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On the Solutions of a Fourth Order Difference Equation

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Abstract

In this paper, we solve and study the global behavior of the well defined solutions of the difference equation

$$x_{n+1}=\frac{x_n x_{n-3}}{Ax_{n-2}+Bx_{n-3}}, \quad n=0,1,...,$$

where $A,B>0$ and the initial values x_{-i}, $i\in\{0,1,2,3\}$ are real numbers.

1. Introduction

In [1], we determined an explicit formula for the solutions of the fourth order difference equation

$$x_{n+1}=\frac{x_n x_{n-2}}{ax_{n-2}+bx_{n-3}},\ n=0,1,...,$$

where a,b are positive real numbers and the initial conditions x_{-3},x_{-2},x_{-1},x_0 are real numbers.
In [2] and [8], we determined the forbidden set and introduced an explicit formula for the solutions of each of the two fourth order difference equations (respectively)

$$x_{n+1}=\frac{ax_n x_{n-2}}{-bx_n+cx_{n-3}},\ n=0,1,...,$$

and

$$x_{n+1}=\frac{ax_n x_{n-2}}{bx_n+cx_{n-3}},\ n=0,1,...,$$

where a,b,c are positive real numbers and the initial conditions x_{-3},x_{-2},x_{-1},x_0 are real numbers.
In [10], the authors studied the qualitative analysis of the fourth order difference equation

$$x_{n+1}=ax_{n-1}+\frac{bx_{n-1}}{cx_{n-1}-dx_{n-3}},\ n=0,1,...,$$

where a,b,c,d are positive real numbers and the initial conditions x_{-3},x_{-2},x_{-1},x_0 are arbitrary real numbers.
In [15], the authors obtained the solutions of the fourth order difference equation

$$x_{n+1}=\frac{x_n x_{n-3}}{x_{n-2}(\pm 1\pm x_n x_{n-3})}, \quad n=0,1,...,$$

where the initial conditions are arbitrary real numbers.
In [24], the author studied the boundedness character of the positive solutions of the fourth order difference equation

$$x_{n+1}=\max\{A,\frac{x_n^p}{x_{n-3}^p}\},\ n=0,1,...,$$

where the parameters A and p are positive real numbers. For more on difference equations (See [3]- [7], [9], [11]- [14], [16]- [23]) and the references therein.

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In this paper, we study the difference equation

$$x_{n+1}=\frac{x_nx_{n-3}}{Ax_{n-2}+Bx_{n-3}},\quad n=0,1,..., \tag{1.1}$$

where $A,B>0$ and the initial values x_{-i}, $i\in\{0,1,2,3\}$ are real numbers.
The transformation

$$z_n=\frac{x_{n-1}}{x_n},\ \text{with}\ z_{-2}=\frac{x_{-3}}{x_{-2}},\ z_{-1}=\frac{x_{-2}}{x_{-1}}\ \text{and}\ z_0=\frac{x_{-1}}{x_0} \tag{1.2}$$

reduces Equation (1.1) into the difference equation

$$z_{n+1}=\frac{A}{z_{n-2}}+B,\ n=0,1,.... \tag{1.3}$$

During this paper, we suppose that

$$\theta_j=\frac{\lambda_+^j-\lambda_-^j}{\sqrt{B^2+4A}},$$

where $\lambda_-=\frac{B}{2}-\frac{\sqrt{B^2+4A}}{2}$ and $\lambda_+=\frac{B}{2}+\frac{\sqrt{B^2+4A}}{2}$, $j=0,1,....$
Let $\mu_l(j)=Ax_l\theta_j+x_{l-1}\theta_{j+1}$, $l\in\{0,-1,-2\}$ and $j=0,1,....$
We give the following Lemma without proof:

Lemma 1.1. *The following identities are true:*

1. $A\theta_j+B\theta_{j+1}=\theta_{j+2},\ j=0,1,....$
2. $A\mu_l(j)+B\mu_l(j+1)=\mu_l(j+2),\ l\in\{0,-1,-2\}$ *and* $j=0,1,....$

