexive real Banach space.*

We need the following lemma in the proof of our main result.

Lemma 2.4. *Let $\varrho = \frac{(1+|\alpha|)\sqrt{b-a}}{2}$. Then, for all $u \in E$,*

$$\max_{t\in[a,b]} |u(t)| \le \varrho\|u\|. \tag{2.1}$$

Proof. For $t \in [a, b]$, from

$$u(t) = \int_a^t u'(s)\mathrm{d}s + C_1$$

it follows that

$$u(a) = C_1.$$

Since $u(a) = 0$, we have

$$C_1 = 0.$$

Thus
$$u(t)=\int_a^t u'(s)\mathrm{d}s. \tag{2.2}$$
Similarly, from
$$u(t)=\int_b^t u'(s)\mathrm{d}s+C_2, \qquad u(b)=\alpha u(\eta)$$
and by (2.2) we have
$$C_2=\alpha u(\eta)=\alpha\int_a^\eta u'(s)\mathrm{d}s.$$
So,
$$u(t)=\int_b^t u'(s)\mathrm{d}s+\alpha\int_a^\eta u'(s)\mathrm{d}s. \tag{2.3}$$
Now, (2.2) and (2.3) imply that
$$|u(t)|\le\int_a^t |u'(s)|\mathrm{d}s$$
and
$$|u(t)|\le\int_b^t |u'(s)|\mathrm{d}s+|\alpha|\int_a^\eta |u'(s)|\mathrm{d}s=\int_t^b |u'(s)|\mathrm{d}s+|\alpha|\int_a^\eta |u'(s)|\mathrm{d}s.$$
Hence,
$$2|u(t)|\le\int_a^b |u'(s)|\mathrm{d}s+|\alpha|\int_a^b |u'(s)|\mathrm{d}s=(1+|\alpha|)\int_a^b |u'(s)|\mathrm{d}s.$$
Then, from Hölder's inequality,
$$|u(t)|\le\frac{1+|\alpha|}{2}\int_a^b |u'(s)|\mathrm{d}s\le\frac{(1+|\alpha|)\sqrt{b-a}}{2}\left(\int_a^b |u'(s)|^2\mathrm{d}s\right)^{\frac{1}{2}}=\varrho\|u\|.$$
This completes the proof of the lemma. □

Corresponding to the functions f, K and h, we introduce the functions $F:[a,b]\times\mathbb{R}\to\mathbb{R}$, $\tilde{K}:[0,+\infty[\to\mathbb{R}$ and $H:\mathbb{R}\to\mathbb{R}$, respectively, as follows
$$F(t,x):=\int_0^x f(t,\xi)\mathrm{d}\xi \quad \text{for all } (t,x)\in[a,b]\times\mathbb{R},$$
$$\tilde{K}(x):=\int_0^x K(\xi)\mathrm{d}\xi \quad \text{for all } x\ge 0$$
and
$$H(x):=\int_0^x h(\xi)\mathrm{d}\xi \quad \text{for all } x\in\mathbb{R}.$$
We say that a function $u\in E$ is a weak solution of the problem $(K^{f,g}_{\lambda,\mu})$ if
$$\begin{aligned}K\Big(\int_a^b |u'(t)|^2\mathrm{d}t\Big)\int_a^b u'(t)v'(t)\mathrm{d}t-\int_a^b h(u(t))v(t)\mathrm{d}t\\ -\lambda\int_a^b f(t,u(t))v(t)\mathrm{d}t-\mu\int_a^b g(t,u(t))v(t)\mathrm{d}t=0\end{aligned}$$
holds for all $v\in E$.

We assume throughout and without further mention, that the following the condition $m>L(b-a)\varrho^2$ holds.

