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Research Article

Invertible skew pairings and crossed products for weak Hopf algebras

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Abstract

In this paper we work with invertible skew pairings for weak bialgebras in a symmetric monoidal category where every idempotent morphism splits. We prove that this kind of skew pairings induces examples of weak distributive laws and therefore they provide weak wreath products. Also we will show that they define weakly comonoidal mutually weak inverse pairs of weak distributive laws and, by the results proved by G. Böhm and J. Gómez-Torrecillas, we obtain weak wreath products that become weak bialgebras with respect to the tensor product coalgebra structure. As an application, we will show that the Drinfel'd double of a finite weak Hopf algebra can be constructed using the weak wreath product associated to an invertible 1-skew pairing.

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1. Introduction

The notions of weak bialgebra and weak Hopf algebra were introduced by G. Böhm, F. Nill and K. Szlachányi in [4] as a generalization of classical bialgebras and Hopf algebras. A weak bialgebra is an algebra-coalgebra where the product does not preserve the unit (dually the coproduct does not preserve the counit). As a consequence, some axioms involved in the definition of bialgebra are replaced for weaker conditions. These changes also affect the axioms related with the antipode in the notion of weak Hopf algebra. The main example of weak Hopf algebra is the groupoid algebra; other interesting examples are the face algebras defined by Hayashi [10, 11] (face algebras are weak bialgebras where the target counital subalgebra is commutative), and generalized Kac algebras by Yamanouchi [25]. Also there exists a relevant connection between weak bialgebras and $\times_R$-bialgebras in

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the sense of Takeuchi [24]. As was proved by P. Schauenburg (see [22]), a weak bialgebra is the same that a $\times_R$-bialgebra where R is a separable algebra. From this point of view face algebras are $\times_R$-bialgebras where R is a commutative separable algebra.

On the other hand, Doi discovered in [7] a contruction to modify the algebra structure of a bialgebra A over a field $\mathbb{F}$ using a convolution invertible two-cocycle σ in A. With the new algebra structure and the original coalgebra structure A is a new bialgebra, denoted by A^{σ}, and if A is a Hopf algebra, so is A^{σ}. In this case, if μ_A is the original product in A and λ_A its antipode, $\mu_{A^{\sigma}}(a\otimes b)=\sigma(a_1\otimes b_1)a_2b_2\sigma^{-1}(a_3\otimes b_3)$, for all $a,b\in A$, and the antipode of A^{σ} is given by $\lambda_{A^{\sigma}}(a)=\sigma(a_1\otimes\lambda_A(a_2))\lambda_A(a_3)\sigma^{-1}(\lambda_A(a_4)\otimes a_5)$ for $a\in A$. The most celebrated example of this construction is the Drinfel'd double of a Hopf algebra H. If H^* is the dual of H and $A=H^{*cop}\otimes H$, the Drinfel'd double $D(H)$ can be obtained as A^{σ} where σ is defined by $\sigma((x\otimes g)\otimes(y\otimes h))=x(1_H)y(g)\varepsilon_H(h)$ for $x,y\in H^*$ and $g,h\in H$. As was pointed by Doi and Takeuchi in [8] "this will be the the shortest description of the multiplication of $D(H)$".

A particular case of alterations of products by two-cocycles are provided by convolution invertible skew pairings on bialgebras. If A and H are bialgebras and $\tau: A\otimes H\to\mathbb{F}$ is a convolution invertible skew pairing, Doi and Takeuchi defined in [8] a new biagebra $A\bowtie_{\tau} H$ in the following way: The morphism $\omega: A\otimes H\otimes A\otimes H\to\mathbb{F}$ defined by $\omega((a\otimes g)\otimes(b\otimes h))=\varepsilon_A(a)\varepsilon_H(h)\tau(b\otimes g)$, for $a,b\in A$ and $g,h\in H$, is a convolution invertible two-cocycle in $A\otimes H$ and $A\bowtie_{\tau} H=(A\otimes H)^{\omega}$. The construction of $A\bowtie_{\tau} H$ also generalizes the Drinfel'd double because for any finite dimensional Hopf algebra H, H^{*cop} and H are skew-paired.

The generalization of the construcction of the Drinfel'd double to the $\times_R$-bialgebra setting using the Doi and Takeuchi viewpoint, (i.e., to get the Drinfel'd double as an example of twisted tensor product $A\bowtie_{\tau} H$ associated to a skew pairing) was proposed by P. Schauenburg in [21] (see also [22]). More concretely, in this paper we can find the definition of skew paring for $\times_R$-bialgebras and, in the sixth section, the author defines the $\times_R$-Hopf algebra version of the Drinfel'd double. In this case, the Drinfel'd double is obtained as a special case of a twisted tensor product $A\bowtie_{\tau} H$ associated to a skew pairing $\tau: A\otimes H\to R$ defined for two $\times_R$-Hopf algebras A and H. Note that in this case the definition of $A\bowtie_{\tau} H$ is a generalization of the one proposed by Doi and Takeuchi for Hopf algebras but in this new setting $A\bowtie_{\tau} H$ is defined directly and is not obtained as a deformation of a product by a convolution invertible two-cocycle (see [21] for the details). In any case, Schauenburg's construction permits to obtain the Drinfel'd double of finite dimensional weak Hopf algebra for the reasons set out in the first paragraph of this introduction.

On the other hand, the Drinfel'd double for finite dimensional weak Hopf algebras over a field have been introduced by A. Nenciu in [17] (see also [2]). In [6] for a finitely generated and projective weak Hopf algebra H over a commutative ring R, S. Caenepeel, Dingguo Wang and Yanmin Yin introduce the Drinfeld double using duality results between entwining structures (see [5]) and smash product structures, and they show that the category of Yetter-Drinfel'd modules is isomorphic to the category of modules over the Drinfeld double. Finally, it should be emphasized that the Drinfel'd double can be obtained as an example of the theory of double crossed products of weak Hopf algebras developed in [3]. In this paper sufficient conditions under which the weak wreath product algebra associated a weak distributive law between weak bialgebras (weak Hopf algebras) becomes a weak bialgebra with respect to the tensor product coalgebra structure are given. In this setting the key to prove the results is the use of pairs of weakly comonoidal weak distributive laws. The theory developed in [3] is capable to describe the Drinfel'd double of a finite dimensional weak Hopf algebra as a weak wreath product algebra with respect to the tensor product coalgebra structure without mention to skew pairings.

The aim of this paper is to prove that in a monoidal setting and with a suitable definition of convolution invertible skew pairing for weak bialgebras it is posssible to define two weak distributive laws Ψ and Φ such that Φ is a weak inverse for Ψ and the pair (Ψ, Φ) is weakly comonoidal. Then, as a consequence, by the results proved in [3] we obtain an example of weak bialgebra where the product is the wreath product algebra and the coproduct is the tensor product coalgebra. For a weak Hopf algebra H we prove that, if H is finite, it is possible to define a convolution invertible skew pairing and the associated weakly monoidal pair of invertible weak distributive laws are exactly the ones that define the Drinfel'd double as proposed in Example 11 of [3]. Then, we prove that there exists a link between the theory of twisted tensor products proposed in [21] for $\times_R$-bialgebras and the general theory of double crossed products of weak bialgebras introduced in [3]. Here the bridge will be the notion of convolution invertible skew pairing for weak bialgebras.

2. Preliminaries

In this paper we will work in a monoidal setting. Following [16], recall that a monoidal category is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called tensor product, an object K of $\mathcal{C}$, called the unit object, and families of natural isomorphisms

$$a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P),$$

$$r_M : M \otimes K \to M, \quad l_M : K \otimes M \to M,$$

in $\mathcal{C}$, called associativity, right unit and left unit constraints, respectively, satisfying the Pentagon Axiom and the Triangle Axiom, i.e.,

$$a_{M,N,P\otimes Q} \circ a_{M\otimes N,P,Q} = (id_M \otimes a_{N,P,Q}) \circ a_{M,N\otimes P,Q} \circ (a_{M,N,P} \otimes id_Q),$$

$$(id_M \otimes l_N) \circ a_{M,K,N} = r_M \otimes id_N,$$

where for each object X in $\mathcal{C}$, id_X denotes the identity morphism of X. A monoidal category is called strict if the associativity, right unit and left unit constraints are identities. It is a well-known fact (see for example [15]) that every non-strict monoidal category is monoidal equivalent to a strict one. Then we can assume without loss of generality that the category is strict. This lets us to treat monoidal categories as if they were strict and, as a consequence, the results proved in an strict setting hold for every non-strict symmetric monoidal category, for example the category of vector spaces over a field $\mathbb{F}$, or the category of left modules over a commutative ring R. For simplicity of notation, given objects M, N, P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

A braiding for a strict monoidal category $\mathcal{C}$ is a natural family of isomorphisms

$$c_{M,N} : M \otimes N \to N \otimes M$$

subject to the conditions

$$c_{M,N\otimes P} = (N \otimes c_{M,P}) \circ (c_{M,N} \otimes P), \quad c_{M\otimes N,P} = (c_{M,P} \otimes N) \circ (M \otimes c_{N,P}).$$

A strict braided monoidal category $\mathcal{C}$ is a strict monoidal category with a braiding. These categories were introduced by Joyal and Street in [12] (see also [13]) motivated by the theory of braids and links in topology. Note that, as a consequence of the definition, the equalities $c_{M,K} = c_{K,M} = id_M$ hold, for all object M of $\mathcal{C}$. If the braiding satisfies that $c_{N,M} \circ c_{M,N} = id_{M\otimes N}$, for all M, N in $\mathcal{C}$, we will say that $\mathcal{C}$ is symmetric. In this case, we call the braiding c a symmetry for the category $\mathcal{C}$.

Throughout this paper $\mathcal{C}$ denotes a strict symmetric monoidal category with tensor product $\otimes$, unit object K and symmetry c. Following [1], we also assume that in $\mathcal{C}$ every idempotent morphism splits, i.e., for any morphism $q : X \to X$ such that $q \circ q = q$ there exist an object Z, called the image of q, and morphisms $i : Z \to X$, $p : X \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$. The morphisms p and i will be called a factorization of q. Note that Z, p and i are unique up to isomorphism. The categories satisfying this property constitute a broad class that includes, among others, the categories with epi-monic decomposition for morphisms and categories with equalizers or coequalizers. For example, complete bornological spaces is a not abelian symmetric monoidal closed category, but it does have coequalizers (see [19]). On the other hand, let Hilb be the category whose objects are complex Hilbert spaces and whose morphisms are the continuous linear maps. Then Hilb is not an abelian and closed category but is a symmetric monoidal category (see [14]) with coequalizers.

An algebra in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in $\mathcal{C}$ and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$ and $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given algebras A and B, $f : A \to B$ is a morphism of algebras if $f \circ \eta_A = \eta_B$ and $\mu_B \circ (f \otimes f) = f \circ \mu_A$. Also, if A, B are algebras in $\mathcal{C}$, the object $A \otimes B$ is an algebra in $\mathcal{C}$ where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. If $A = (A, \eta_A, \mu_A)$ is an algebra so is $A^{op} = (A, \eta_A, \mu_A \circ c_{A,A})$.

A coalgebra in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$ and $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If D and E are coalgebras in $\mathcal{C}$, $f : D \to E$ is a morphism of coalgebras if $\varepsilon_E \circ f = \varepsilon_D$, and $(f \otimes f) \circ \delta_D = \delta_E \circ f$. Moreover, if D, E are coalgebras in $\mathcal{C}$, the object $D \otimes E$ is a coalgebra in $\mathcal{C}$ where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$. If $D = (D, \varepsilon_D, \delta_D)$ is a coalgebra so is $D^{cop} = (D, \varepsilon_D, c_{D,D} \circ \delta_D)$.

If A is an algebra, B a coalgebra and $f : B \to A$, $g : B \to A$ are morphisms, we define the convolution product by $f * g = \mu_A \circ (f \otimes g) \circ \delta_B$. The morphism $f : B \to A$ is convolution invertible if there exists $f^{-1} : B \to A$ such that $f * f^{-1} = f^{-1} * f = \varepsilon_B \otimes \eta_A$.

Let A be an algebra. The pair (M, ϕ_M) is a right A-module if M is an object in $\mathcal{C}$ and $\phi_M : M \otimes A \to M$ is a morphism in $\mathcal{C}$ satisfying $\phi_M \circ (M \otimes \eta_A) = id_M$, $\phi_M \circ (\phi_M \otimes A) = \phi_M \circ (M \otimes \mu_A)$. Given two right A-modules (M, ϕ_M) and (N, ϕ_N), $f : M \to N$ is a morphism of right A-modules if $\phi_N \circ (f \otimes A) = f \circ \phi_M$. In a similar way we can define the notions of left A-module (we denote the left action by φ_M) and morphism of left A-modules.

By weak bialgebras we understand the monoidal version of the notion of weak bialgebra introduced in [4], as a generalization of classical bialgebras. Here we recall the definition.

Definition 2.1. A weak bialgebra H is an object in $\mathcal{C}$ with an algebra structure (H, η_H, μ_H) and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}$,

(a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$,
$= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$,

(a3) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$
$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.

If moreover, there exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) satisfying:

(a4) $id_H * \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,

(a5) $\lambda_H * id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,

(a6) $\lambda_H * id_H * \lambda_H = \lambda_H$,

we will say that H is a weak Hopf algebra.