2. Solution of Equation (1.1)

In this section, we shall give two invariant sets and introduce the solution of Equation (1.1).
Consider the sets

$$D_+=\{(u_0,u_{-1},u_{-2},u_{-3})\in\mathbb{R}^4:-\frac{u_0}{(\lambda_+/A)^3}=\frac{u_{-1}}{(\lambda_+/A)^2}=-\frac{u_{-2}}{\lambda_+/A}=u_{-3}\}$$

and

$$D_-=\{(u_0,u_{-1},u_{-2},u_{-3})\in\mathbb{R}^4:-\frac{u_0}{(\lambda_-/A)^3}=\frac{u_{-1}}{(\lambda_-/A)^2}=-\frac{u_{-2}}{\lambda_-/A}=u_{-3}\}.$$

Theorem 2.1. *The two sets D_+ and D_- are invariant sets for Equation* (1.1).

Proof. Let $(x_0,x_{-1},x_{-2},x_{-3})\in D_+$. We show that $(x_n,x_{n-1},x_{n-2},x_{n-3})\in D_+$ for each $n\in\mathbb{N}$. The proof is by induction on n.
The point $(x_0,x_{-1},x_{-2},x_{-3})\in D_+$ implies

$$-\frac{x_0}{(\lambda_+/A)^3}=\frac{x_{-1}}{(\lambda_+/A)^2}=-\frac{x_{-2}}{\lambda_+/A}=x_{-3}\}.$$

Now for $n=1$, we have

$$x_1=\frac{x_0x_{-3}}{Ax_{-2}+Bx_{-3}}=\frac{-(\lambda_+/A)^2x_{-2}(A/\lambda_+)x_{-2}}{Ax_{-2}-B(A/\lambda_+)x_{-2}}$$
$$=-\frac{1}{A^2}\frac{\lambda_+x_{-2}}{1-B/\lambda_+}=-\frac{1}{(A/\lambda_+)^3}x_{-2}.$$

Then we have

$$-\frac{x_1}{(\lambda_+/A)^3}=\frac{x_0}{(\lambda_+/A)^2}=-\frac{x_{-1}}{\lambda_+/A}=x_{-2}.$$

This implies that $(x_1,x_0,x_{-1},x_{-2})\in D_+$.
Suppose now that for a certain $n\in\mathbb{N}$, $(x_n,x_{n-1},x_{n-2},x_{n-3})\in D_+$. That is

$$-\frac{x_n}{(\lambda_+/A)^3}=\frac{x_{n-1}}{(\lambda_+/A)^2}=-\frac{x_{n-2}}{\lambda_+/A}=x_{n-3}.$$

Then

$$x_{n+1}=\frac{x_nx_{n-3}}{Ax_{n-2}+Bx_{n-3}}=\frac{-(\lambda_+/A)^2x_{n-2}(A/\lambda_+)x_{n-2}}{Ax_{n-2}-B(A/\lambda_+)x_{n-2}}$$
$$=-\frac{1}{A^2}\frac{\lambda_+x_{n-2}}{1-B/\lambda_+}=-\frac{1}{(A/\lambda_+)^3}x_{n-2}.$$

Then we have

$$-\frac{x_{n+1}}{(\lambda_+/A)^3}=\frac{x_n}{(\lambda_+/A)^2}=-\frac{x_{n-1}}{\lambda_+/A}=x_{n-2}.$$

This means that the point $(x_{n+1},x_n,x_{n-1},x_{n-2})\in D_+$. Therefore, D_+ is an invariant set for Equation (1.1).
By similar way, we can show that D_- is an invariant set for Equation (1.1).
This completes the proof. □

Theorem 2.2. *Let $\{x_n\}_{n=-3}^{\infty}$ be a well defined solution of Equation (1.1). Then*

$$x_n = \begin{cases} \frac{\nu}{\mu_{-2}(\frac{n+2}{3})\mu_{-1}(\frac{n-1}{3})\mu_0(\frac{n-1}{3})}, & n = 1,4,..., \\ \frac{\nu}{\mu_{-2}(\frac{n+1}{3})\mu_{-1}(\frac{n+1}{3})\mu_0(\frac{n-2}{3})}, & n = 2,5,..., \\ \frac{\nu}{\mu_{-2}(\frac{n}{3})\mu_{-1}(\frac{n}{3})\mu_0(\frac{n}{3})}, & n = 3,6,..., \end{cases} \tag{2.1}$$

where $\nu = x_0x_{-1}x_{-2}x_{-3}$.