Now for every $u\in E$, we define
$$\Phi(u)=\frac{1}{2}\tilde{K}(\|u\|^2)-\int_a^b H(u(t))\mathrm{d}t \tag{2.4}$$
$$J(u)=\int_a^b F(t,u(t))\mathrm{d}t \tag{2.5}$$

and

$$\Psi(u)=\int_a^b G(t,u(t))\mathrm{d}t. \tag{2.6}$$

Standard arguments show that $I:\Phi-\mu\Psi-\lambda J$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u\in E$ given by

$$\begin{aligned} I'(u)(v)=&K\Big(\int_a^b |u'(t)|^2\mathrm{d}t\Big)\int_a^b u'(t)v'(t)\mathrm{d}t-\int_a^b h(u(t))v(t)\mathrm{d}t\\ &-\lambda\int_a^b f(t,u(t))v(t)\mathrm{d}t-\mu\int_a^b g(t,u(t))v(t)\mathrm{d}t \end{aligned}$$

for all $u,v\in E$. Hence, the critical points of the functional I are the weak solutions of $(K^{f,g}_{\lambda,\mu})$.

We need the following proposition in the proofs of our main results.

Proposition 2.5. *Let $J:X\longrightarrow X^*$ be the operator defined by*

$$J(u)(v)=K\Big(\int_a^b |u'(t)|^2\mathrm{d}t\Big)\int_a^b u'(t)v'(t)\mathrm{d}t-\int_a^b h(u(t))v(t)\mathrm{d}t$$

for every $u,v\in E$. Then, J admits a continuous inverse on E^*.

Proof. Recalling the assumption (2.1) we have

$$\begin{aligned} J(u)(u)&=K\Big(\int_a^b |u'(t)|^2\mathrm{d}t\Big)\int_a^b |u'(t)|^2\mathrm{d}t-\int_a^b h(u(t))u(t)\mathrm{d}t\\ &\geq m\|u\|^2-L\int_a^b |u(t)|^2\mathrm{d}t\\ &\geq \Big(m-L(b-a)\varrho^2\Big)\|u\|^2. \end{aligned}$$

Since $m>L(b-a)\varrho^2$, this follows that J is coercive. Owing to our assumptions on the data, one has

$$\langle J(u)-J(v),u-v\rangle\geq C\|u-v\|^2>0$$

for some $C>0$ for every $u,v\in X$, which means that J is strictly monotone. Consequently, thanks to Minty-Browder theorem [50], the operator J is a surjection and admits an inverse mapping. Thus it is sufficient to show that J^{-1} is continuous. For this, let $(v_n)_{n=1}^{\infty}$ be a sequence in X^* such that $v_n\to v$ in E^*. Let u_n and u in E such that

$$J^{-1}(v_n)=u_n \quad\text{and}\quad J^{-1}(v)=u.$$

By the coercivity of J, we conclude that the sequence (u_n) is bounded in the reflexive space E. For a subsequence, we have $u_n\rightharpoonup\widehat{u}$ in E, which implies

$$\lim_{n\to+\infty}\langle J(u_n)-J(u),u_n-\widehat{u}\rangle=\lim_{n\to+\infty}\langle f_n-f,u_n-\widehat{u}\rangle=0.$$

Therefore, by the continuity of J, we have

$$u_n\to\widehat{u}\quad\text{in } E\quad\text{and}\quad J(u_n)\to J(\widehat{u})=J(u)\quad\text{in } E^*.$$

Moreover, since J is an injection, we conclude that $u=\widehat{u}$. □

3. Main results

In this section, we formulate our main results.

Let

$$\lambda_1=\inf\left\{\frac{\tilde{K}(\|u\|^2)-2\int_a^b H(u(t))\mathrm{d}t}{2\int_a^b F(t,u(t))\mathrm{d}t}\;:\;u\in E\,,\int_a^b F(t,u(t))\mathrm{d}t>0\right\}$$

and $\lambda_2 = \frac{1}{\max\{0,\lambda_0,\lambda_\infty\}}$ where

$$\lambda_0 = \limsup_{|u|\to 0} \frac{2\int_a^b F(t,u(t))\mathrm{d}t}{\tilde{K}(\|u\|^2) - 2\int_a^b H(u(t))\mathrm{d}t}$$

and

$$\lambda_\infty = \limsup_{\|u\|\to \infty} \frac{2\int_a^b F(t,u(t))\mathrm{d}t}{\tilde{K}(\|u\|^2) - 2\int_a^b H(u(t))\mathrm{d}t}.$$