If H is a weak bialgebra and we define the morphisms Π_H^L (target), Π_H^R (source), $\overline{\Pi}_H^L$ and $\overline{\Pi}_H^R$ by

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

it is straightforward to show that they are idempotent. Also, they satisfy the equalities

$$\Pi_H^L \circ \overline{\Pi}_H^L = \Pi_H^L, \quad \Pi_H^L \circ \overline{\Pi}_H^R = \overline{\Pi}_H^R, \quad \Pi_H^R \circ \overline{\Pi}_H^L = \overline{\Pi}_H^L, \quad \Pi_H^R \circ \overline{\Pi}_H^R = \Pi_H^R, \tag{2.1}$$

$$\overline{\Pi}_H^L \circ \Pi_H^L = \overline{\Pi}_H^L, \quad \overline{\Pi}_H^L \circ \Pi_H^R = \Pi_H^R, \quad \overline{\Pi}_H^R \circ \Pi_H^L = \Pi_H^L, \quad \overline{\Pi}_H^R \circ \Pi_H^R = \overline{\Pi}_H^R, \tag{2.2}$$

and moreover the identities

$$(H \otimes \Pi_H^L) \circ \delta_H \circ \Pi_H^L = \delta_H \circ \Pi_H^L, \quad (\Pi_H^R \otimes H) \circ \delta_H \circ \Pi_H^R = \delta_H \circ \Pi_H^R, \tag{2.3}$$

$$(H \otimes \overline{\Pi}_H^R) \circ \delta_H \circ \overline{\Pi}_H^R = \delta_H \circ \overline{\Pi}_H^R, \quad (\overline{\Pi}_H^L \otimes H) \circ \delta_H \circ \overline{\Pi}_H^L = \delta_H \circ \overline{\Pi}_H^L, \tag{2.4}$$

$$\mu_H \circ (H \otimes \Pi_H^L) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H), \tag{2.5}$$

$$(H \otimes \Pi_H^L) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H), \tag{2.6}$$

$$\mu_H \circ (\Pi_H^R \otimes H) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H), \tag{2.7}$$

$$(\Pi_H^R \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)), \tag{2.8}$$

$$\mu_H \circ (\overline{\Pi}_H^R \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes \delta_H), \tag{2.9}$$

$$\mu_H \circ (H \otimes \overline{\Pi}_H^L) = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (\delta_H \otimes H), \tag{2.10}$$

$$(\overline{\Pi}_H^L \otimes H) \circ \delta_H = (H \otimes \mu_H) \circ ((\delta_H \circ \eta_H) \otimes H), \tag{2.11}$$

$$(H \otimes \overline{\Pi}_H^R) \circ \delta_H = (\mu_H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)). \tag{2.12}$$

Note that by the previous identities it is easy to prove that

$$\delta_H \circ \eta_H = (\Pi_H^R \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \Pi_H^L) \circ \delta_H \circ \eta_H \tag{2.13}$$
$$= (H \otimes \overline{\Pi}_H^R) \circ \delta_H \circ \eta_H = (\overline{\Pi}_H^L \otimes H) \circ \delta_H \circ \eta_H,$$

and that

$$\varepsilon_H \circ \mu_H = \varepsilon_H \circ \mu_H \circ (\Pi_H^R \otimes H) = \varepsilon_H \circ \mu_H \circ (H \otimes \Pi_H^L) \tag{2.14}$$
$$= \varepsilon_H \circ \mu_H \circ (\overline{\Pi}_H^R \otimes H) = \varepsilon_H \circ \mu_H \circ (H \otimes \overline{\Pi}_H^L).$$

On the other hand,

$$H^{op} = (H, \eta_H, \mu_H \circ c_{H,H}, \varepsilon_H, \delta_H)$$

and

$$H^{cop} = (H, \eta_H, \mu_H, \varepsilon_H, c_{H,H} \circ \delta_H)$$

are weak bialgebras in $\mathcal{C}$. Therefore so is

$$(H^{op})^{cop} = (H, \eta_H, \mu_H \circ c_{H,H}, \varepsilon_H, c_{H,H} \circ \delta_H).$$

Finally, note that

$$\Pi_{H^{op}}^L = \overline{\Pi}_H^R, \quad \Pi_{H^{op}}^R = \overline{\Pi}_H^L, \tag{2.15}$$

and

$$\Pi_{H^{cop}}^L = \overline{\Pi}_H^L, \quad \Pi_{H^{cop}}^R = \overline{\Pi}_H^R. \tag{2.16}$$

If H is a weak Hopf algebra in $\mathcal{C}$, the antipode λ_H is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant, i.e.,

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H, \tag{2.17}$$

$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H. \tag{2.18}$$

Also the morphisms Π_H^L, Π_H^R, $\overline{\Pi}_H^L$, $\overline{\Pi}_H^R$ satisfy the equalities

$$\Pi_H^L = id_H * \lambda_H, \quad \Pi_H^R = \lambda_H * id_H, \quad \Pi_H^L * id_H = id_H, \quad \Pi_H^R * \lambda_H = \lambda_H, \tag{2.19}$$

$$\Pi_H^L = \lambda_H \circ \overline{\Pi}_H^L = \overline{\Pi}_H^R \circ \lambda_H, \quad \Pi_H^R = \overline{\Pi}_H^L \circ \lambda_H = \lambda_H \circ \overline{\Pi}_H^R. \tag{2.20}$$

If the antipode λ_H of H is an isomorphism, H^{op} and H^{cop} are weak Hopf algebras in $\mathcal{C}$ with antipode $\lambda_{H^{op}} = \lambda_{H^{cop}} = \lambda_H^{-1}$. Then, under these conditions, $(H^{op})^{cop}$ is a weak Hopf algebra with antipode $\lambda_{(H^{op})^{cop}} = \lambda_H$.

Definition 2.2. An object P in $\mathcal{C}$ is said to be finite if there exists P^* in $\mathcal{C}$ such that

$$(P \otimes -, P^* \otimes -, \alpha_P, \beta_P)$$

is an adjoint pair. In this case the object P^* will be called the dual of P.

Definition 2.3. Let H be a finite weak Hopf algebra in $\mathcal{C}$. We define the dual weak Hopf algebra of H by

$$H^* = (H^*, \eta_{H^*}, \mu_{H^*}, \varepsilon_{H^*}, \delta_{H^*}, \lambda_{H^*})$$

where

$$\eta_{H^*} = (H^* \otimes \varepsilon_H) \circ \alpha_H(K),$$

$$\mu_{H^*} = (H^* \otimes (\beta_H(K) \circ (H \otimes \beta_H(K) \otimes H^*) \circ (\delta_H \otimes H^* \otimes H^*))) \circ (\alpha_H(K) \otimes H^* \otimes H^*),$$

$$\varepsilon_{H^*} = \beta_H(K) \circ (\eta_H \otimes H^*),$$

$$\delta_{H^*} = (H^* \otimes H^* \otimes (\beta_H(K) \circ (\mu_H \otimes H^*))) \circ (((H^* \otimes \alpha_H(K) \otimes H) \circ \alpha_H(K)) \otimes H^*),$$

$$\lambda_{H^*} = (H^* \otimes \beta_H(K)) \circ (H^* \otimes \lambda_H \otimes H^*) \circ (\alpha_H(K) \otimes H^*).$$

From now on, if H is a finite weak Hopf algebra such that the antipode is an isomorphism, we will denote by $\widehat{H}$ the weak Hopf algebra $((H^*)^{op})^{cop}$. Therefore $\widehat{H}$ is defined by

$$\widehat{H} = (H^*, \eta_{\widehat{H}}, \mu_{\widehat{H}}, \varepsilon_{\widehat{H}}, \delta_{\widehat{H}}, \lambda_{\widehat{H}})$$

where, $\eta_{\widehat{H}} = \eta_{H^*}$, $\varepsilon_{\widehat{H}} = \varepsilon_{H^*}$, $\lambda_{\widehat{H}} = \lambda_{H^*}$ and

$$\mu_{\widehat{H}} = (H^* \otimes (\beta_H(K) \circ (H \otimes \beta_H(K) \otimes H^*) \circ ((c_{H,H} \circ \delta_H) \otimes H^* \otimes H^*))) \circ (\alpha_H(K) \otimes H^* \otimes H^*),$$

$$\delta_{\widehat{H}} = (H^* \otimes H^* \otimes (\beta_H(K) \circ ((\mu_H \circ c_{H,H}) \otimes H^*))) \circ (((H^* \otimes \alpha_H(K) \otimes H) \circ \alpha_H(K)) \otimes H^*).$$

Note that

$$\widehat{H}^{cop} = (H^*)^{op}. \tag{2.21}$$

Example 2.4. As group algebras and their duals are the natural examples of Hopf algebras, groupoid algebras and their duals provide examples of weak Hopf algebras. Recall that a groupoid G is simply a category in which every morphism is an isomorphism. In this example, we consider finite groupoids, i.e., groupoids with a finite number of objects. The set of objects of G will be denoted by G_0 and the set of morphisms by G_1. The identity morphism on $x \in G_0$ will also be denoted by id_x and for a morphism $\sigma : x \to y$ in G_1, we write $s(\sigma)$ and $t(\sigma)$, respectively, for the source and the target of σ.

Let G be a groupoid, and let R be a commutative ring. The groupoid algebra is the direct product

$$H = \bigoplus_{\sigma \in G_1} R\sigma$$

with the product of two morphisms being equal to their composition if the latter is defined and 0 otherwise, i.e., $\mu_H(\sigma \otimes_R \tau) = \sigma \circ \tau$ if $s(\sigma) = t(\tau)$ and $\mu_H(\sigma \otimes_R \tau) = 0$ if $s(\sigma) \neq t(\tau)$. The unit element is $1_H = \sum_{x \in G_0} id_x$. The algebra H is a cocommutative weak Hopf algebra (i.e., $c_{H,H} \circ \delta_H = \delta_H$), with coproduct δ_H, counit ε_H and antipode λ_H given by the formulas:

$$\delta_H(\sigma) = \sigma \otimes_R \sigma, \quad \varepsilon_H(\sigma) = 1, \quad \lambda_H(\sigma) = \sigma^{-1}.$$

In this case the target and source morphisms are

$$\Pi_H^L(\sigma) = id_{t(\sigma)}, \quad \Pi_H^R(\sigma) = id_{s(\sigma)}$$

and $\lambda_H \circ \lambda_H = id_H$, i.e. the antipode is involutory.

If G_1 is finite, then H is free of a finite rank as a R-module. Hence H is finite as object in the category R-Mod and

$$H^* = Hom_R(H,R) = \bigoplus_{\sigma\in G_1} Rf_\sigma$$

is a commutative weak Hopf algebra (i.e., $\mu_H \circ c_{H,H} = \mu_H$) with involutory antipode where $\beta_H(R)(\tau \otimes_R f_\sigma) = \delta_{\tau,\sigma}$.

The weak Hopf algebra structure of H^* is given by the formulas

$$1_{H^*} = \sum_{\sigma\in G_1} f_\sigma, \qquad \mu_{H^*}(f_\sigma \otimes_R f_\tau) = \delta_{\sigma,\tau} f_\sigma,$$

$$\varepsilon_{H^*}(f_\sigma) = \begin{cases} 1 & \text{if} \quad \sigma = id_x \\ 0 & \text{if} \quad \sigma \neq id_x \end{cases}, \qquad \delta_{H^*}(f_\sigma) = \sum_{s(\sigma)=s(\omega)} f_\omega \otimes_R f_{\sigma\circ\omega^{-1}},$$

$$\lambda_{H^*}(f_\sigma) = f_{\sigma^{-1}}.$$

Note that in this case $1_{\widehat{H}} = 1_{H^*}$, $\mu_{\widehat{H}} = \mu_{H^*}$, $\varepsilon_{\widehat{H}} = \varepsilon_{H^*}$, $\lambda_{\widehat{H}} = \lambda_{H^*}$ and

$$\delta_{\widehat{H}}(f_\sigma) = \sum_{s(\sigma)=s(\omega)} f_{\sigma\circ\omega^{-1}} \otimes_R f_\omega.$$

Therefore, in the conditions of this example, $\widehat{H}^{cop} = H^*$.

3. Invertible skew pairings for weak bialgebras

The notion of skew pairing for weak bialgebras was introduced in [22] as a restriction of the corresponding definition for $\times_R$-algebras discussed in Section 5 of [21]. In the following definition we introduce the notions of 1-skew pairing and 2-skew pairing for a pair of weak bialgebras. The axioms involved in the definition of 1-skew pairing are exactly the ones we can find in the classical notion of skew pairing of bialgebras (see Definition 7.7.7 of [20]). The second of these definitions, i.e. the definition of 2-skew pairing, is exactly the one introduced by P. Schauenburg in [22] with the name of skew pairing between weak bialgebras.

Definition 3.1. Let A and B be weak bialgebras in $\mathcal{C}$. We define a 1-skew pairing between A and B (or a 1-skew pairing for short) as a morphism

$$\mathfrak{s} : A\otimes B \to K$$

in $\mathcal{C}$ satisfying

$$\mathfrak{s}\circ(A\otimes\mu_B) = (\mathfrak{s}\otimes\mathfrak{s})\circ(A\otimes c_{A,B}\otimes B)\circ((c_{A,A}\circ\delta_A)\otimes B\otimes B), \tag{3.1}$$

$$\mathfrak{s}\circ(\mu_A\otimes B) = (\mathfrak{s}\otimes\mathfrak{s})\circ(A\otimes c_{A,B}\otimes B)\circ(A\otimes A\otimes\delta_B), \tag{3.2}$$

$$\mathfrak{s}\circ(\eta_A\otimes B) = \varepsilon_B, \tag{3.3}$$

$$\mathfrak{s}\circ(A\otimes\eta_B) = \varepsilon_A. \tag{3.4}$$

In the following we will denote with $\mathsf{Sk}_1(A,B)$ the set of 1-skew pairings between A and B.

Let S and T be weak bialgebras in $\mathcal{C}$. Similarly, a 2-skew pairing between S and T (or a 2-skew pairing for short) is a morphism $\mathfrak{t} : S\otimes T\to K$ in $\mathcal{C}$ satisfying

$$\mathfrak{t}\circ(\mu_S\otimes T) = (\mathfrak{t}\otimes\mathfrak{t})\circ(S\otimes c_{S,T}\otimes T)\circ(S\otimes S\otimes(c_{T,T}\circ\delta_T)), \tag{3.5}$$

$$\mathfrak{t}\circ(S\otimes\mu_T) = (\mathfrak{t}\otimes\mathfrak{t})\circ(S\otimes c_{S,T}\otimes T)\circ(\delta_S\otimes T\otimes T), \tag{3.6}$$

and (3.3), (3.4) for $A = S$ and $B = T$.

In the following we will denote with $\mathsf{Sk}_2(S,T)$ the set of 2-skew pairings between S and T.