Proof. We can write the given solution (2.1) as

$$x_{3m+1} = \frac{\nu}{\mu_{-2}(m+1)\mu_{-1}(m)\mu_0(m)},$$

$$x_{3m+2} = \frac{\nu}{\mu_{-2}(m+1)\mu_{-1}(m+1)\mu_0(m)},$$

and

$$x_{3m+3} = \frac{\nu}{\mu_{-2}(m+1)\mu_{-1}(m+1)\mu_0(m+1)}.$$

When $m = 0$,

$$\begin{aligned} x_1 &= \frac{\nu}{\mu_{-2}(1)\mu_{-1}(0)\mu_0(0)} = \frac{\nu}{(Ax_{-2}+Bx_{-3})x_{-2}x_{-1}} \\ &= \frac{x_0x_{-3}}{Ax_{-2}+Bx_{-3}}, \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{\nu}{\mu_{-2}(1)\mu_{-1}(1)\mu_0(0)} = \frac{\nu}{(Ax_{-2}+Bx_{-3})(Ax_{-1}+Bx_{-2})x_{-1}} \\ &= \frac{x_1x_{-2}}{Ax_{-1}+Bx_{-2}}, \end{aligned}$$

and

$$\begin{aligned} x_3 &= \frac{\nu}{\mu_{-2}(1)\mu_{-1}(1)\mu_0(1)} = \frac{\nu}{(Ax_{-2}+Bx_{-3})(Ax_{-1}+Bx_{-2})(Ax_0+Bx_{-1})} \\ &= \frac{x_0x_{-3}}{Ax_{-2}+Bx_{-3}} \frac{x_{-2}x_{-1}}{(Ax_{-1}+Bx_{-2})(Ax_0+Bx_{-1})} \\ &= \frac{x_1x_{-2}}{Ax_{-1}+Bx_{-2}} \frac{x_{-1}}{Ax_0+Bx_{-1}} = \frac{x_2x_{-1}}{Ax_0+Bx_{-1}}. \end{aligned}$$

Suppose that the result is true for $m > 0$.
Then

$$\begin{aligned} \frac{x_{3m}x_{3m-3}}{Ax_{3m-2}+Bx_{3m-3}} &= \frac{\frac{\nu}{\mu_{-2}(m)\mu_{-1}(m)\mu_0(m)} \frac{\nu}{\mu_{-2}(m-1)\mu_{-1}(m-1)\mu_0(m-1)}}{\frac{A\nu}{\mu_{-2}(m)\mu_{-1}(m-1)\mu_0(m-1)} + \frac{B\nu}{\mu_{-2}(m-1)\mu_{-1}(m-1)\mu_0(m-1)}} \\ &= \frac{\frac{\nu}{\mu_{-1}(m)\mu_0(m)}}{A\mu_{-2}(m-1)+B\mu_{-2}(m)}. \end{aligned}$$

Using Lemma (1.1), we have

$$A\mu_{-2}(m-1)+B\mu_{-2}(m) = \mu_{-2}(m+1).$$

Then

$$\begin{aligned} &\frac{x_{3m}x_{3m-3}}{Ax_{3m-2}+Bx_{3m-3}} \\ &= \frac{\frac{\nu}{\mu_{-1}(m)\mu_0(m)}}{A\mu_{-2}(m-1)+B\mu_{-2}(m)} \\ &= \frac{\nu}{\mu_{-2}(m+1)\mu_{-1}(m)\mu_0(m)} = x_{3m+1}. \end{aligned}$$

Similarly we can show that

$$\frac{x_{3m+1}x_{3m-2}}{Ax_{3m-1}+Bx_{3m-2}} = x_{3m+2} \text{ and } \frac{x_{3m+2}x_{3m-1}}{Ax_{3m}+Bx_{3m-1}} = x_{3m+3}.$$