Theorem 3.1. *Assume that*

($\mathcal{A}_1$) *there exists a constant $\varepsilon > 0$ such that*

$$\max\left\{\limsup_{u\to 0} \frac{\max_{t\in[a,b]} F(t,u(t))}{|u|^2}, \limsup_{|u|\to\infty} \frac{\max_{t\in[a,b]} F(t,u(t))}{|u|^2}\right\} < \varepsilon;$$

($\mathcal{A}_2$) *there exists a function $w \in E$ such that*

$$\tilde{K}(\|w\|^2) - 2\int_a^b H(w(t))\mathrm{d}t \neq 0$$

and

$$\varrho(b-a)\varepsilon < \frac{(m - L(b-a)\varrho^2)\int_a^b F(t,w(t))\mathrm{d}t}{\tilde{K}(\|w\|^2) - 2\int_a^b H(w(t))\mathrm{d}t}.$$

Then, for each compact interval $[c,d] \subset (\lambda_1,\lambda_2)$, there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ and every L^1-Carathéodory function $g : [a,b]\times\mathbb{R}\to\mathbb{R}$ there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the problem $(K^{f,g}_{\lambda,\mu})$ has at least three weak solutions whose norms in E are less than R.

Proof. Take $X = E$. Clearly, X is a separable and uniformly convex Banach space. Let the functionals Φ, J and Ψ be as given in (2.4), (2.5) and (2.6), respectively. The functional Φ is C^1, and its derivative admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous. Indeed, putting $S(u) = \int_a^b H(u(t))\mathrm{d}t$ for $u \in X$, for any $u_n \in X$ with $u_n \to u$ weakly in X, since S and $\tilde{K}$ are continuous on X, one has

$$\begin{aligned}\liminf_{n\to\infty} \Phi(u_n) &= \lim_{n\to\infty} \frac{1}{2}\tilde{K}(\|u_n\|^2) - \lim_{n\to\infty} S(u_n)\\ &= \frac{1}{2}\tilde{K}(\|u\|^2) - S(u) = \Phi(u).\end{aligned}$$

Taking into account that h is a Lipschitz continues function with Lipschitz constant $L > 0$ and $h(0) = 0$, and $m \le K(s) \le M$ for all $s \in [0,+\infty[$, from (2.4) we have

$$\frac{m - L(b-a)\varrho^2}{2}\|u\|^2 \le \Phi(u) \le \frac{M + L(b-a)\varrho^2}{2}\|u\|^2 \tag{3.1}$$

for all $u \in X$ and bearing the condition ($\mathcal{I}$) in mind, it follows $\lim_{\|u\|\to+\infty}\Phi(u) = +\infty$, namely Φ is coercive. Moreover, let A be a bounded subset of X. That is, there exist constants $c > 0$, such that $\|u\| \le c$ for each $u \in A$. Then, by (3.1) we have

$$|\Phi(u)| \le \frac{M + L(b-a)\varrho^2}{2}c^2.$$

Hence Φ is bounded on each bounded subset of X. Furthermore, $\Phi \in \mathcal{W}_X$. Indeed, let the sequence $\{u_k\}_{k=1}^\infty \subset X$, $u \subset X$, $u_k \rightharpoonup u$ and $\liminf_{k\to\infty}\Phi(u_k) \le \Phi(u)$. Since the functions H is continuous, one has

$$\liminf_{k\to\infty} \frac{1}{2}\tilde{K}(\|u_k\|^2) \le \frac{1}{2}\tilde{K}(\|u\|^2).$$