Proposition 3.2. *Let X, Y be weak bialgebras. If $\mathfrak{s} : X\otimes Y\to K$ is in $\mathsf{Sk}_1(X,Y)$ then $\mathfrak{s}$ belongs to $\mathsf{Sk}_2(X^{cop},Y^{cop})$. Similarly, if $\mathfrak{r} : X\otimes Y\to K$ is in $\mathsf{Sk}_2(X,Y)$ then $\mathfrak{r}$ belongs to $\mathsf{Sk}_1(X^{cop},Y^{cop})$.*

Proof. The proof follows directly from Definition 3.1. □

Proposition 3.3. *Let A and B be weak bialgebras in $\mathcal{C}$ and let $\mathfrak{s} : A\otimes B\to K$ be a 1-skew pairing. The following identities hold:*

$$\mathfrak{s}\circ(\Pi_A^L\otimes B)=\mathfrak{s}\circ(A\otimes\overline{\Pi}_B^R), \tag{3.7}$$

$$\mathfrak{s}\circ(\Pi_A^R\otimes B)=\mathfrak{s}\circ(A\otimes\overline{\Pi}_B^L), \tag{3.8}$$

$$\mathfrak{s}\circ(\overline{\Pi}_A^L\otimes B)=\mathfrak{s}\circ(A\otimes\Pi_B^L), \tag{3.9}$$

$$\mathfrak{s}\circ(\overline{\Pi}_A^R\otimes B)=\mathfrak{s}\circ(A\otimes\Pi_B^R). \tag{3.10}$$

Proof. The proof for the first identity is the following:

$\mathfrak{s}\circ(\Pi_A^L\otimes B)$

$=\mathfrak{s}\circ((((\mathfrak{s}\circ(\mu_A\otimes\eta_B))\otimes A)\circ(A\otimes c_{A,A})\circ((\delta_A\circ\eta_A)\otimes A))\otimes B)$ (Definition of Π_A^L and (3.4))

$=(((\mathfrak{s}\otimes\mathfrak{s})\circ(A\otimes c_{A,B}\otimes B)\circ(A\otimes A\otimes(\delta_B\circ\eta_B)))\otimes\mathfrak{s})\circ(A\otimes c_{A,A}\otimes B)\circ((\delta_A\circ\eta_A)\otimes A\otimes B)$ ((3.2))

$=(\mathfrak{s}\otimes\mathfrak{s})\circ(A\otimes c_{A,B}\otimes B)\circ((c_{A,A}\circ\delta_A\circ\eta_A)\otimes B\otimes B)\circ(B\otimes B\otimes\mathfrak{s})\circ(B\otimes c_{A,B}\otimes B)$ $\circ(B\otimes A\otimes(\delta_B\circ\eta_B))\circ c_{A,B}$ (naturality of c)

$=\mathfrak{s}\circ(\eta_A\otimes\mu_B)\circ(B\otimes B\otimes\mathfrak{s})\circ(B\otimes c_{A,B}\otimes B)\circ(B\otimes A\otimes(\delta_B\circ\eta_B))\circ c_{A,B}$ ((3.2))

$=\mathfrak{s}\circ(A\otimes(((\varepsilon_B\circ\mu_B)\otimes B)\circ(B\otimes(\delta_B\circ\eta_B)))$ ((3.3) and naturality of c)

$=\mathfrak{s}\circ(A\otimes\overline{\Pi}_B^R)$ (definition of $\overline{\Pi}_B^R$).

On the other hand,

$\mathfrak{s}\circ(\Pi_A^R\otimes B)$

$=\mathfrak{s}\circ(((A\otimes(\mathfrak{s}\circ(\mu_A\otimes\eta_B)))\circ(c_{A,A}\otimes A)\circ(A\otimes(\delta_A\circ\eta_A)))\otimes B)$ (definition of Π_A^R and (3.4))

$=\mathfrak{s}\circ(((A\otimes\mathfrak{s}\otimes\mathfrak{s})\circ(c_{A,A}\otimes c_{A,B}\otimes B)\circ(A\otimes(\delta_A\circ\eta_A)\otimes(\delta_B\circ\eta_B)))\otimes B)$ ((3.2))

$=(\mathfrak{s}\otimes\mathfrak{s})\circ(A\otimes c_{A,B}\otimes B)\circ((c_{A,A}\circ\delta_A\circ\eta_A)\otimes B\otimes B)\circ(\mathfrak{s}\otimes B\otimes B)\circ(A\otimes(\delta_B\circ\eta_B))\otimes B)$ (naturality of c)

$=\mathfrak{s}\circ(\eta_A\otimes\mu_B)\circ(\mathfrak{s}\otimes B\otimes B)\circ(A\otimes(\delta_B\circ\eta_B))\otimes B)$ ((3.1))

$=\mathfrak{s}\circ(A\otimes((B\otimes(\varepsilon_B\circ\mu_B))\circ((\delta_B\circ\eta_B)\otimes B)))$ ((3.3))

$=\mathfrak{s}\circ(A\otimes\overline{\Pi}_B^L)$ (definition of $\overline{\Pi}_B^L$),

and then (3.8) holds. The proofs for (3.9) and (3.10) are similar and the details are left to the reader. □

Similarly, we can prove the following result.

Proposition 3.4. *Let S and T be weak bialgebras in $\mathcal{C}$ and let $\mathfrak{t}$ be a 2-skew pairing. The following identities hold:*

$$\mathfrak{t}\circ(\Pi_S^L\otimes T)=\mathfrak{t}\circ(S\otimes\overline{\Pi}_T^L), \tag{3.11}$$

$$\mathfrak{t}\circ(\Pi_S^R\otimes T)=\mathfrak{t}\circ(T\otimes\overline{\Pi}_T^R), \tag{3.12}$$

$$\mathfrak{t}\circ(\overline{\Pi}_S^L\otimes T)=\mathfrak{t}\circ(S\otimes\Pi_T^R), \tag{3.13}$$

$$\mathfrak{t}\circ(\overline{\Pi}_S^R\otimes T)=\mathfrak{t}\circ(S\otimes\Pi_T^L). \tag{3.14}$$

Definition 3.5. Let A and B be weak bialgebras in $\mathcal{C}$. If $\mathfrak{s} : A\otimes B \to K$ is a 1-skew pairing, we will say that $\mathfrak{s}$ is convolution invertible if there exists a 2-skew pairing $\mathfrak{s}^{-1} : A\otimes B \to K$ in $\mathcal{C}$ such that

$$\mathfrak{s} * \mathfrak{s}^{-1} = \mathfrak{s} \circ (\Pi_A^L \otimes B) = \mathfrak{s}^{-1} \circ (A \otimes \Pi_B^R), \tag{3.15}$$

$$\mathfrak{s}^{-1} * \mathfrak{s} = \mathfrak{s} \circ (\Pi_A^R \otimes B) = \mathfrak{s}^{-1} \circ (A \otimes \Pi_B^L), \tag{3.16}$$

$$\mathfrak{s}^{-1} * \mathfrak{s} * \mathfrak{s}^{-1} = \mathfrak{s}^{-1}. \tag{3.17}$$

Note that when $\mathfrak{s}^{-1}$ exists it is unique. Indeed, if $\mathfrak{t} : A \otimes B \to K$ is a 2-skew pairing satisfying the equalities of (3.15), (3.16) and (3.17)

$$\mathfrak{t} = (\mathfrak{t}*\mathfrak{s})*\mathfrak{t} = (\mathfrak{s}\circ(\Pi_A^R\otimes B))*\mathfrak{t} = \mathfrak{s}^{-1}*(\mathfrak{s}*\mathfrak{t}^{-1}) = \mathfrak{s}^{-1}*(\mathfrak{s}\circ(\Pi_A^L\otimes B)) = \mathfrak{s}^{-1}*(\mathfrak{s}*\mathfrak{s}^{-1}) = \mathfrak{s}^{-1}.$$

In addition,

$$\mathfrak{s} * \mathfrak{s}^{-1} * \mathfrak{s} = \mathfrak{s} \tag{3.18}$$

holds because, by (3.2) and (2.19),

$$\mathfrak{s} * \mathfrak{s}^{-1} * \mathfrak{s} = \mathfrak{s} \circ ((\Pi_A^L * id_A) \otimes B) = \mathfrak{s}.$$

Following the same pattern we can define the notion of convolution invertible 2-skew pairing.

Definition 3.6. We will say that a 2-skew pairing $\mathfrak{t} : S \otimes T \to K$ is convolution invertible if there exists a 1-skew pairing $\mathfrak{t}^{-1} : S \otimes T \to K$ in $\mathcal{C}$ such that

$$\mathfrak{t} * \mathfrak{t}^{-1} = \mathfrak{t} \circ (S \otimes \Pi_T^L) = \mathfrak{t}^{-1} \circ (\Pi_S^R \otimes T), \tag{3.19}$$

$$\mathfrak{t}^{-1} * \mathfrak{t} = \mathfrak{t} \circ (S \otimes \Pi_T^R) = \mathfrak{t}^{-1} \circ (\Pi_S^L \otimes T), \tag{3.20}$$

$$\mathfrak{t}^{-1} * \mathfrak{t} * \mathfrak{t}^{-1} = \mathfrak{t}^{-1}. \tag{3.21}$$

As in the previous case, the inverse of $\mathfrak{t}$ is unique and $\mathfrak{t} * \mathfrak{t}^{-1} * \mathfrak{t} = \mathfrak{t}$ holds.

Proposition 3.7. *Let A and B be weak bialgebras in $\mathcal{C}$ and let $\mathfrak{s}$ be a 1-skew pairing. Then, if $\mathfrak{s} : A \otimes B \to K$ is convolution invertible with inverse $\mathfrak{s}^{-1}$, $\mathfrak{s}^{-1} : A \otimes B \to K$ is a convolution invertible 2-skew pairing with inverse $(\mathfrak{s}^{-1})^{-1} = \mathfrak{s}$.*

Proof. The proof follows easily from the above definition. □

Proposition 3.8. *If A is a weak Hopf algebra with antipode λ_A and B is a weak bialgebra, then, any given 1-skew pairing $\mathfrak{s} : A\otimes B \to K$ is convolution invertible with inverse 2-skew pairing*

$$\mathfrak{s}^{-1} = \mathfrak{s} \circ (\lambda_A \otimes B). \tag{3.22}$$

Proof. Indeed, by (2.17), (3.1), (3.2), the naturality of c and $c^2 = id$ we obtain

$$\mathfrak{s}^{-1} \circ (\mu_A \otimes B) = (\mathfrak{s}^{-1} \otimes \mathfrak{s}^{-1}) \circ (A \otimes c_{A,B} \otimes B) \circ (A \otimes A \otimes (c_{B,B} \circ \delta_B)), \tag{3.23}$$

$$\mathfrak{s}^{-1} \circ (A \otimes \mu_B) = (\mathfrak{s}^{-1} \otimes \mathfrak{s}^{-1}) \circ (A \otimes c_{A,B} \otimes B) \circ (\delta_A \otimes B \otimes B), \tag{3.24}$$

and by (2.18), (3.3) and (3.4)

$$\mathfrak{s}^{-1} \circ (\eta_A \otimes B) = \varepsilon_B, \tag{3.25}$$

$$\mathfrak{s}^{-1} \circ (A \otimes \eta_B) = \varepsilon_A. \tag{3.26}$$

On the other hand, by (3.2) and the identities of (2.19) we have:

$$\mathfrak{s} * \mathfrak{s}^{-1} = \mathfrak{s} \circ ((id_A * \lambda_A) \otimes B) = \mathfrak{s} \circ (\Pi_A^L \otimes B),$$

$$\mathfrak{s}^{-1} * \mathfrak{s} = \mathfrak{s} \circ ((\lambda_A * id_A) \otimes B) = \mathfrak{s} \circ (\Pi_A^R \otimes B),$$

and

$$\mathfrak{s}^{-1} * \mathfrak{s} * \mathfrak{s}^{-1} = \mathfrak{s} \circ ((\Pi_A^R * \lambda_A) \otimes B) = \mathfrak{s} \circ (\lambda_A \otimes B) = \mathfrak{s}^{-1}.$$

Finally, by Proposition 3.4 and (2.20) the following identities

$$\mathfrak{s} \circ (\Pi_A^L \otimes B) = \mathfrak{s}^{-1} \circ (A \otimes \Pi_B^R), \tag{3.27}$$

$$\mathfrak{s}\circ(\Pi_A^R\otimes B)=\mathfrak{s}^{-1}\circ(A\otimes\Pi_B^L), \tag{3.28}$$

hold, and thus $\mathfrak{s}$ is convolution invertible with inverse $\mathfrak{s}^{-1}=\mathfrak{s}\circ(\lambda_A\otimes B)$. □

Corollary 3.9. *Let A, B be weak Hopf algebras and let $\mathfrak{s}$ be an invertible 1-skew pairing. If the antipode of B is an isomorphism, then the following identities hold:*

$$\mathfrak{s}^{-1}=\mathfrak{s}\circ(A\otimes\lambda_B^{-1}), \tag{3.29}$$

$$\mathfrak{s}=\mathfrak{s}\circ(\lambda_A\otimes\lambda_B), \tag{3.30}$$

Proof. Let us prove (3.29). Indeed, firstly $\mathfrak{s}*(\mathfrak{s}\circ(A\otimes\lambda_B^{-1}))=\mathfrak{s}*\mathfrak{s}^{-1}$ because

$\mathfrak{s}*(\mathfrak{s}\circ(A\otimes\lambda_B^{-1}))$
$=(\mathfrak{s}\otimes\mathfrak{s})\circ(A\otimes c_{A,B}\otimes B)\circ((c_{A,A}\circ\delta_A)\otimes(c_{B,B}\circ(B\otimes\lambda_B^{-1})\circ\delta_B))$ (naturality of c and $c^2=id$)
$=\mathfrak{s}\circ(A\otimes(\mu_B\circ c_{B,B}\circ(B\otimes\lambda_B^{-1})\circ\delta_B))$ ((3.1))
$=\mathfrak{s}\circ(A\otimes\Pi^L_{B^{op}})$ (definition of $\Pi^L_{B^{op}}$)
$=\mathfrak{s}\circ(A\otimes\overline{\Pi}^R_B)$ ((2.15))
$=\mathfrak{s}\circ(\Pi_A^L\otimes B)$ ((3.7))
$=\mathfrak{s}*\mathfrak{s}^{-1}$ ((3.15)).

We continue in this fashion to obtain the identity $(\mathfrak{s}\circ(A\otimes\lambda_B^{-1}))*\mathfrak{s}=\mathfrak{s}^{-1}*\mathfrak{s}$:

$(\mathfrak{s}\circ(A\otimes\lambda_B^{-1}))*\mathfrak{s}$
$=(\mathfrak{s}\otimes\mathfrak{s})\circ(A\otimes c_{A,B}\otimes B)\circ((c_{A,A}\circ\delta_A)\otimes(c_{B,B}\circ(\lambda_B^{-1}\otimes B)\circ\delta_B))$ (naturality of c and $c^2=id$)
$=\mathfrak{s}\circ(A\otimes(\mu_B\circ c_{B,B}\circ(\lambda_B^{-1}\otimes B)\circ\delta_B))$ ((3.1))
$=\mathfrak{s}\circ(A\otimes\Pi^R_{B^{op}})$ (definition of $\Pi^R_{B^{op}}$)
$=\mathfrak{s}\circ(A\otimes\overline{\Pi}^L_B)$ ((2.16))
$=\mathfrak{s}\circ(\Pi_A^R\otimes B)$ ((3.8))
$=\mathfrak{s}^{-1}*\mathfrak{s}$ ((3.16)).