This completes the proof. □

3. Global behavior of Equation (1.1)

In this section, we introduce the forbidden set and determine the global behavior of Equation (1.1). Clear that, if $x_0 = 0$ and $x_{-1}x_{-2}x_{-3} \neq 0$, then x_4 is undefined. If $x_{-1} = 0$ and $x_0x_{-2}x_{-3} \neq 0$, then x_7 is undefined. If $x_{-2} = 0$ and $x_0x_{-1}x_{-3} \neq 0$, then x_6 is undefined. Finally, if $x_{-3} = 0$ and $x_0x_{-1}x_{-2} \neq 0$, then x_5 is undefined.
The following result provides the forbidden set of Equation (1.1).

Theorem 3.1. *The forbidden set of equation (1.1) as*

$$F = \bigcup_{i=0}^{3}\{(u_0,u_{-1},u_{-2},u_{-3}) \in \mathbb{R}^4 : u_{-i} = 0\} \cup \bigcup_{m=1}^{\infty}\{(u_0,u_{-1},u_{-2},u_{-3}) \in \mathbb{R}^4 : u_0 = -\frac{u_{-1}}{A}\frac{\theta_{m+1}}{\theta_m}\} \cup$$
$$\bigcup_{m=1}^{\infty}\{(u_0,u_{-1},u_{-2},u_{-3}) \in \mathbb{R}^4 : u_{-1} = -\frac{u_{-2}}{A}\frac{\theta_{m+1}}{\theta_m}\} \cup \bigcup_{m=1}^{\infty}\{(u_0,u_{-1},u_{-2},u_{-3}) \in \mathbb{R}^4 : u_{-2} = -\frac{u_{-3}}{A}\frac{\theta_{m+1}}{\theta_m}\}.$$

Theorem 3.2. *Assume that $\{x_n\}_{n=-3}^{\infty}$ is a well defined solution of Equation (1.1). Then the following statements are true:*

1. *If $A+B > 1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ converges to zero.*
2. *If $A+B < 1$, then the solution $\{x_n\}_{n=-3}^{\infty}$ is unbounded.*

Proof. We can write $\theta_j = \lambda_+^j \dfrac{(1-(\frac{\lambda_-}{\lambda_+})^j)}{\sqrt{B^2+4A}}$.

1. If $A+B > 1$, then $\lambda_+ > 1$. That is $\theta_m \to \infty$ as $m \to \infty$. This implies that

$$x_{3m} = \frac{\nu}{\mu_{-2}(m)\mu_{-1}(m)\mu_0(m)} \to 0 \text{ as } m \to \infty.$$

Similarly, we can show that

$$x_{3m+1} \to 0 \text{ as } m \to \infty \text{ and } x_{3m+2} \to 0 \text{ as } m \to \infty.$$

For (2), it is enough to note that $\lambda_+ < 1$ when $A+B < 1$. This completes the proof.

□

Theorem 3.3. *Assume that $A+B = 1$, then every well defined solution $\{x_n\}_{n=-3}^{\infty}$ of Equation (1.1) converges to a finite limit.*

Proof. When $A+B = 1$, we have $\lambda_+ = 1$. Then

$$\mu_{-j}(m) = Ax_{-j}\theta_m + x_{-j-1}\theta_{m+1} \to \frac{Ax_{-j}+x_{-j-1}}{\sqrt{B^2+4A}} \text{ as } m \to \infty,\ j \in \{0,1,2\}.$$

This implies that

$$x_{3m} = \frac{\nu}{\mu_{-2}(m)\mu_{-1}(m)\mu_0(m)} \to$$
$$\frac{\nu(B^2+4A)^{\frac{3}{2}}}{(Ax_0+x_{-1})(Ax_{-1}+x_{-2})(Ax_{-2}+x_{-3})} \text{ as } m \to \infty.$$

Similarly, we have that

$$x_{3m+1} \to \frac{\nu(B^2+4A)^{\frac{3}{2}}}{(Ax_0+x_{-1})(Ax_{-1}+x_{-2})(Ax_{-2}+x_{-3})} \text{ as } m \to \infty,$$

and

$$x_{3m+2} \to \frac{\nu(B^2+4A)^{\frac{3}{2}}}{(Ax_0+x_{-1})(Ax_{-1}+x_{-2})(Ax_{-2}+x_{-3})} \text{ as } m \to \infty.$$