Now by continuity of $\tilde{K}$, from Remark 2.1 the functional $u \to \tilde{K}(\|u_k\|^2)$ belongs to the class $\mathcal{W}_X$. Thus $\{u_k\}_{k=1}^{\infty}$ has a subsequence converging strongly to u. Therefore, $\Phi \in \mathcal{W}_X$. The functionals J and Ψ are two C^1-functionals with compact derivatives. Moreover, Φ has a strict local minimum 0 with $\Phi(0) = J(0) = 0$.
In view of $(\mathcal{A}_1)$, there exist τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(t,u) \leq \varepsilon |u|^2 \tag{3.2}$$

for every $t \in [a,b]$ and every u with $|u| \in [0,\tau_1) \cup (\tau_2, \infty)$. Since $F(t,u)$ is continuous on $[a,b]\times\mathbb{R}$, it is bounded on $t \in [a,b]$ and $|u| \in [\tau_1, \tau_2]$. Thus we can choose $\delta > 0$ and $\upsilon > 2$ such that

$$F(t,u) \leq \varepsilon |u|^2 + \delta |u|^{\upsilon}$$

for all $(t,u) \in [a,b]\times\mathbb{R}$. So, by (2.1), we have

$$J(u) \leq \varrho^2 \varepsilon (b-a)\|u\|^2 + \varrho^{\upsilon}\delta(b-a)\|u\|^{\upsilon} \tag{3.3}$$

for all $u \in X$. Hence, from (3.3) we have

$$\limsup_{|u|\to 0} \frac{J(u)}{\Phi(u)} \leq \frac{2\varrho^2(b-a)\varepsilon}{m - L(b-a)\varrho^2}. \tag{3.4}$$

Moreover, by using (3.2), for each $u \in X \setminus \{0\}$, we obtain

$$\begin{aligned}
\frac{J(u)}{\Phi(u)} &= \frac{\int_{|u|\leq\tau_2} F(t,u(t))}{\Phi(u)} + \frac{\int_{|u|>\tau_2} F(t,u(t))}{\Phi(u)} \\
&\leq \frac{\sup_{t\in[a,b],|u|\in[0,\tau_2]} F(t,u)}{\Phi(u)} + \frac{\varrho^2(b-a)\varepsilon}{\Phi(u)} \\
&\leq \frac{2\sup_{t\in[a,b],|u|\in[0,\tau_2]} F(t,u)}{\|u\|^2} + \frac{2\varrho^2(b-a)\varepsilon}{m - L(b-a)\varrho^2}.
\end{aligned}$$

So, we get

$$\limsup_{\|u\|\to \infty} \frac{J(u)}{\Phi(u)} \leq \frac{2\varrho^2(b-a)\varepsilon}{m - L(b-a)\varrho^2}. \tag{3.5}$$

In view of (3.4) and (3.5), we have

$$\rho = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to 0} \frac{J(u)}{\Phi(u)}\right\} \leq \frac{2\varrho^2(b-a)\varepsilon}{m - L(b-a)\varrho^2}. \tag{3.6}$$

Assumption $(\mathcal{A}_2)$ in conjunction with (3.6) yields

$$\begin{aligned}
\sigma &= \sup_{u\in\Phi^{-1}(0,\infty)} \frac{J(u)}{\Phi(u)} = \sup_{X\setminus\{0\}} \frac{J(u)}{\Phi(u)} \geq \frac{\int_a^b F(t,w(t))}{\Phi(w(t))} \\
&= \frac{2\int_a^b F(t,w(t))}{\tilde{K}(\|w\|^2) - 2\int_a^b H(w(t))\mathrm{d}t} > \frac{2\varrho^2(b-a)\varepsilon}{m - L(b-a)\varrho^2} \geq \rho.
\end{aligned}$$

Thus, all the hypotheses of Theorem 2.2 are satisfied. Clearly, $\lambda_1 = \frac{1}{\beta}$ and $\lambda_2 = \frac{1}{\alpha}$. Then, using Theorem 2.2, for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$, there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ and every L^1-Carathéodory function $g : [a,b]\times\mathbb{R} \to \mathbb{R}$ there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the problem $(K_{\lambda,\mu}^{f,g})$ has at least three weak solutions whose norms in X are less than R. $\square$

Remark 3.2. If f, g are non-negative, then the strong maximum principle ensures that the weak solutions the problem $(K_{\lambda,\mu}^{f,g})$ are non-negative.