On the other hand,

$(\mathfrak{s}\circ(A\otimes\lambda_B^{-1}))*\mathfrak{s}*(\mathfrak{s}\circ(A\otimes\lambda_B^{-1}))$
$=(\mathfrak{s}\circ(\Pi_A^R\otimes B))*(\mathfrak{s}\circ(A\otimes\lambda_B^{-1}))$ ($(\mathfrak{s}\circ(A\otimes\lambda_B^{-1}))*\mathfrak{s}=\mathfrak{s}\circ(\Pi_A^R\otimes B)$)
$=(\mathfrak{s}\otimes\mathfrak{s})\circ(A\otimes c_{A,B}\otimes B)\circ(\delta_A\otimes((\overline{\Pi}^L_B\otimes\lambda_B^{-1})\circ\delta_B))$ ((3.8) and naturality of c)
$=(\mathfrak{s}\otimes\mathfrak{s})\circ(A\otimes c_{A,B}\otimes B)\circ((c_{A,A}\circ\delta_A)\otimes(c_{B,B}\circ(\overline{\Pi}^L_B\otimes\lambda_B^{-1})\circ\delta_B))$ (naturality of c and $c^2=id$)
$=\mathfrak{s}\circ(A\otimes(c_{B,B}\circ(\overline{\Pi}^L_B\otimes\lambda_B^{-1})\circ\delta_B))$ ((3.1))
$=\mathfrak{s}\circ(A\otimes(\Pi^R_{B^{op}}*\lambda_{B^{op}}))$ ((2.16))
$=\mathfrak{s}\circ(A\otimes\lambda_{B^{op}})$ ((2.19) for B^{op})
$=\mathfrak{s}\circ(A\otimes\lambda_B^{-1})$ (definition of $\lambda_{B^{op}}$).

Therefore,

$$\mathfrak{s}\circ(A\otimes\lambda_B^{-1})=(\mathfrak{s}\circ(A\otimes\lambda_B^{-1}))*\mathfrak{s}*(\mathfrak{s}\circ(A\otimes\lambda_B^{-1}))=(\mathfrak{s}\circ(A\otimes\lambda_B^{-1}))*\mathfrak{s}*\mathfrak{s}^{-1}=\mathfrak{s}^{-1}*\mathfrak{s}*\mathfrak{s}^{-1}=\mathfrak{s}^{-1}.$$

Finally, the proof of (3.30) folllows from (3.29) because

$$\mathfrak{s}\circ(\lambda_A\otimes\lambda_B)=\mathfrak{s}^{-1}\circ(A\otimes\lambda_B)=\mathfrak{s}\circ(A\otimes(\lambda_B^{-1}\circ\lambda_B))=\mathfrak{s}.$$

□

Proposition 3.10. *If T is a weak Hopf algebra with antipode λ_T and S is a weak bialgebra any 2-skew pairing $\mathfrak{t}:S\otimes T\to K$ is convolution invertible with inverse 1-skew pairing*

$$\mathfrak{t}^{-1}=\mathfrak{t}\circ(S\otimes\lambda_T). \tag{3.31}$$

Proof. The proof is similar to the one developed for Proposition 3.10. □

As a consequence of the previous proposition, we have the 2-skew version of Corollary 3.9.

Corollary 3.11. *Let S, T be weak Hopf algebras and let $\mathfrak{t}$ be an invertible 2-skew pairing. If the antipode of S is an isomorphism, then the following identities hold:*

$$\mathfrak{t}^{-1} = \mathfrak{t} \circ (\lambda_S^{-1} \otimes T), \tag{3.32}$$

$$\mathfrak{t} = \mathfrak{t} \circ (\lambda_S \otimes \lambda_T). \tag{3.33}$$

Example 3.12. Let H be a finite weak Hopf algebra in $\mathcal{C}$ satisfying that the antipode is an isomorphism. The morphism

$$\mathfrak{s} = \beta_H(K) \circ c_{H^*,H} : \widehat{H}^{cop} \otimes H \to K$$

is an invertible 1-skew pairing. Indeed, first note that by the naturality of the braiding, $c^2 = id$ and the properties of the adjunction the following identity holds:

$$c_{H^*,H} = (H \otimes H^* \otimes \mathfrak{s}) \circ (H \otimes c_{H^*,H^*} \otimes H) \otimes ((c_{H^*,H} \circ \alpha_H) \otimes H^* \otimes H). \tag{3.34}$$

Then,

$$\begin{aligned}
&(\mathfrak{s} \otimes \mathfrak{s}) \circ (H^* \otimes c_{H^*,H} \otimes H) \circ ((c_{H^*,H^*} \circ \delta_{\widehat{H}^{cop}}) \otimes H \otimes H) \\
&= (\beta_H(K) \otimes \mathfrak{s}) \circ (H \otimes \delta_{\widehat{H}} \otimes H) \circ (c_{H^*,H} \otimes H) \ \text{(naturality of } c \text{ and definition of } \delta_{\widehat{H}}) \\
&= \beta_H(K) \circ (\mu_H \otimes H^*) \circ (H \otimes ((H \otimes H^* \otimes \mathfrak{s}) \circ (H \otimes c_{H^*,H^*} \otimes H) \otimes ((c_{H^*,H} \circ \alpha_H) \otimes H^* \otimes H))) \\
&\quad \circ (c_{H^*,H} \otimes H) \ \text{(naturality of the braiding and properties of the adjunction)} \\
&= \beta_H(K) \circ (\mu_H \otimes H^*) \circ (H \otimes c_{H^*,H}) \circ (c_{H^*,H} \otimes H) \ ((3.34)) \\
&= \mathfrak{s} \circ (H^* \otimes \mu_H) \ \text{(naturality of } c),
\end{aligned}$$

and (3.1) holds. On the other hand, $\mathfrak{s}$ satisfies (3.2) because if

$$p = (\mathfrak{s} \otimes \mathfrak{s}) \circ (H^* \otimes c_{H^*,H} \otimes H) \circ (H^* \otimes H^* \otimes (c_{H,H} \circ \delta_H))$$

we have

$$\begin{aligned}
&\mathfrak{s} \circ (\mu_{\widehat{H}^{cop}} \otimes H) \\
&= \mathfrak{s} \circ (H^* \otimes p \otimes H) \circ (c_{H^*,H^*} \otimes H^* \otimes H \otimes H) \circ (H^* \otimes c_{H^*,H^*} \otimes H \otimes H) \\
&\quad \circ (c_{H^*,H^*} \otimes \alpha_H(K) \otimes H) \text{(naturality of } c \text{ and } c^2 = id) \\
&= (\mathfrak{s} \otimes p) \circ (H^* \otimes c_{H^*,H} \otimes H^* \otimes H) \circ (c_{H^*,H^*} \otimes c_{H^*,H} \otimes H) \circ (H^* \otimes c_{H^*,H^*} \otimes c_{H,H}) \\
&\quad \circ (c_{H^*,H^*} \otimes \alpha_H(K) \otimes H) \ \text{(naturality of } c) \\
&= p \circ (c_{H^*,H^*} \otimes ((\mathfrak{s} \otimes H) \circ (H^* \otimes c_{H,H}) \circ (\alpha_H(K) \otimes H)) \ \text{(naturality of } c) \\
&= (\mathfrak{s} \otimes \mathfrak{s}) \circ (H^* \otimes c_{H^*,H} \otimes H) \circ (c_{H^*,H^*} \otimes (c_{H,H} \circ \delta_H)) \ \text{(naturality of } c \text{ and properties of the} \\
&\quad \text{adjunction)} \\
&= (\mathfrak{s} \otimes \mathfrak{s}) \circ c_{H^* \otimes H, H^* \otimes H} \circ (H^* \otimes c_{H^*,H} \otimes H) \circ (H^* \otimes H^* \otimes \delta_H) \ \text{(naturality of } c \text{ and } c^2 = id) \\
&= (\mathfrak{s} \otimes \mathfrak{s}) \circ (H^* \otimes c_{H^*,H} \otimes H) \circ (H^* \otimes H^* \otimes \delta_H) \ \text{(naturality of } c).
\end{aligned}$$

Finally, the equalities (3.3) and (3.4) follow easily. By Proposition 3.8 the inverse of $\mathfrak{s}$ is

$$\mathfrak{s}^{-1} = \beta_H(K) \circ c_{H^*,H} \circ (\lambda_{\widehat{H}^{cop}} \otimes H)$$

and then

$$\mathfrak{s}^{-1} = \beta_H(K) \circ c_{H^*,H} \circ (H^* \otimes \lambda_H^{-1}).$$

In the particular case of the groupoid algebra (see Example 2.4) we have

$$\mathfrak{s}(f_\sigma \otimes_R \sigma) = \delta_{\sigma,\tau}, \quad \mathfrak{s}^{-1}(f_\sigma \otimes_R \sigma) = \delta_{\sigma^{-1},\tau}.$$

Proposition 3.13. *Let X, Y be weak bialgebras. If $\mathfrak{s} : X \otimes Y \to K$ is a 1-skew pairing, the morphism $\mathfrak{r} = \mathfrak{s} \circ c_{Y,X} : Y \otimes X \to K$ is a 2-skew pairing. Then the map*

$$F : \mathsf{Sk}_1(X, Y) \to \mathsf{Sk}_2(Y, X)$$

defined by $F(\mathfrak{s}) = \mathfrak{r}$ is a bijection with inverse $F^{-1}(\mathfrak{t}) = \mathfrak{t} \circ c_{X,Y}$. Moreover, if $\mathfrak{s}$ is convolution invertible with inverse $\mathfrak{s}^{-1}$, so is $\mathfrak{r}$ with inverse $\mathfrak{r}^{-1} = \mathfrak{s}^{-1} \circ c_{Y,X}$.

Proof. The proof follows easily from the definitions, the naturality of c and $c^2 = id$. □

Example 3.14. By example 3.12 we know that if H be a finite weak Hopf algebra in $\mathcal{C}$ satisfying that the antipode is an isomorphism, the morphism $\mathfrak{s} = \beta_H(K) \circ c_{H^*,H} : \widehat{H}^{cop} \otimes H \to K$ belongs to $\mathsf{Sk}_1(\widehat{H}^{cop}, H)$. Therefore, by the previous result $\mathfrak{r} = \beta_H(K)$ is in $\mathsf{Sk}_2(H, \widehat{H}^{cop})$.

4. Invertible skew pairings and double crossed products

In this section we prove that invertible 1-skew pairings induce examples of weak distributive laws (see [5], [23]) and therefore weak wreath products. Also if the skew pairing has inverse we will show that it is possible to construct a weakly comonoidal mutually weak inverse pair of weak distributive laws. Then, by the results proved in [3], we have a weak wreath product that becomes a weak bialgebra with respect to the tensor product coalgebra structure. As a particular case, we will show that the Drinfel'd double of a finite weak Hopf algebra can be constructed using the weak wreath product associated to an invertible skew pairing.

Definition 4.1. Let A and B be algebras in $\mathcal{C}$. A weak distributive law in $\mathcal{C}$ is a morphism

$$\Psi : A \otimes B \to B \otimes A$$

in $\mathcal{C}$ subject to the following conditions:

$$\Psi \circ (\mu_A \otimes B) = (B \otimes \mu_A) \circ (\Psi \otimes A) \circ (A \otimes \Psi), \tag{4.1}$$

$$\Psi \circ (\eta_A \otimes B) = (\mu_B \otimes A) \circ (B \otimes \Psi) \circ (B \otimes \eta_A \otimes \eta_B), \tag{4.2}$$

$$\Psi \circ (A \otimes \mu_B) = (\mu_B \otimes A) \circ (B \otimes \Psi) \circ (\Psi \otimes B), \tag{4.3}$$

$$\Psi \circ (A \otimes \eta_B) = (B \otimes \mu_A) \circ (\Psi \otimes A) \circ (\eta_A \otimes \eta_B \otimes A). \tag{4.4}$$

It is a well-known fact that (4.2) and (4.4) can be replaced by

$$(B \otimes \mu_A) \circ (\Psi \otimes A) \circ (\eta_A \otimes B \otimes A) = (\mu_B \otimes A) \circ (B \otimes \Psi) \circ (B \otimes A \otimes \eta_B). \tag{4.5}$$

As a consequence, the morphism

$$\nabla_\Psi : B \otimes A \to A \otimes B$$

defined by

$$\nabla_\Psi = (B \otimes \mu_A) \circ (\Psi \otimes A) \circ (\eta_A \otimes B \otimes A)$$

or, thanks to (4.5), by

$$\nabla_\Psi = (\mu_B \otimes A) \circ (B \otimes \Psi) \circ (B \otimes A \otimes \eta_B),$$

is idempotent. Note that by (4.1)

$$\nabla_\Psi \circ \Psi = \Psi \tag{4.6}$$

holds for any weak distributive law Ψ.

In the following we will denote by $B \times A$ the image of ∇_Ψ and by $p_\Psi : B \otimes A \to B \times A$, $i_\Psi : B \times A \to B \otimes A$ the morphisms such that $i_\Psi \circ p_\Psi = \nabla_\Psi$ and $p_\Psi \circ i_\Psi = id_{B\times A}$.

On the other hand, if $\Psi : A \otimes B \to B \otimes A$ is a weak distributive law, the object $B \times A$ is an algebra with the weak wreath product defined by

$$\mu_{B\times A} = p_\Psi \circ (\mu_B \otimes \mu_A) \circ (B \otimes \Psi \otimes A) \circ (i_\Psi \otimes i_\Psi)$$

and unit $\eta_{B\times A} = p_\Psi \circ (\eta_B \otimes \eta_A)$. This kind of product is an example of weak crossed product in the sense of [9].