Therefore, the solution $\{x_n\}_{n=-3}^{\infty}$ of Equation (1.1) converges to

$$\frac{\nu(B^2+4A)^{\frac{3}{2}}}{(Ax_0+x_{-1})(Ax_{-1}+x_{-2})(Ax_{-2}+x_{-3})} \text{ as } m \to \infty.$$

This completes the proof. □

Example (1) Figure 1. shows that, if $A = 0.2$, $B = 0.4$ $(A+B < 1)$, then a solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) with $x_{-3} = 3$, $x_{-2} = 2$, $x_{-1} = -1$ and $x_0 = 3$ is unbounded.
Example (2) Figure 2. shows that, if $A = 1.6$, $B = 0.3$ $(A+B > 1)$, then a solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) with $x_{-3} = 3$, $x_{-2} = 2$, $x_{-1} = -1$ and $x_0 = 3$ converges to zero.
Example (3) Figure 3. shows that, if $A = 0.62$, $B = 0.38$ $(A+B = 1)$, then a solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) with $x_{-3} = -1$, $x_{-2} = 1.2$, $x_{-1} = 2.5$ and $x_0 = 1.7$ converges to

$$\frac{\nu(B^2+4A)^{\frac{3}{2}}}{(Ax_0+x_{-1})(Ax_{-1}+x_{-2})(Ax_{-2}+x_{-3})} \simeq 8.666.$$

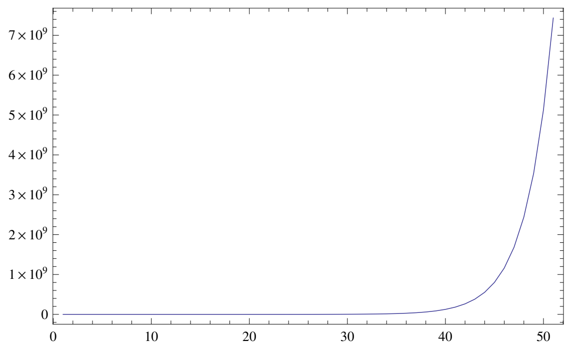

Figure 3.1: $x_{n+1} = \dfrac{x_n x_{n-3}}{0.2x_{n-2} + 0.4x_{n-3}}$

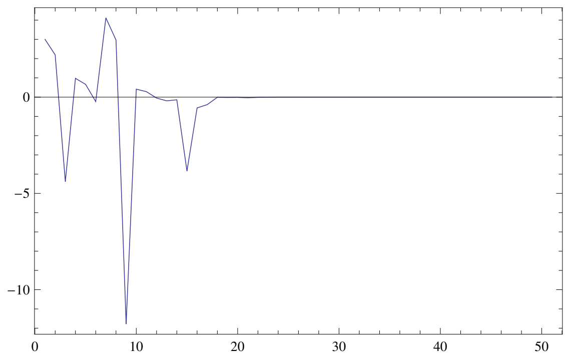

Figure 3.2: $x_{n+1} = \dfrac{x_n x_{n-3}}{1.6x_{n-2} + 0.3x_{n-3}}$

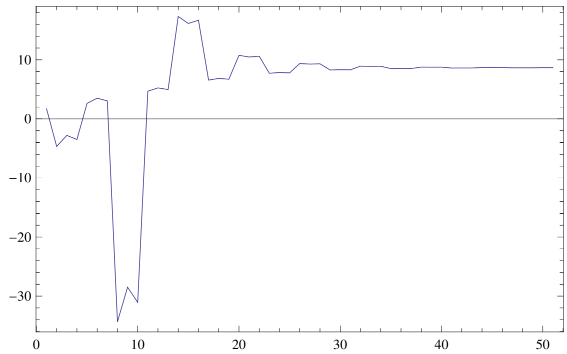

Figure 3.3: $x_{n+1} = \dfrac{x_n x_{n-3}}{0.62x_{n-2} + 0.38x_{n-3}}$

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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