The another announced application of Theorem 2.2 reads as follows:

Theorem 3.3. *Assume that*

$$\max\left\{\limsup_{u\to 0}\frac{\max_{t\in[a,b]}F(t,u(t))}{|u|^2},\ \limsup_{|u|\to\infty}\frac{\max_{t\in[a,b]}F(t,u(t))}{|u|^2}\right\}\le 0 \tag{3.7}$$

and

$$\sup_{u\in E}\frac{\int_a^b F(t,u(t))\mathrm{d}t}{\tilde{K}(\|u\|^2)-2\int_a^b H(u(t))\mathrm{d}t}>0. \tag{3.8}$$

Then, for each compact interval $[c,d]\subset(\lambda_1,\infty)$ there exists $R>0$ with the following property: for every $\lambda\in[c,d]$ and every L^1-Carathéodory function $g:[a,b]\times\mathbb{R}\to\mathbb{R}$ there exists $\gamma>0$ such that for each $\mu\in[0,\gamma]$, the problem $(K^{f,g}_{\lambda,\mu})$ has at least three weak solutions whose norms in E are less than R.

Proof. In view of (3.7), there exist an arbitrary $\varepsilon>0$ and numbers τ_1,τ_2 with $0<\tau_1<\tau_2$ such that

$$F(t,u)\le\varepsilon|u|^2$$

for every $t\in[a,b]$ and every u with $|u|\in[0,\tau_1)\cup(\tau_2,\infty)$. Since $F(t,u)$ is an L^1-Carathéodory function on $[a,b]\times\mathbb{R}$, it is bounded on $t\in[a,b]$ and $|u|\in[\tau_1,\tau_2]$. Thus we can choose $\delta>0$ and $\upsilon>2$ in a manner that

$$F(t,u)\le\varepsilon|u|^2+\delta|u|^\upsilon$$

for all $(t,u)\in[a,b]\times\mathbb{R}$. So, by the same process in proof of Theorem 3.1 we have Relations (3.4) and (3.5). Since ε is arbitrary, (3.4) and (3.5) give

$$\max\left\{0,\ \limsup_{\|u\|\to\infty}\frac{J(u)}{\Phi(u)},\ \limsup_{u\to 0}\frac{J(u)}{\Phi(u)}\right\}\le 0.$$

Then, with the notation of Theorem 2.2, we have $\rho=0$. By (3.8), we also have $\sigma>0$. In this case clearly $\lambda_1=\frac{1}{\sigma}$ and $\lambda_2=\infty$. Thus, by using Theorem 2.2 the conclusion is achieved. □

Remark 3.4. Theorem 1.1 immediately follows from Theorem 3.3.

Remark 3.5. In Assumption $(\mathcal{A}_2)$ if we choose w by setting

$$w(t)=\begin{cases}\frac{d}{\eta-a}(t-a), & t\in[a,\eta),\\ d, & t\in[\eta,\frac{b+\eta}{2}],\\ d\left(\frac{2(\alpha-1)}{b-\eta}t-\frac{\alpha(b+\eta)-2b}{b-\eta}\right), & t\in(\frac{b+\eta}{2},b].\end{cases} \tag{3.9}$$

then it becomes:

$(\mathcal{A}_2')$ there exists a positive constant d such that

$$\tilde{K}(Pd)-2\int_a^b H(w(t))\mathrm{d}t\neq 0$$

and

$$\varrho(b-a)\varepsilon<\frac{(m-L(b-a)\varrho^2)\int_a^b F(t,w(t))\mathrm{d}t}{\tilde{K}(Pd)-2\int_a^b H(w(t))\mathrm{d}t}$$

where

$$P:=\frac{1}{\eta-a}+\frac{2(\alpha-1)^2}{b-\eta}. \tag{3.10}$$

It is clear that $w\in E$ and

$$\|w\|^2=\int_a^b|w'(t)|^2\mathrm{d}t=\int_a^\eta\left(\frac{d}{\eta-a}\right)^2\mathrm{d}t+\int_{\frac{b+\eta}{2}}^b\left(\frac{2d(\alpha-1)}{b-\eta}\right)^2\mathrm{d}t$$
$$=\left(\frac{1}{\eta-a}+\frac{2(\alpha-1)^2}{b-\eta}\right)d^2=Pd^2.$$