Definition 4.2. We will say that a weak distributive law $\Psi : A\otimes B\to B\otimes A$ in $\mathcal{C}$ has a weak inverse if there exists a weak distributive law $\Phi : B\otimes A\to A\otimes B$ in $\mathcal{C}$ satisfying

$$\nabla_{\Psi} = \Psi\circ\Phi, \tag{4.7}$$

$$\nabla_{\Phi} = \Phi\circ\Psi. \tag{4.8}$$

If Ψ admits Φ as a weak inverse we have the identities:

$$\Phi\circ\Psi\circ\Phi = \Phi, \quad \Psi\circ\Phi\circ\Psi = \Psi, \tag{4.9}$$

and then

$$\Phi\circ\nabla_{\Psi} = \Phi, \quad \Psi\circ\nabla_{\Phi} = \Psi, \tag{4.10}$$

hold.

The following notion was introduced in [3] and was the link beteween the weak wreath product and the tensor coproduct to obtain a tensor product weak bialgebra.

Definition 4.3. Let A, B be algebras-coalgebras in $\mathcal{C}$. Let (Ψ,Φ) be a pair of weak distributive laws such that $\Phi : B\otimes A\to A\otimes B$ is a weak inverse for $\Psi : A\otimes B\to B\otimes A$. We say that (Ψ,Φ) is weakly comonoidal if the following identities hold:

$$(\nabla_{\Psi}\otimes B\otimes A)\circ\delta_{B\otimes A}\circ\Psi = (\Psi\otimes\Psi)\circ\delta_{A\otimes B} = (B\otimes A\otimes\nabla_{\Psi})\circ\delta_{B\otimes A}\circ\Psi, \tag{4.11}$$

$$(\nabla_{\Phi}\otimes A\otimes B)\circ\delta_{A\otimes B}\circ\Phi = (\Phi\otimes\Phi)\circ\delta_{B\otimes A} = (A\otimes B\otimes\nabla_{\Phi})\circ\delta_{A\otimes B}\circ\Phi, \tag{4.12}$$

$$(\varepsilon_B\otimes\varepsilon_A)\circ\Psi = (\varepsilon_A\otimes\varepsilon_B)\circ\nabla_{\Phi}. \tag{4.13}$$

Note that (4.13) is equivalent to

$$(\varepsilon_A\otimes\varepsilon_B)\circ\Phi = (\varepsilon_B\otimes\varepsilon_A)\circ\nabla_{\Psi}. \tag{4.14}$$

Remark 4.4. Note that if (Ψ,Φ) is a pair of weak distributive laws such that Φ is a weak inverse for Ψ, then (Φ,Ψ) is a pair of weak distributive laws such that Ψ is a weak inverse for Φ. Thus (Ψ,Φ) is weakly comonoidal if and only if so is (Φ,Ψ).

By Theorem 4 of [3], if A and B are weak bialgebras in $\mathcal{C}$ and (Ψ,Φ) is a pair of weakly comonoidal weak distributive laws, the object $B\times A$ is a weak bialgebra where the algebra structure is the one defined by the weak wreath product and the coalgebra structure has coproduct

$$\delta_{B\times A} = (p_{\Psi}\otimes p_{\Psi})\circ\delta_{B\otimes A}\circ i_{\Psi}$$

and counit

$$\varepsilon_{B\times A} = (\varepsilon_B\otimes\varepsilon_A)\circ i_{\Psi}.$$

Moreover, if A and B are weak Hopf algebras, $B\times A$ is a weak Hopf algebra (see Theorem 5 of [3]) with antipode

$$\lambda_{B\times A} = p_{\Psi}\circ\Psi\circ(\lambda_A\otimes\lambda_B)\circ c_{B,A}\circ i_{\Psi}.$$

Lemma 4.5. *Let X, Y be weak bialgebras and let $\mathfrak{u} : X\otimes Y\to K$ be a convolution invertible 1-skew pairing. The following assertions hold.*

(i) $(X,\phi_X^{\mathfrak{u}} = (X\otimes\mathfrak{u})\circ(\delta_X\otimes Y))$ *is a right Y-module.*
(ii) $(X,\phi_X^{\mathfrak{u}^{-1}} = (X\otimes\mathfrak{u}^{-1})\circ(\delta_X\otimes Y))$ *is a right Y^{op}-module.*
(iii) $(Y,\varphi_Y^{\mathfrak{u}} = (\mathfrak{u}\otimes Y)\circ(X\otimes\delta_Y))$ *is a left X^{op}-module.*
(iv) $(Y,\varphi_Y^{\mathfrak{u}^{-1}} = (\mathfrak{u}^{-1}\otimes Y)\circ(X\otimes\delta_Y))$ *is a left X-module.*
(v) $\varphi_Y^{\mathfrak{u}}\circ(\phi_X^{\mathfrak{u}}\otimes Y) = (\mathfrak{u}\otimes Y)\circ(X\otimes((\mu_Y\otimes Y)\circ(Y\otimes\delta_Y)))$.
(vi) $\overline{\Pi}_Y^R = \varphi_Y^{\mathfrak{u}}\circ(\phi_X^{\mathfrak{u}}\otimes Y)\circ(\eta_X\otimes Y\otimes\eta_Y)$.
(vii) $\phi_X^{\mathfrak{u}^{-1}}\circ(X\otimes\varphi_Y^{\mathfrak{u}^{-1}}) = (X\otimes\mathfrak{u}^{-1})\circ(((X\otimes\mu_X)\circ(\delta_X\otimes X))\otimes Y)$.
(viii) $\overline{\Pi}_X^L = \phi_X^{\mathfrak{u}^{-1}}\circ(X\otimes\varphi_Y^{\mathfrak{u}^{-1}})\circ(\eta_X\otimes X\otimes\eta_Y)$.
(ix) $\mu_X\circ(\phi_X^{\mathfrak{u}}\otimes\phi_X^{\mathfrak{u}})\circ(X\otimes c_{X,Y}\otimes Y)\circ(X\otimes X\otimes\delta_Y) = \phi_X^{\mathfrak{u}}\circ(\mu_X\otimes Y)$.
(x) $\mu_Y\circ(\varphi_Y^{\mathfrak{u}^{-1}}\otimes\varphi_Y^{\mathfrak{u}^{-1}})\circ(X\otimes c_{X,Y}\otimes Y)\circ(\delta_X\otimes Y\otimes Y) = \varphi_Y^{\mathfrak{u}^{-1}}\circ(X\otimes\mu_Y)$.

(xi) $\phi_X^{\mathfrak{u}^{-1}} \circ (\phi_X^{\mathfrak{u}} \otimes Y) = (\mathfrak{u}^{-1} \otimes \mathfrak{u}) \circ (X \otimes c_{X,Y} \otimes Y) \circ (((X \otimes \delta_X) \circ \delta_X) \otimes c_{Y,Y})$.

(xii) $\phi_X^{\mathfrak{u}} \circ (\phi_X^{\mathfrak{u}^{-1}} \otimes Y) = (\mathfrak{u} \otimes \mathfrak{u}^{-1}) \circ (X \otimes c_{X,Y} \otimes Y) \circ (((X \otimes \delta_X) \circ \delta_X) \otimes c_{Y,Y})$

(xiii) $\varphi_Y^{\mathfrak{u}^{-1}} \circ (X \otimes \varphi_Y^{\mathfrak{u}}) = (\mathfrak{u} \otimes \mathfrak{u}^{-1}) \circ (X \otimes c_{X,Y} \otimes Y) \circ (c_{X,X} \otimes ((Y \otimes \delta_Y) \circ \delta_Y))$.

(xiv) $\varphi_Y^{\mathfrak{u}} \circ (X \otimes \varphi_Y^{\mathfrak{u}^{-1}}) = (\mathfrak{u}^{-1} \otimes \mathfrak{u}) \circ (X \otimes c_{X,Y} \otimes Y) \circ (c_{X,X} \otimes ((Y \otimes \delta_Y) \circ \delta_Y))$.

Proof. We begin by proving (i). First note that by (3.4) for $\mathfrak{u}$ we have

$$\phi_X^{\mathfrak{u}} \circ (X \otimes \eta_Y) = (X \otimes (\mathfrak{u} \circ (X \otimes \eta_Y))) \circ \delta_X = (X \otimes \varepsilon_X) \circ \delta_X = id_X.$$

Also,

$$\begin{aligned}
&\phi_X^{\mathfrak{u}} \circ (X \otimes \mu_Y) \\
&= (X \otimes ((\mathfrak{u} \otimes \mathfrak{u}) \circ (X \otimes c_{X,Y} \otimes Y) \circ ((c_{X,X} \circ \delta_X) \otimes Y \otimes Y))) \circ (\delta_X \otimes Y \otimes Y) \text{ ((3.1) for } \mathfrak{u}) \\
&= \phi_X^{\mathfrak{u}} \circ (\phi_X^{\mathfrak{u}} \otimes Y) \text{ (coassociativity of } \delta_X \text{ and naturality of } c).
\end{aligned}$$

Therefore $(X, \phi_X^{\mathfrak{u}})$ is a right Y-module and (i) holds. On the other hand, by (3.4) for $\mathfrak{u}^{-1}$ we obtain

$$\phi_X^{\mathfrak{u}^{-1}} \circ (X \otimes \eta_Y) = id_X.$$

Moreover,

$$\begin{aligned}
&\phi_X^{\mathfrak{u}^{-1}} \circ (X \otimes \mu_Y^{op}) \\
&= (X \otimes ((\mathfrak{u}^{-1} \otimes \mathfrak{u}^{-1}) \circ (X \otimes c_{X,Y} \otimes Y) \circ (\delta_X \otimes Y \otimes Y))) \circ (\delta_X \otimes c_{Y,Y}) \text{ ((3.6) for } \mathfrak{u}^{-1}) \\
&= (X \otimes ((\mathfrak{u}^{-1} \otimes \mathfrak{u}^{-1}) \circ (X \otimes c_{X,Y} \otimes Y) \circ ((c_{X,X} \circ \delta_X) \otimes Y \otimes Y))) \circ (\delta_X \otimes Y \otimes Y) \\
&\quad \text{(naturality of } c \text{ and } c^2 = id) \\
&= \phi_X^{\mathfrak{u}^{-1}} (\phi_X^{\mathfrak{u}^{-1}} \otimes Y) \text{ (coassociativity of } \delta_X \text{ and naturality of } c).
\end{aligned}$$

Then, (ii) holds. Similarly we can prove (iii) and (iv). The proof for (v) is the following:

$$\begin{aligned}
&\varphi_Y^{\mathfrak{u}} \circ (\phi_X^{\mathfrak{u}} \otimes Y) \\
&= (((\mathfrak{u} \otimes \mathfrak{u}) \circ (X \otimes c_{X,Y} \otimes Y) \circ ((c_{X,X} \circ \delta_X) \otimes Y \otimes Y)) \otimes Y) \circ (X \otimes Y \otimes \delta_Y) \text{ (naturality of } c) \\
&= (\mathfrak{u} \otimes Y) \circ (X \otimes ((\mu_Y \otimes Y) \circ (Y \otimes \delta_Y))) \text{ ((3.2) for } \mathfrak{u}).
\end{aligned}$$

As a consequence, composing with $\eta_X \otimes Y \otimes Y$ in the two sides of v), by (3.3) for $\mathfrak{u}$, we have

$$\varphi_Y^{\mathfrak{r}} \circ ((\phi_X^{\mathfrak{u}} \circ (\eta_X \otimes Y)) \otimes Y) = ((\varepsilon_Y \circ \mu_Y) \otimes Y) \circ (Y \otimes \delta_Y)$$

and then, by (2.9) for Y,

$$\varphi_Y^{\mathfrak{r}} \circ ((\phi_X^{\mathfrak{u}} \circ (\eta_X \otimes Y)) \otimes Y) = \mu_Y \circ (\overline{\Pi}_Y^R \otimes Y)$$

holds. Thus, composing in this last equality with $Y \otimes \eta_Y$ we obtain (vi). The proofs of (vii) and (viii) follow a similar pattern and we left the details to the reader.

The equality (ix) follows by

$$\begin{aligned}
&\mu_X \circ (\phi_X^{\mathfrak{u}} \otimes \phi_X^{\mathfrak{u}}) \circ (X \otimes c_{X,Y} \otimes Y) \circ (X \otimes X \otimes \delta_Y) \\
&= (\mu_X \otimes \mathfrak{u} \otimes \mathfrak{u}) \circ (X \otimes c_{X,X} \otimes c_{X,Y} \otimes Y) \circ (\delta_X \otimes \delta_X \otimes \delta_Y) \text{ (naturality of } c) \\
&= (X \otimes \mathfrak{u}) \circ (((\mu_X \otimes \mu_X) \circ \delta_{X \otimes X}) \otimes Y) \text{ ((3.2) for } \mathfrak{u}) \\
&= \phi_X^{\mathfrak{u}} \circ (\mu_X \otimes Y) \text{ ((a1) of Definition 2.1 for } X).
\end{aligned}$$

and by a similar proof we get (x). Finally the identities (xi)-(xiv) follow from the naturality of c and the coassociativity of δ_X and δ_Y. □

Theorem 4.6. *Let X, Y be weak bialgebras and let $\mathfrak{u} : X \otimes Y \to K$ be a convolution invertible 1-skew pairing. Then,*

$$\Psi = (\varphi_Y^{\mathfrak{u}^{-1}} \otimes \phi_X^{\mathfrak{u}}) \circ \delta_{X \otimes Y} : X \otimes Y \to Y \otimes X$$

is a weak distributive law.

Proof. We first prove (4.1). Indeed,

$$(Y \otimes \mu_X) \circ (\Psi \otimes X) \circ (X \otimes \Psi)$$

$= (\varphi_Y^{\mathfrak{u}^{-1}} \otimes (\mu_X \circ (\phi_X^{\mathfrak{u}} \otimes X))) \circ (X \otimes c_{X,Y} \otimes Y \otimes X) \circ (\delta_X \otimes (\delta_Y \circ \varphi_Y^{\mathfrak{u}^{-1}}) \otimes \phi_X^{\mathfrak{u}}) \circ (X \otimes \delta_{X\otimes Y})$ (definition of Ψ)
$= (\varphi_Y^{\mathfrak{u}^{-1}} \otimes (\mu_X \circ (\phi_X^{\mathfrak{u}} \otimes X))) \circ (X \otimes c_{X,Y} \otimes Y \otimes X) \circ (\delta_X \otimes ((\varphi_Y^{\mathfrak{u}^{-1}} \otimes Y) \circ (X \otimes \delta_Y)) \otimes \phi_X^{\mathfrak{u}}) \circ (X \otimes \delta_{X\otimes Y})$ (coassociativity of δ_Y)
$= ((\varphi_Y^{\mathfrak{u}^{-1}} \circ (\mu_X \otimes Y)) \otimes (\mu_X \circ (\phi_X^{\mathfrak{u}} \otimes \phi_X^{\mathfrak{u}}) \circ (X \otimes c_{X,Y} \otimes Y) \circ (X \otimes X \otimes \delta_Y))) \circ \delta_{X\otimes X\otimes Y}$ (naturality of c, coassociativity of δ_Y and iv) of Lemma 4.5)
$= ((\varphi_Y^{\mathfrak{u}^{-1}} \circ (\mu_X \otimes Y)) \otimes (\phi_X^{\mathfrak{u}} \circ (\mu_X \otimes Y))) \circ \delta_{X\otimes X\otimes Y}$ ((ix) of Lemma 4.5)
$= (\varphi_Y^{\mathfrak{u}^{-1}} \otimes \phi_X^{\mathfrak{u}}) \circ (X \otimes c_{X,Y} \otimes Y) \circ (((\mu_X \otimes \mu_X) \circ \delta_{X\otimes X}) \otimes \delta_Y)$ (naturality of c)
$= \Psi \circ (\mu_X \otimes Y)$ ((a1) of Definition 2.1 for X).

The proof for (4.2) follows a similar pattern changing the coassociativity of δ_Y by the coassociativity of δ_X, (iv) of Lemma 4.5 by (i), (ix) of Lemma 4.5 by (x) and Definition 2.1 for X and Y.

To prove (4.3) and (4.4), first note that, by the naturality of c, $c^2 = id$ and the coassociativity of δ_Y, the identity

$$\Psi = (\mathfrak{u}^{-1} \otimes ((c_{X,Y} \otimes \mathfrak{u}) \circ \delta_{X\otimes Y})) \circ \delta_{X\otimes Y} \tag{4.15}$$

holds. Then, we get (4.3) because:

$(\mu_Y \otimes X) \circ (Y \otimes \Psi) \circ (Y \otimes \eta_X \otimes \eta_Y)$
$= (\mathfrak{u}^{-1} \otimes \mu_Y \otimes X) \circ (X \otimes c_{Y,Y} \otimes c_{X,Y} \otimes \mathfrak{u}) \circ (c_{Y,X} \otimes c_{X,Y} \otimes c_{X,Y} \otimes Y)$
$\circ (Y \otimes X \otimes X \otimes c_{X,Y} \otimes Y \otimes Y \otimes Y) \circ (Y \otimes ((X \otimes \delta_X) \circ \delta_X \circ \eta_X) \otimes ((Y \otimes \mu_Y \otimes Y)$
$\circ ((\delta_Y \circ \eta_Y) \otimes (\delta_Y \circ \eta_Y))))$ ((4.15), naturality of c and (a3) of Definition 2.1)
$= (\mathfrak{u}^{-1} \otimes c_{X,Y}) \otimes (X \otimes c_{X,Y} \otimes \mathfrak{u} \otimes Y) \circ (X \otimes X \otimes c_{X,Y} \otimes c_{Y,Y})$
$\circ (X \otimes X \otimes X \otimes ((Y \otimes (\mu_Y \circ (Y \otimes \mu_Y)) \otimes Y) \circ (c_{Y,Y} \otimes Y \otimes Y \otimes Y)$
$\circ (Y \otimes ((\delta_Y \circ \eta_Y) \otimes (\delta_Y \circ \eta_Y))))) \circ (X \otimes X \otimes c_{Y,X}) \circ ((X \otimes c_{Y,X} \otimes X) \circ (c_{Y,X} \otimes X \otimes X)$
$\circ (Y \otimes ((X \otimes \delta_X) \circ \delta_X \circ \eta_X))$ (naturality of c and $c^2 = id$)
$= (\mathfrak{u}^{-1} \otimes c_{X,Y}) \otimes (X \otimes c_{X,Y} \otimes \mathfrak{u} \otimes Y) \circ (X \otimes X \otimes c_{X,Y} \otimes c_{Y,Y})$
$\circ (X \otimes X \otimes X \otimes ((Y \otimes (\mu_Y \circ (\mu_Y \otimes Y)) \otimes Y) \circ (c_{Y,Y} \otimes Y \otimes Y \otimes Y)$
$\circ (Y \otimes ((\delta_Y \circ \eta_Y) \otimes (\delta_Y \circ \eta_Y))))) \circ (X \otimes X \otimes c_{Y,X}) \circ (X \otimes c_{Y,X} \otimes X) \circ (c_{Y,X} \otimes X \otimes X)$
$\circ (Y \otimes ((X \otimes \delta_X) \circ \delta_X \circ \eta_X))$ (associativity of μ_Y)
$= (\mathfrak{u}^{-1} \otimes c_{X,Y}) \otimes (X \otimes c_{X,Y} \otimes \mathfrak{u} \otimes Y) \circ (X \otimes X \otimes c_{X,Y} \otimes c_{Y,Y}) \circ$
$(X \otimes X \otimes X \otimes ((\Pi_Y^R \otimes ((Y \otimes \overline{\Pi}_Y^R) \circ \delta_Y)) \circ \delta_Y)) \circ (X \otimes X \otimes c_{Y,X}) \circ (X \otimes c_{Y,X} \otimes X)$
$\circ (c_{Y,X} \otimes X \otimes X) \circ (Y \otimes ((X \otimes \delta_X) \circ \delta_X \circ \eta_X))$ ((2.8) and (2.12))
$= ((\mathfrak{u}^{-1} \circ (X \otimes \Pi_Y^R)) \otimes c_{X,Y}) \otimes (X \otimes c_{X,Y} \otimes (\mathfrak{u} \circ (X \otimes \overline{\Pi}_Y^R)) \otimes Y) \circ (X \otimes X \otimes c_{X,Y} \otimes c_{Y,Y})$
$\circ (((X \otimes \delta_X) \circ \delta_X \circ \eta_X) \otimes ((Y \otimes \delta_Y) \circ \delta_Y))$ (naturality of c)
$= ((\mathfrak{u}^{-1} \circ (\overline{\Pi}_X^L \otimes Y)) \otimes c_{X,Y}) \otimes (X \otimes c_{X,Y} \otimes (\mathfrak{u} \circ (\Pi_X^L \otimes Y)) \otimes Y) \circ (X \otimes X \otimes c_{X,Y} \otimes c_{Y,Y})$
$\circ (((X \otimes \delta_X) \circ \delta_X \circ \eta_X) \otimes ((Y \otimes \delta_Y) \circ \delta_Y))$ ((3.7) and (3.13))
$= (\mathfrak{u}^{-1} \otimes c_{X,Y}) \otimes (X \otimes c_{X,Y} \otimes \mathfrak{u} \otimes Y) \circ (X \otimes X \otimes c_{X,Y} \otimes c_{Y,Y})$
$\circ (((\overline{\Pi}_X^L \otimes ((X \otimes \Pi_X^L) \circ \delta_X)) \circ \delta_X \circ \eta_X) \otimes ((Y \otimes \delta_Y) \circ \delta_Y))$ (naturality of c)
$= (\mathfrak{u}^{-1} \otimes c_{X,Y}) \otimes (X \otimes c_{X,Y} \otimes \mathfrak{u} \otimes Y) \circ (X \otimes X \otimes c_{X,Y} \otimes c_{Y,Y})$
$\circ (((X \otimes \delta_X) \circ \delta_X \circ \eta_X) \otimes ((Y \otimes \delta_Y) \circ \delta_Y))$ (naturality of c, coassociativity of δ_X and (2.13))
$= \Psi \circ (\eta_X \otimes Y)$ (naturality of c, $c^2 = id$ and (4.15)).

Finally, (4.4) follows by

$(Y \otimes \mu_X) \circ (\Psi \otimes X) \circ (\eta_X \otimes \eta_Y \otimes X)$
$= (c_{X,Y} \otimes \mathfrak{u}) \circ (\mu_X \otimes \mathfrak{u}^{-1} \otimes c_{X,Y} \otimes Y) \circ (X \otimes c_{X,X} \otimes c_{X,Y} \otimes Y \otimes Y)$
$\circ (c_{X,X} \otimes c_{X,X} \otimes Y \otimes Y \otimes Y) \circ (((X \otimes \mu_X \otimes X) \circ ((\delta_X \circ \eta_X) \otimes (\delta_X \circ \eta_X))) \otimes ((c_{Y,X} \otimes Y \otimes Y)$
$\circ (Y \otimes c_{Y,X} \otimes Y) \circ (Y \otimes Y \otimes c_{Y,X}) \circ (((Y \otimes \delta_Y) \circ \delta_Y) \otimes X)))$ ((a3) of Definition 2.1, naturality of c, $c^2 = id$ and (4.15))
$= (c_{X,Y} \otimes \mathfrak{u}) \circ (X \otimes \mathfrak{u}^{-1} \otimes c_{X,Y} \otimes Y) \circ (c_{X,X} \otimes c_{X,Y} \otimes Y \otimes Y)$
$\circ (((X \otimes ((\mu_X \otimes X) \circ (\mu_X \otimes c_{X,X}))) \circ ((\delta_X \circ \eta_X) \otimes (\delta_X \circ \eta_X) \otimes X)) \otimes Y \otimes Y \otimes Y)$
$\circ (c_{Y,X} \otimes Y \otimes Y) \circ (Y \otimes c_{Y,X} \otimes Y) \circ (Y \otimes Y \otimes c_{Y,X}) \circ (((Y \otimes \delta_Y) \circ \delta_Y \circ \eta_Y) \otimes X)$ (naturality

$$
\begin{aligned}
&\text{of } c \text{ and } c^2 = id)\\
&= (c_{X,Y} \otimes \mathfrak{u}) \circ (X \otimes \mathfrak{u}^{-1} \otimes c_{X,Y} \otimes Y) \circ (c_{X,X} \otimes c_{X,Y} \otimes Y \otimes Y)\\
&\quad \circ(((X \otimes ((\mu_X \otimes X) \circ (X \otimes \mu_X \otimes X) \circ (X \otimes X \otimes c_{X,X})))\\
&\quad \circ((\delta_X \circ \eta_X) \otimes (\delta_X \circ \eta_X) \otimes X)) \otimes Y \otimes Y \otimes Y) \circ (c_{Y,X} \otimes Y \otimes Y) \circ (Y \otimes c_{Y,X} \otimes Y)\\
&\quad \circ(Y \otimes Y \otimes c_{Y,X}) \circ (((Y \otimes \delta_Y) \circ \delta_Y \circ \eta_Y) \otimes X) \ \text{(associativity of } \mu_X)\\
&= (c_{X,Y} \otimes \mathfrak{u}) \circ (X \otimes \mathfrak{u}^{-1} \otimes c_{X,Y} \otimes Y) \circ (c_{X,X} \otimes c_{X,Y} \otimes Y \otimes Y)\\
&\quad \circ(((((\overline{\Pi}^L_X \otimes X) \circ \delta_X) \otimes \Pi^L_X) \circ \delta_X) \otimes Y \otimes Y \otimes Y) \circ (c_{Y,X} \otimes Y \otimes Y) \circ (Y \otimes c_{Y,X} \otimes Y)\\
&\quad \circ(Y \otimes Y \otimes c_{Y,X}) \circ (((Y \otimes \delta_Y) \circ \delta_Y \circ \eta_Y) \otimes X) \ \text{((2.6) and (2.11))}\\
&= (c_{X,Y} \otimes (\mathfrak{u} \circ (\Pi^L_X \otimes Y))) \circ (X \otimes (\mathfrak{u}^{-1} \circ (\overline{\Pi}^L_X \otimes Y)) \otimes c_{X,Y} \otimes Y) \circ (c_{X,X} \otimes c_{X,Y} \otimes Y \otimes Y)\\
&\quad \circ(((\delta_X \otimes X) \circ \delta_X) \otimes ((Y \otimes \delta_Y) \circ \delta_Y \circ \eta_Y)) \ \text{(naturality of } c)\\
&= (c_{X,Y} \otimes (\mathfrak{u} \circ (X \otimes \overline{\Pi}^R_Y))) \circ (X \otimes (\mathfrak{u}^{-1} \circ (X \otimes \Pi^R_Y)) \otimes c_{X,Y} \otimes Y) \circ (c_{X,X} \otimes c_{X,Y} \otimes Y \otimes Y)\\
&\quad \circ(((\delta_X \otimes X) \circ \delta_X) \otimes ((Y \otimes \delta_Y) \circ \delta_Y \circ \eta_Y)) \ \text{((3.7) and (3.13))}\\
&= (c_{X,Y} \otimes \mathfrak{u}) \circ (X \otimes \mathfrak{u}^{-1} \otimes c_{X,Y} \otimes Y) \circ (c_{X,X} \otimes c_{X,Y} \otimes Y \otimes Y)\\
&\quad \circ(((\delta_X \otimes X) \circ \delta_X) \otimes ((\Pi^R_Y \otimes ((Y \otimes \overline{\Pi}^R_Y) \circ \delta_Y)) \circ \delta_Y \circ \eta_Y)) \ \text{(naturality of } c)\\
&= (c_{X,Y} \otimes \mathfrak{u}) \circ (X \otimes \mathfrak{u}^{-1} \otimes c_{X,Y} \otimes Y) \circ (c_{X,X} \otimes c_{X,Y} \otimes Y \otimes Y)\\
&\quad \circ(((\delta_X \otimes X) \circ \delta_X) \otimes ((Y \otimes \delta_Y) \circ \delta_Y \circ \eta_Y)) \ \text{(naturality of } c\text{, coassociativity of } \delta_Y \text{ and (2.13))}\\
&= \Psi \circ (X \otimes \eta_Y) \ \text{(naturality of } c,\ c^2 = id \text{ and (4.15))}.
\end{aligned}
$$

□

Theorem 4.7. *Let A, B be weak bialgebras and let $\mathfrak{s} : A \otimes B \to K$ be a convolution invertible 1-skew pairing. Then,*

$$\Psi = (\varphi_B^{\mathfrak{s}^{-1}} \otimes \phi_A^{\mathfrak{s}}) \circ \delta_{A\otimes B} : A \otimes B \to B \otimes A,$$

$$\Phi = c_{B,A} \circ (\varphi_B^{\mathfrak{s}} \otimes \phi_A^{\mathfrak{s}^{-1}}) \circ \delta_{A\otimes B} \circ c_{B,A} : B \otimes A \to A \otimes B,$$

are weak distributive laws such that Φ is a weak inverse for ψ.