By using (2.4) and (3.1) we have

$$\frac{m - L(b-a)\varrho^2}{2} Pd^2 \leq \Phi(w) \leq \frac{M + L(b-a)\varrho^2}{2} Pd^2.$$

Now, we point out some results in which the function f has separated variables. To be precise, consider the following problem

$$\begin{cases} -K(\int_a^b |u'(t)|^2 \mathrm{d}t) u''(t) = \ \lambda\theta(t) f(u(t)) + \mu g(t, u(t)) + h(u(t)), \ t \in (a,b), \\ u(a) = 0, \ u(b) = \alpha u(\eta) \end{cases} \qquad (K_{\lambda,\mu}^{f,\theta,g})$$

where $\theta : [a,b] \to \mathbb{R}$ is a non-negative and non-zero continuous function, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ is as introduced in the problem $(K_{\lambda,\mu}^{f,g})$ in Introduction.

Set $F(t,x) = \theta(t)F(x)$ for every $(t,x) \in [a,b] \times \mathbb{R}$ where

$$F(x) = \int_0^x f(\xi) \mathrm{d}\xi$$

for all $x \in \mathbb{R}$. The following existence results are consequences of Theorem 3.1.

Theorem 3.6. *Assume that*

($\mathcal{A}_1'$) *there exists a constant* $\varepsilon > 0$ *such that*

$$\sup_{t \in [a,b]} \theta(t). \max \left\{ \limsup_{u \to 0} \frac{F(u)}{|u|^2}, \limsup_{|u| \to \infty} \frac{F(u)}{|u|^2} \right\} < \varepsilon;$$

($\mathcal{A}_2''$) *there exists a positive constant* d *such that*

$$\tilde{K}(Pd) - 2\int_a^b H(w(t)) \mathrm{d}t \neq 0$$

and

$$\varrho(b-a)\varepsilon < \frac{(m - L(b-a)\varrho^2) \int_a^b F(t, w(t)) \mathrm{d}t}{\tilde{K}(Pd) - 2\int_a^b H(w(t)) \mathrm{d}t}.$$

where w *and* P *are given as in* (3.9) *and* (3.10)*, respectively.*

Then, for each compact interval $[c,d] \subset (\lambda_3, \lambda_4)$ *where* λ_3 *and* λ_4 *are the same as* λ_1 *and* λ_2*, but* $\int_a^b F(t,u(t))\mathrm{d}t$ *replaced by* $\int_a^b \theta(t) F(u(t)) \mathrm{d}t$*, respectively, there exists* $R > 0$ *with the following property: for every* $\lambda \in [c,d]$ *and every* L^1*-Carathéodory function* $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ *there exists* $\gamma > 0$ *such that for each* $\mu \in [0,\gamma]$*, the problem* $(K_{\lambda,\mu}^{f,\theta,g})$ *has at least three weak solutions whose norms in* E *are less than* R*.*

Theorem 3.7. *Assume that there exists a positive constant* d *such that*

$$\tilde{K}(Pd) - 2\int_a^b H(w(t)) \mathrm{d}t > 0 \text{ and } \int_a^b \theta(t) F(w(t)) \mathrm{d}t > 0 \tag{3.11}$$

where w *and* P *are given as in* (3.9) *and* (3.10)*, respectively. Moreover, suppose that*

$$\limsup_{u \to 0} \frac{f(u)}{|u|} = \limsup_{|u| \to \infty} \frac{f(u)}{|u|} = 0. \tag{3.12}$$

Then, for each compact interval $[c,d] \subset (\lambda_3, \infty)$ *where* λ_3 *is the same as* λ_1 *but* $\int_a^b F(t,u(t))\mathrm{d}t$ *replaced by* $\int_a^b \theta(t)F(u(t))\mathrm{d}t$*, there exists* $R > 0$ *with the following property: for every* $\lambda \in [c,d]$ *and every* L^1*-Carathéodory function* $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ *there exists* $\gamma > 0$ *such that for each* $\mu \in [0,\gamma]$*, the problem* $(K_{\lambda,\mu}^{f,\theta,g})$ *has at least three weak solution whose norms in* E *are less than* R*.*

Proof. We easily observe that from (3.12) the assumption ($\mathcal{A}_1'$) is satisfied for every $\varepsilon > 0$. Moreover, using (3.11), by choosing $\varepsilon > 0$ small enough one can drive the assumption ($\mathcal{A}_2''$). Hence, the conclusion follows from Theorem 3.6. $\square$

Remark 3.8. We observe that, in our results, no asymptotic conditions on f and g are needed and only algebraic conditions on f are imposed to guarantee the existence of solutions.