Proof. The morphism Ψ is a weak distributive law by Theorem 4.6. By the naturality of c and $c^2 = id$ we have

$$\Phi = (\varphi_{A^{cop}}^{\mathfrak{r}^{-1}} \otimes \phi_{B^{cop}}^{\mathfrak{r}}) \circ \delta_{B^{cop}\otimes A^{cop}}$$

where $\mathfrak{r} = \mathfrak{s} \circ c_{B,A}$ and $\mathfrak{r}^{-1} = \mathfrak{s}^{-1} \circ c_{B,A}$. By Proposition 3.13, $\mathfrak{r}$ is convolution invertible and belongs to $\mathsf{Sk}_2(B, A)$. Then, by Proposition 3.2, $\mathfrak{r}$ is in $\mathsf{Sk}_1(B^{cop}, A^{cop})$ and is convolution invertible. Therefore we can apply Theorem 4.6 for $X = B^{cop}$, $Y = A^{cop}$ and $\mathfrak{u} = \mathfrak{r}$ and then Φ is a weak distributive law for the weak bialgebras B^{cop} and A^{cop}. Thus Φ is a weak distributive law for the weak bialgebras B and A. Note that by the naturality of c and $c^2 = id$ we have

$$\Phi = (\mathfrak{s} \otimes ((A \otimes B \otimes \mathfrak{s}^{-1}) \circ \delta_{A\otimes B})) \circ \delta_{A\otimes B} \circ c_{B,A}. \tag{4.16}$$

To prove the identities (4.7) and (4.8), first we need to show that the following equalities hold:

$$\Psi \circ \Phi = ((\mathfrak{s} \circ (\Pi^L_A \otimes B)) \otimes B \otimes A \otimes (\mathfrak{s} \circ (\Pi^L_A \otimes B))) \circ (A \otimes \delta_B \otimes \delta_A \otimes B) \circ \delta_{A\otimes B} \circ c_{B,A}. \tag{4.17}$$

$$\Phi \circ \Psi = ((\mathfrak{s} \circ (\Pi^R_A \otimes B)) \otimes A \otimes B \otimes (\mathfrak{s} \circ (\Pi^R_A \otimes B))) \circ (A \otimes \delta_{B\otimes A} \otimes B) \circ \delta_{A\otimes B}. \tag{4.18}$$

Indeed, in one hand

$$
\begin{aligned}
&\Psi \circ \Phi\\
&= (\varphi_B^{\mathfrak{s}^{-1}} \otimes \phi_A^{\mathfrak{s}}) \circ (A \otimes c_{A,B} \otimes B) \circ ((\delta_A \circ \phi_A^{\mathfrak{s}^{-1}}) \otimes (\delta_B \circ \varphi_B^{\mathfrak{s}})) \circ \delta_{A^{cop}\otimes B^{cop}} \circ c_{B,A}\\
&\quad \text{(naturality of } c \text{ and } c^2 = id))\\
&= (\varphi_B^{\mathfrak{s}^{-1}} \otimes \phi_A^{\mathfrak{s}}) \circ (A \otimes c_{A,B} \otimes B) \circ (((A \otimes \phi_A^{\mathfrak{s}^{-1}}) \circ (\delta_A \otimes B)) \otimes ((\varphi_B^{\mathfrak{s}} \otimes B) \circ (A \otimes \delta_B)))\\
&\quad \circ \delta_{A^{cop}\otimes B^{cop}} \circ c_{B,A} \ \text{(coassociativity of } \delta_A \text{ and } \delta_B)\\
&= ((\varphi_B^{\mathfrak{s}^{-1}} \circ (A \otimes \varphi_B^{\mathfrak{s}})) \otimes (\phi_A^{\mathfrak{s}} \circ (\phi_A^{\mathfrak{s}^{-1}} \otimes B)) \circ (A \otimes A \otimes c_{A,B} \otimes B \otimes B) \circ (A \otimes c_{A,A} \otimes c_{B,B} \otimes B)\\
&\quad \circ(\delta_A \otimes A \otimes B \otimes \delta_B) \circ ((c_{A,A} \circ \delta_A) \otimes (c_{B,B} \circ \delta_B)) \circ c_{B,A} \ \text{(naturality of } c \text{ and } c^2 = id)\\
&= (((\mathfrak{s} \otimes \mathfrak{s}^{-1}) \circ (A \otimes c_{B,A} \otimes B) \circ (A \otimes A \otimes \delta_B)) \otimes B \otimes A \otimes ((\mathfrak{s} \otimes \mathfrak{s}^{-1}) \circ (A \otimes c_{A,B} \otimes B)\\
&\quad \circ(\delta_A \otimes B \otimes B))) \circ (A \otimes A \otimes \delta_B \otimes \delta_A \otimes B \otimes B) \circ (c_{A,A} \otimes c_{A,B} \otimes c_{B,B}) \circ (A \otimes c_{A,A} \otimes c_{B,B} \otimes B)
\end{aligned}
$$

$\circ(\delta_A\otimes A\otimes B\otimes \delta_B)\circ((c_{A,A}\circ\delta_A)\otimes(c_{B,B}\circ\delta_B))\circ c_{B,A}$ ((xii), (xiii) of Lemma 4.5 and $c^2=id$)
$=((\mathfrak{s}*\mathfrak{s}^{-1})\otimes B\otimes A\otimes(\mathfrak{s}*\mathfrak{s}^{-1}))\circ(A\otimes\delta_B\otimes\delta_A\otimes B)\circ\delta_{A\otimes B}\circ c_{B,A}$ (naturality of c, $c^2=id$ and coassociativity of δ_A and δ_B)
$=((\mathfrak{s}\circ(\Pi_A^L\otimes B))\otimes B\otimes A\otimes(\mathfrak{s}\circ(\Pi_A^L\otimes B)))\circ(A\otimes\delta_B\otimes\delta_A\otimes B)\circ\delta_{A\otimes B}\circ c_{B,A}$ ((3.15)).

and, on the other hand,

$\Phi\circ\Psi$
$=c_{B,A}\circ(\varphi_B^{\mathfrak{s}}\otimes\phi_A^{\mathfrak{s}^{-1}})\circ(A\otimes c_{A,B}\otimes B)\circ((\delta_A\circ\phi_A^{\mathfrak{s}})\otimes(\delta_B\circ\varphi_B^{\mathfrak{s}^{-1}}))\circ\delta_{A^{cop}\otimes B^{cop}}$ (naturality of c and $c^2=id$))
$=c_{B,A}\circ(\varphi_B^{\mathfrak{s}}\otimes\phi_A^{\mathfrak{s}^{-1}})\circ(A\otimes c_{A,B}\otimes B)\circ(((A\otimes\phi_A^{\mathfrak{s}})\circ(\delta_A\otimes B))\otimes((\varphi_B^{\mathfrak{s}^{-1}}\otimes B)\circ(A\otimes\delta_B)))$ $\circ\delta_{A^{cop}\otimes B^{cop}}$ (coassociativity of δ_A and δ_B)
$=c_{B,A}\circ((\varphi_B^{\mathfrak{s}}\circ(A\otimes\varphi_B^{\mathfrak{s}^{-1}}))\otimes(\phi_A^{\mathfrak{s}^{-1}}\circ(\phi_A^{\mathfrak{s}}\otimes B))\circ(A\otimes A\otimes c_{A,B}\otimes B\otimes B)$ $\circ(A\otimes c_{A,A}\otimes c_{B,B}\otimes B)\circ(\delta_A\otimes A\otimes B\otimes\delta_B)\circ((c_{A,A}\circ\delta_A)\otimes(c_{B,B}\circ\delta_B))$ (naturality of c and $c^2=id$)
$=(((\mathfrak{s}^{-1}\otimes\mathfrak{s})\circ(A\otimes c_{B,A}\otimes B)\circ(A\otimes A\otimes\delta_B))\otimes c_{B,A}\otimes((\mathfrak{s}^{-1}\otimes\mathfrak{s})\circ(A\otimes c_{B,A}\otimes B)$ $\circ(\delta_A\otimes B\otimes B)))\circ(A\otimes A\otimes\delta_B\otimes\delta_A\otimes B\otimes B)\circ(A\otimes A\otimes c_{A,B}\otimes B\otimes B)\circ(((A\otimes\delta_A)$ $\circ\delta_A)\otimes((B\otimes c_{B,B})\circ(c_{B,B}\otimes B)\circ(B\otimes\delta_B)\circ c_{B,B}\circ\delta_B))$ ((xiv), (xi) of Lemma 4.5 and $c^2=id$)
$=((\mathfrak{s}^{-1}*\mathfrak{s})\otimes c_{B,A}\otimes(\mathfrak{s}^{-1}*\mathfrak{s}))\circ(A\otimes\delta_B\otimes\delta_A\otimes B)\circ\delta_{A\otimes B}$ (naturality of c, $c^2=id$ and coassociativity of δ_A and δ_B)
$=((\mathfrak{s}\circ(\Pi_A^R\otimes B))\otimes A\otimes B\otimes(\mathfrak{s}\circ(\Pi_A^R\otimes B)))\circ(A\otimes\delta_{B\otimes A}\otimes B)\circ\delta_{A\otimes B}$ ((3.16)),

therefore

∇_Ψ
$=(\mu_B\otimes A)\circ(B\otimes((\mathfrak{u}^{-1}\otimes((c_{A,B}\otimes\mathfrak{u})\circ\delta_{A\otimes B}))\circ\delta_{A\otimes B}\circ(A\otimes\eta_B)))$ ((4.15))
$=c_{B,A}\circ(A\otimes\mu_B\otimes\mathfrak{u})\circ(c_{B,A}\otimes c_{A,B}\otimes B)\circ(B\otimes\mathfrak{u}^{-1}\otimes\delta_A\otimes B\otimes B)\circ(B\otimes A\otimes c_{A,B}\otimes B\otimes B)$ $\circ(B\otimes\delta_A\otimes((B\otimes\mu_B\otimes B)\circ((\delta_B\circ\eta_B)\otimes(\delta_B\circ\eta_B))$ (naturality of c and (a3) of Definition 2.1)
$=(\mathfrak{u}^{-1}\otimes c_{A,B}\otimes\mathfrak{u})\circ(A\otimes c_{A,B}\otimes c_{A,B}\otimes B)\circ(A\otimes A\otimes c_{A,B}\otimes B\otimes B)$ $\circ(((A\otimes\delta_A)\circ\delta_A)\otimes((B\otimes(\mu_B\circ(B\otimes\mu_B))\otimes B)\circ(c_{B,B}\otimes B\otimes B\otimes B)$ $\circ(B\otimes(\delta_B\circ\eta_B)\otimes(\delta_B\circ\eta_B))))\circ c_{B,A}$ (naturality of c and $c^2=id$)
$=(\mathfrak{u}^{-1}\otimes c_{A,B}\otimes\mathfrak{u})\circ(A\otimes c_{A,B}\otimes c_{A,B}\otimes B)\circ(A\otimes A\otimes c_{A,B}\otimes B\otimes B)$ $\circ(((A\otimes\delta_A)\circ\delta_A)\otimes((B\otimes(\mu_B\circ(\mu_B\otimes B))\otimes B)\circ(c_{B,B}\otimes B\otimes B\otimes B)$ $\circ(B\otimes(\delta_B\circ\eta_B)\otimes(\delta_B\circ\eta_B))))\circ c_{B,A}$ (associativity of μ_B)
$=(c_{A,B}\otimes\mathfrak{u})\circ(A\otimes c_{A,B}\otimes B)\circ(((\mathfrak{u}^{-1}\otimes\delta_A)\circ(A\otimes c_{A,B}))\otimes B\otimes B)$ $\circ(\delta_A\otimes((\Pi_B^R\otimes((B\otimes\overline{\Pi}_B^R)\circ\delta_B))\circ\delta_B))\circ c_{B,A}$ (naturality of c and (2.13))
$=((\mathfrak{s}^{-1}\circ(A\otimes\Pi_B^R))\otimes B\otimes A\otimes(\mathfrak{s}\circ(A\otimes\overline{\Pi}_B^R)))\circ(A\otimes\delta_B\otimes\delta_A\otimes B)\circ\delta_{A\otimes B}\circ c_{B,A}$ (naturality of c)
$=((\mathfrak{s}\circ(\Pi_A^L\otimes B))\otimes B\otimes A\otimes(\mathfrak{s}\circ(\Pi_A^L\otimes B)))\circ(A\otimes\delta_B\otimes\delta_A\otimes B)\circ\delta_{A\otimes B}\circ c_{B,A}$ ((3.7) and (3.15))
$=\Psi\circ\Phi$ ((4.17)),

and

∇_Φ
$=(\mu_A\otimes B\otimes\mathfrak{s}^{-1})\circ(A\otimes((\mathfrak{s}\otimes\delta_{A\otimes B})\circ\delta_{A\otimes B}\circ(\eta_A\otimes B)))$ (naturality of c and (4.16)))
$=(\mathfrak{s}\otimes\mu_A\otimes B\otimes\mathfrak{s}^{-1})\circ(A\otimes c_{A,B}\otimes A\otimes B\otimes A\otimes B)\circ(c_{A,A}\otimes c_{A,B}\otimes c_{A,B}\otimes B)$ $\circ(A\otimes A\otimes A\otimes c_{A,B}\otimes B)\circ(A\otimes((A\otimes\delta_A)\circ\delta_A\circ\eta_A)\otimes((B\otimes\delta_B)\circ\delta_B))$ (naturality of c)
$=(\mathfrak{s}\otimes\mu_A\otimes B\otimes\mathfrak{s}^{-1})\circ(A\otimes c_{A,B}\otimes A\otimes B\otimes A\otimes B)\circ(c_{A,A}\otimes c_{A,B}\otimes c_{A,B}\otimes B)$ $\circ(A\otimes A\otimes A\otimes c_{A,B}\otimes B)\circ(A\otimes((A\otimes\mu_A\otimes A)\circ((\delta_A\circ\eta_A)\otimes(\delta_A\circ\eta_A))\otimes((B\otimes\delta_B)\circ\delta_B))$ ((a3) of Definition 2.1)

$= (\mathfrak{s}\otimes A\otimes B\otimes \mathfrak{s}^{-1})\circ (A\otimes c_{A,B}\otimes c_{A,B}\otimes B)\circ (A\otimes A\otimes c_{A,B}\otimes B\otimes B)$
$\circ(((A\otimes(\mu_A\circ(A\otimes\mu_A))\otimes A)\circ(c_{A,A}\otimes A\otimes A\otimes A)\circ(A\otimes((\delta_A\circ\eta_A)\otimes(\delta_A\circ\eta_A))))$
$\otimes((B\otimes\delta_B)\circ\delta_B))$ (naturality of c)
$= (\mathfrak{s}\otimes A\otimes B\otimes \mathfrak{s}^{-1})\circ (A\otimes c_{A,B}\otimes c_{A,B}\otimes B)\circ (A\otimes A\otimes c_{A,B}\otimes B\otimes B)$
$\circ(((A\otimes(\mu_A\circ(\mu_A\otimes A))\otimes A)\circ(c_{A,A}\otimes A\otimes A\otimes A)\circ(A\otimes((\delta_A\circ\eta_A)\otimes(\delta_A\circ\eta_A)))))$
$\otimes((B\otimes\delta_B)\circ\delta_B))$ (associativity of μ_A)
$= (\mathfrak{s}\otimes A\otimes B\otimes \mathfrak{s}^{-1})\circ (A\otimes c_{A,B}\otimes c_{A,B}\otimes B)\circ (A\otimes A\otimes c_{A,B}\otimes B\otimes B)$
$\circ(((\Pi_A^R\otimes((A\otimes\overline{\Pi}_A^R)\circ\delta_A))\circ\delta_A)\otimes((B\otimes\delta_B)\circ\delta_B))$ ((2.13))
$= ((\mathfrak{s}\circ(\Pi_A^R\otimes B))\otimes A\otimes B\otimes(\mathfrak{s}^{-1}\circ(\overline{\Pi}_A^R\otimes B)))\circ(A\otimes B\otimes\delta_{A\otimes B})\circ\delta_{A\otimes B}$ (naturality of of c)
$= ((\mathfrak{s}\circ(\Pi_A^R\otimes B))\otimes A\otimes B\otimes(\mathfrak{s}\circ(\Pi_A^R\otimes B)))\circ(A\otimes\delta_{B\otimes A}\otimes B)\circ\delta_{A\otimes B}$ ((3.14), (3.16), naturality of c, $c^2=id$ and coassociativity of δ_B)
$= \Phi\circ\Psi$ ((4.18)),

Thus (4.7) and (4.8) hold. □

Theorem 4.8. *Let A, B be weak bialgebras, let $\mathfrak{s}: A\otimes B\to K$ be a convolution invertible 1-skew pairing and let Ψ and Φ be the weak distributive laws introduced in Theorem 4.6. Then, the pair (Ψ,Φ) is weakly comonoidal.*

Proof. We begin by proving (4.11). Indeed, first note that

$(\nabla_\Psi\otimes B\otimes A)\circ\delta_{B\otimes A}\circ\Psi$
$= ((\mathfrak{s}^{-1}*\mathfrak{s}*\mathfrak{s}^{-1})\otimes c_{A,B}\otimes(((\mathfrak{s}*\mathfrak{s}^{-1})\otimes c_{A,B})\circ\delta_{A\otimes B})\otimes\mathfrak{s})\circ(\delta_{A\otimes B}\otimes\delta_{A\otimes B})\circ\delta_{A\otimes B}$
(naturality of c, $c^2=id$ and coassociativity of $\delta_{A\otimes B}$)
$= (\mathfrak{s}^{-1}\otimes c_{A,B}\otimes(((\mathfrak{s}*\mathfrak{s}^{-1})\otimes c_{A,B})\circ\delta_{A\otimes B})\otimes\mathfrak{s})\circ(\delta_{A\otimes B}\otimes\delta_{A\otimes B})\circ\delta_{A\otimes B}$ ((3.17))
$= (\Psi\otimes\Psi)\circ\delta_{A\otimes B}$ (coassociativity of $\delta_{A\otimes B}$),

and on the other hand,

$(B\otimes A\otimes\nabla_\Psi)\circ\delta_{B\otimes A}\circ\Psi$
$= (\mathfrak{s}^{-1}\otimes c_{A,B}\otimes(((\mathfrak{s}*\mathfrak{s}^{-1})\otimes c_{A,B})\circ\delta_{A\otimes B})\otimes(\mathfrak{s}*\mathfrak{s}^{-1}*\mathfrak{s}))\circ(\delta_{A\otimes B}\otimes\delta_{A\otimes B})\circ\delta_{A\otimes B}$
(naturality of c, $c^2=id$ and coassociativity of $\delta_{A\otimes B}$)
$= (\mathfrak{s}^{-1}\otimes c_{A,B}\otimes(((\mathfrak{s}*\mathfrak{s}^{-1})\otimes c_{A,B})\circ\delta_{A\otimes B})\otimes\mathfrak{s})\circ(\delta_{A\otimes B}\otimes\delta_{A\otimes B})\circ\delta_{A\otimes B}$ ((3.18))
$= (\Psi\otimes\Psi)\circ\delta_{A\otimes B}$ (coassociativity of $\delta_{A\otimes B}$).

Therefore, (4.11) holds. The proof for (4.12) follows from

$(\nabla_\Phi\otimes A\otimes B)\circ\delta_{A\otimes B}\circ\Phi$
$= ((\mathfrak{s}*\mathfrak{s}^{-1}*\mathfrak{s})\otimes A\otimes B\otimes(((\mathfrak{s}^{-1}*\mathfrak{s})\otimes A\otimes B)\circ\delta_{A\otimes B})\otimes\mathfrak{s}^{-1})\circ(\delta_{A\otimes B}\otimes\delta_{A\otimes B})\circ\delta_{A\otimes B}\circ c_{B,A}$
(coassociativity of $\delta_{A\otimes B}$)
$= (\mathfrak{s}\otimes A\otimes B\otimes\otimes(((\mathfrak{s}^{-1}*\mathfrak{s})\otimes A\otimes B)\circ\delta_{A\otimes B})\otimes\mathfrak{s}^{-1})\circ(\delta_{A\otimes B}\otimes\delta_{A\otimes B})\circ\delta_{A\otimes B}\circ c_{B,A}$
((3.18))
$= (\Phi\otimes\Phi)\circ\delta_{B\otimes A}$ (naturality of c, $c^2=id$ and coassociativity of $\delta_{A\otimes B}$),

and

$(A\otimes B\otimes\nabla_\Phi)\circ\delta_{A\otimes B}\circ\Phi$
$= (\mathfrak{s}\otimes A\otimes B\otimes(((\mathfrak{s}^{-1}*\mathfrak{s})\otimes A\otimes B)\circ\delta_{A\otimes B})\otimes(\mathfrak{s}^{-1}*\mathfrak{s}*\mathfrak{s}^{-1}))\circ(\delta_{A\otimes B}\otimes\delta_{A\otimes B})\circ\delta_{A\otimes B}\circ c_{B,A}$
(coassociativity of $\delta_{A\otimes B}$)
$= (\mathfrak{s}\otimes A\otimes B\otimes\otimes(((\mathfrak{s}^{-1}*\mathfrak{s})\otimes A\otimes B)\circ\delta_{A\otimes B})\otimes\mathfrak{s}^{-1})\circ(\delta_{A\otimes B}\otimes\delta_{A\otimes B})\circ\delta_{A\otimes B}\circ c_{B,A}$
((3.17))
$= (\Phi\otimes\Phi)\circ\delta_{B\otimes A}$ (naturality of c, $c^2=id$ and coassociativity of $\delta_{A\otimes B}$).

Finally, by naturality of c, properties of the counits and (3.17)

$$(\varepsilon_B\otimes\varepsilon_A)\circ\Psi=(\mathfrak{s}*\mathfrak{s}^{-1})\circ c_{B,A}=(\mathfrak{s}*\mathfrak{s}^{-1}*\mathfrak{s}*\mathfrak{s}^{-1})\circ c_{B,A}=(\varepsilon_B\otimes\varepsilon_A)\circ\nabla_\Phi,$$

and then (4.13) holds.

□

Corollary 4.9. *Let A, B be weak bialgebras and let $\mathfrak{w} : A\otimes B \to K$ be a convolution invertible 2-skew pairing. Then,*

$$\Gamma = (\varphi_B^{\mathfrak{w}} \otimes \phi_A^{\mathfrak{w}^{-1}}) \circ \delta_{A\otimes B} : A\otimes B \to B\otimes A,$$

$$\Upsilon = c_{B,A} \circ (\varphi_B^{\mathfrak{w}} \otimes \phi_A^{\mathfrak{w}^{-1}}) \circ \delta_{A\otimes B} \circ c_{B,A} : B\otimes A \to A\otimes B,$$

are weak distributive laws such that Υ is a weak inverse for Γ and the pair (Γ, Υ) is weakly comoidal.

Proof. The proof follows from Proposition 3.7 and Theorems 4.6, 4.7 and 4.8. □

Example 4.10. Let H be a finite weak Hopf algebra in $\mathcal{C}$ satisfying that the antipode is an isomorphism. If $A = \widehat{H}^{cop} = (H^*)^{op}$ and $B = H$, the morphism

$$\mathfrak{s} = \beta_H(K) \circ c_{H^*,H} : (H^*)^{op} \otimes H \to K$$

is convolution invertible in $\mathsf{Sk}_1((H^*)^{op}, H)$ with inverse

$$\mathfrak{s}^{-1} = \mathfrak{s} \circ (H^* \otimes \lambda_H^{-1}).$$

Therefore, by Theorems 4.6, 4.7 and 4.8, the pair (Ψ, Φ) where

$$\Psi = (\varphi_H^{\mathfrak{s}^{-1}} \otimes \phi_{(H^*)^{op}}^{\mathfrak{s}}) \circ \delta_{(H^*)^{op}\otimes H} : (H^*)^{op} \otimes H \to H \otimes (H^*)^{op},$$

$$\Phi = c_{H,(H^*)^{op}} \circ (\varphi_H^{\mathfrak{s}} \otimes \phi_{(H^*)^{op}}^{\mathfrak{s}^{-1}}) \circ \delta_{(H^*)^{op}\otimes H} \circ c_{H,(H^*)^{op}} : H \otimes (H^*)^{op} \to (H^*)^{op} \otimes H,$$

are weak distributive laws such that Φ is a weak inverse for ψ. Moreover, (Ψ, Φ) is a pair of weakly comonoidal weak distributive laws. Then, by Remark 4.4 (Φ, Ψ) is the pair of weakly comonoidal weak distributive laws that induces the Drinfel'd double

$$D(H) = Im(\nabla_\Phi) = (H^*)^{op} \times H = \widehat{H}^{cop} \times H$$

as was pointed in [3] (see also ([18])).

In the conditions of Example 2.4, $D(H) = Im(\nabla_\Phi) = H^* \times H$. Then, using that

$$\varphi_H^{\mathfrak{s}^{-1}}(f_\sigma \otimes_R \tau) = \begin{cases} \tau & \text{if} \quad \tau = \sigma^{-1} \\ 0 & \text{if} \quad \tau \neq \sigma^{-1} \end{cases}, \qquad \phi_{H^*}^{\mathfrak{s}}(f_\sigma \otimes_R \tau) = \begin{cases} f_{\tau^{-1}\circ\sigma} & \text{if} \quad t(\tau) = t(\sigma) \\ 0 & \text{if} \quad t(\tau) \neq t(\sigma) \end{cases}$$

and

$$\varphi_H^{\mathfrak{s}}(f_\sigma \otimes_R \tau) = \begin{cases} \tau & \text{if} \quad \tau = \sigma \\ 0 & \text{if} \quad \tau \neq \sigma \end{cases}, \qquad \phi_{H^*}^{\mathfrak{s}^{-1}}(f_\sigma \otimes_R \tau) = \begin{cases} f_{\tau\circ\sigma} & \text{if} \quad s(\tau) = t(\sigma) \\ 0 & \text{if} \quad s(\tau) \neq t(\sigma) \end{cases}$$

we obtain the expressions for Ψ and Φ in the following way:

$$\Psi(f_\sigma \otimes_R \tau) = \begin{cases} \tau \otimes_R f_{\tau^{-1}\circ\sigma\circ\tau} & \text{if} \quad \sigma \text{ is a cycle on } t(\tau) \\ 0 & \text{otherwise} \end{cases},$$

$$\Phi(\tau \otimes_R f_\sigma) = \begin{cases} f_{\tau\circ\sigma\circ\tau^{-1}} \otimes_R \tau & \text{if} \quad \sigma \text{ is a cycle on } s(\tau) \\ 0 & \text{otherwise} \end{cases}.$$

Therefore,

$$\nabla_\Phi(f_\sigma \otimes_R \tau) = \begin{cases} f_\sigma \otimes_R \tau & \text{if} \quad \sigma \text{ is a cycle on } t(\tau) \\ 0 & \text{otherwise} \end{cases},$$

and then

$$D(H) = \langle \{ f_\sigma \otimes_R \tau, \quad \sigma \text{ is a cycle on } t(\tau) \} \rangle.$$

As a consequence, the algebraic structure as weak Hopf algebra of $D(H)$ is:

$$\eta_{D(H)}(1_{\mathbb{F}}) = \sum_{\omega \text{ cycle on } x} f_\omega \otimes_R id_x,$$

$$\mu_{D(H)}(f_\sigma \otimes_R \tau \otimes_R f_\theta \otimes_R \pi) = \begin{cases} f_\sigma \otimes_R \tau \circ \pi & \text{if} \quad \sigma = \tau \circ \theta \circ \tau^{-1} \text{ and } s(\tau) = t(\pi) \\ \\ 0 & \text{otherwise} \end{cases},$$

$$\varepsilon_{D(H)}(f_\sigma \otimes_R \tau) = \begin{cases} 1 & \text{if} \quad \sigma = id_{t(\tau)} \\ \\ 0 & \text{otherwise} \end{cases},$$

$$\delta_{D(H)}(f_\sigma \otimes_R \tau) = \sum_{\omega \text{ cycle on } t(\tau)} f_\omega \otimes_R \tau \otimes_R f_{\sigma\circ\omega^{-1}} \otimes_R \tau$$

and

$$\lambda_{D(H)}(f_\sigma \otimes_R \tau) = f_{\tau^{-1}\circ\sigma^{-1}\circ\tau} \otimes_R \tau^{-1}.$$

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