Finally, we present the following example to illustrate Theorem 3.7.

Example 3.9. Let $a = 0$, $b = 1$, $\eta = \frac{1}{2}$, $\alpha = 2$, $\theta(t) = e^t$ for all $t \in [0, 1]$, $K(x) = 3 + \cos x$ for all $x \in \mathbb{R}$,

$$f(x) = \begin{cases} \sin^2 x, & \text{if } x < 0, \\ 1 - \cos x, & \text{if } x \geq 0 \end{cases}$$

and

$$h(x) = -\frac{1}{4} \arctan(e^x) \quad \text{for all}\, x \in \mathbb{R}.$$

By simple computation, we obtain $m = 2$, $M = 3$, f is non-negative, $L = \frac{1}{2}$, $\varrho = \frac{3}{2}$,

$$m = 2 > \frac{9}{8} = L(b - a)\varrho^2$$

and $P = 6$. Choosing $d = 1$, we have

$$w(t) = \begin{cases} 2t, & t \in [0, \frac{1}{2}), \\ 1, & t \in [\frac{1}{2}, \frac{3}{4}] \\ 4t - 2, & t \in (\frac{3}{4}, 1]. \end{cases}$$

Thus $w(t) \geq 0$ for all $t \in [0, 1]$,

$$\begin{aligned} \tilde{K}(Pd) - 2\int_a^b H(w(t))\mathrm{d}t = &\int_0^6 (3 + \cos(\xi))\mathrm{d}\xi \\ &+ 2\int_0^1 \int_0^{w(t)} \arctan(e^{\xi})\mathrm{d}\xi\mathrm{d}t > 0, \end{aligned}$$

$$\int_a^b \theta(t)F(w(t))\mathrm{d}t = \int_0^1 e^t \int_0^{w(t)} f(\xi)\mathrm{d}\xi\mathrm{d}t > 0$$

and

$$\lim_{u \to 0} \frac{f(u)}{|u|} = \lim_{u \to \infty} \frac{f(u)}{|u|} = 0.$$

Hence, by applying Theorem 3.7 for each compact interval $[c, d] \subset (0, \infty)$, there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that, for each $\mu \in [0, \gamma]$, the problem

$$\begin{cases} -\left(3 + \cos(\int_0^1 |u'(t)|^2\mathrm{d}t)\right) u''(t) = \lambda f(u(t)) + \mu g(u(t)) & \\ -\frac{1}{4}\arctan(e^{u(t)}), & t \in (0, 1), \\ u(0) = 0,\ u(1) = 2u(\frac{1}{2}) & \end{cases}$$

has at least three non-negative weak solutions in the space

$$E_{\frac{1}{2}} = \left\{ u \in W^{1,2}(0, 1) :\ u(0) = 0,\, u(1) = 2u(\frac{1}{2}) \right\}.$$

Remark 3.10. We point out that the same statements of the above given results can be obtained by considering

$$K(x) = b_1 + b_2 x, \qquad \text{for } x \in [\iota, \kappa]$$

where b_1, b_2, ι and κ are positive numbers. In fact, in this special case, we have

$$\tilde{K}(x) = \int_0^x (b_1 + b_2\xi)\mathrm{d}\xi = \frac{(b_1 + b_2 x)^2}{2b_2} - \frac{b_1^2}{2b_2} \qquad \text{for } x \in [\iota, \kappa],$$

$$m_0 = b_1 + b_2\iota \text{ and } m_1 = b_1 + b_2\kappa.$$

Arguing as in the proof of Theorems 3.1, the existence of at least three weak solutions can be obtained.

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